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THE EXTENDED GENERALIZED INVERSE WEIBULL DISTRIBUTION AND ITS APPLICATIONS

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Abstract. We propose a new model called the odd log-logistic exponentiated inverse Weibull distribution which generalizes the exponentiated inverse Weibull distribution and other ten known and unknown lifetime models. Various properties of the new model are explored. The maximum likelihood method is used to estimate the model parameters. We compare the flexibility of the proposed model with other related distributions by means of two real data sets.

1. Introduction

A random variable (rv) Z has the generalized Inverse Weibull (GIW) distribution with three parameters β (power parameter), a and b if it has probability density function (pdf) and cdf given by

$$\pi_{(\beta,a,b)}(z)|_{(\beta,a,b>0)}^{(z\geq 0)} = \beta b a^b z^{-(b+1)} \mathrm{e}^{-\beta\left(\frac{a}{z}\right)^b}, \tag{1}$$

and

$$\Pi_{(\beta,a,b)}(z)\ |_{(\beta,a,b>0)}^{(z\geq 0)} = \mathrm{e}^{-\beta\left(\frac{a}{z}\right)^b}, \tag{2}$$

respectively, where a is a scale parameter, β and b is a shape parameters. For $\beta = 1$ we have the standard IW model also known as complementary Weibull or reciprocal Weibull (see [5] and [13]). For $\beta = 1$ and $b = 2$ we have the Inverse Rayleigh (IR). For $\beta = 1$ and $b = 1$ we have Inverse exponential (IE) and for $b = 2$ we have GIR and for $b = 1$ we have GIE (for more details about the GIW model see [8] and for more details about the IW distribution and its applications, see [11], [9], [15], [14], [16], [21], [3], [12], [2]).

[7] defined the cdf of the odd log logistic (OLL) family by

$$F_{(\theta,\boldsymbol{\psi})}(x)\ |_{(\theta>0)}^{(x>0)} = \frac{\Pi_{\boldsymbol{\psi}}(x)^{\theta}}{\Pi_{\boldsymbol{\psi}}(x)^{\theta} + \overline{\Pi}_{\boldsymbol{\psi}}(x)^{\theta}}, \tag{3}$$

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the corresponding pdf becomes

$$f_{(\theta,\boldsymbol{\psi})}(x)\ |_{(\theta>0)}^{(x>0)} = \theta \frac{\pi_{\boldsymbol{\psi}}(x)\left[\Pi_{\boldsymbol{\psi}}(x)\overline{\Pi}(x,\boldsymbol{\psi})\right]^{-1+\theta}}{\left[\Pi_{\boldsymbol{\psi}}(x)^{\theta}+\overline{\Pi}_{\boldsymbol{\psi}}(x)^{\theta}\right]^{2}}, \tag{4}$$

where θ is the shape parameter and $\boldsymbol{\psi}$ is the parameters vector of the baseline model. To this end, we use equations (1), (2), (3) and (4) to obtain the four parameter OLLGIW density in (5). A rv X is said to have the OLLGIW distribution if its cdf and pdf are given by

$$F_{(\theta,\beta,a,b)}(x)|_{(\theta,\beta,a,b>0)}^{(x\geq 0)} = \frac{\mathrm{e}^{-\beta\theta\left(\frac{a}{x}\right)^{b}}}{\mathrm{e}^{-\beta\theta\left(\frac{a}{x}\right)^{b}}+\left[1-\mathrm{e}^{-\beta\left(\frac{a}{x}\right)^{b}}\right]^{\theta}} \tag{5}$$

and

$$\begin{aligned} f_{(\theta,\beta,a,b)}(x)|_{(\theta,\beta,a,b>0)}^{(x\geq 0)} &= \theta\beta b a^{b} x^{-(b+1)}\mathrm{e}^{-\beta\left(\frac{a}{x}\right)^{b}} \\ &\quad\times\left\{\mathrm{e}^{-\beta\left(\frac{a}{x}\right)^{b}}\left[1-\mathrm{e}^{-\beta\left(\frac{a}{x}\right)^{b}}\right]\right\}^{-1+\theta} \\ &\quad\times\left\{\mathrm{e}^{-\beta\theta\left(\frac{a}{x}\right)^{b}}+\left[1-\mathrm{e}^{-\beta\left(\frac{a}{x}\right)^{b}}\right]^{\theta}\right\}^{-2}, \end{aligned} \tag{6}$$

respectively. Table 1 proves that our OLLGIW model generalizes eleven known and unknown lifetime models.

Table 1: Submodels of the OLLGIW distribution.

θ	β	a	b	Reduced model	Author
θ	β	a	1	OLLGIE	New
θ	β	a	2	OLLGIR	New
θ	1	a	b	OLLIW	[19]
θ	1	a	1	OLLIE	[19]
θ	1	a	2	OLLIR	[19]
1	β	a	b	GIW	[8]
1	β	a	1	GIE	[15]
1	β	a	2	GIR	[15]
1	1	a	b	IW	[6]
1	1	a	1	IE	[10]
1	1	a	2	IR	[18]

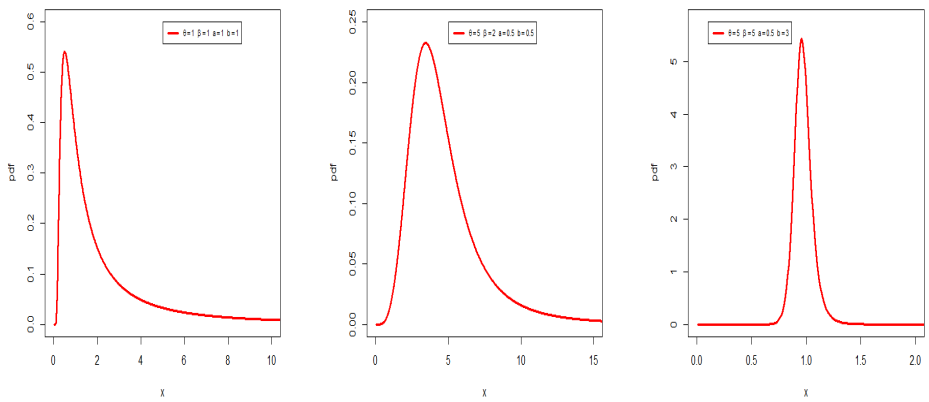

Figure 1: Plots of the OLLGIW pdf.

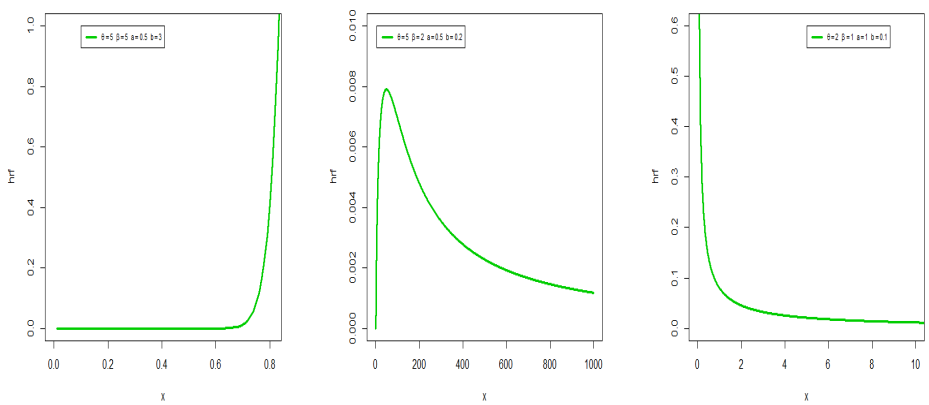

Figure 2: Plots of the OLLGIW hrf.

The main justification for the practicability of OLLGIW model is based on introducing a new flexible extensions of GIW distribution with four parameters, we are motivated to introduce the OLLGIW distribution because it contains a number aforementioned of known lifetime sub models like OLLEIE, OLLEIR, OLLIW, OLLIE, OLLIR, GIW, IW, EIE, EIR, IE and IR (see Table 1). Also it exhibits the increasing, upside-down and decreasing hrfs (see Figure 2). The OLLGIW distribution can be expressed as a double linear mixture of IW density (see the mixture representation). The OLLGIW distribution can be viewed as a good model for fitting the symmetric, right-skewed and unimodal data. The OLLGIW distribution outperforms several of the well known distributions with respect to two real data applications (see Section 4).

The rest of the paper is organized as follows: In Section 2, we introduce some of mathematical properties for the new model. In Section 3, the maximum likelihood method is discussed to estimate the model parameters. Two applications to real

data sets prove empirically the importance of the new model in Section 4. Finally, some conclusions are given in Section 5.

2. Mathematical Properties

2.1. **Useful representation.** We provide a very useful linear representation for the OLLGIW density function. First, we use a power series for the quantity

$$\left[\mathrm{e}^{-\beta\left(\frac{a}{x}\right)^b}\right]^\theta |_{(\theta>0 \text{ real})}$$

given by

$$\left[\mathrm{e}^{-\beta\left(\frac{a}{x}\right)^b}\right]^\theta = \sum_{k=0}^{\infty} a_k \left[\mathrm{e}^{-\beta\left(\frac{a}{x}\right)^b}\right]^k, \tag{7}$$

where

$$a_k = \sum_{j=k}^{\infty} (-1)^{k+j} \binom{\theta}{j}\binom{j}{k}.$$

For any real $\theta > 0$, we consider the generalized binomial expansion

$$\left[1-\mathrm{e}^{-\beta\left(\frac{a}{x}\right)^b}\right]^\theta = \sum_{k=0}^{\infty} (-1)^k \binom{\theta}{k} \left[\mathrm{e}^{-\beta\left(\frac{a}{x}\right)^b}\right]^k. \tag{8}$$

Inserting (7) and (8) in equation (5), we obtain

$$F(x) = \frac{\sum\limits_{k=0}^{\infty} a_k \left[\mathrm{e}^{-\beta\left(\frac{a}{x}\right)^b}\right]^k}{\sum\limits_{k=0}^{\infty} b_k \left[\mathrm{e}^{-\beta\left(\frac{a}{x}\right)^b}\right]^k},$$

where

$$b_k = a_k + (-1)^k \binom{\theta}{k},$$

the ratio of the two power series can be expressed as

$$F(x) = \sum_{k=0}^{\infty} \zeta_k \left[\mathrm{e}^{-\beta\left(\frac{a}{x}\right)^b}\right]^k = \sum_{k=0}^{\infty} \zeta_k \Pi_{(k\beta,a,b)}(x), \tag{9}$$

where

$$\Pi_{(k\beta,a,b)}(x) = \left[\Pi_{(\beta,a,b)}(x)\right]^k = \mathrm{e}^{-k\beta\left(\frac{a}{z}\right)^b}$$

is the IW cdf with scale parameter $a\,(k\beta)^{\frac{1}{b}}$ and shape parameter b, and the coefficients ζ_k's (for $k \geq 0$) are determined from the recurrence equation

$$\zeta_k = \frac{1}{b_0}\left(a_k + \frac{1}{b_0}\sum_{r=1}^{k} b_r\ \zeta_{k-r}\right).$$

By differentiating (9), the pdf of X can be expressed as

$$f(x) = \sum_{k=0}^{\infty} \zeta_{1+k} \pi_{((1+k)\beta,a,b)}(x), \tag{10}$$

where

$$\pi_{((1+k)\beta,a,b)}(x) = (1+k)\,\beta b a^b z^{-(b+1)} \mathrm{e}^{-(1+k)\beta\left(\frac{a}{z}\right)^b}$$

is the IW density with scale parameter $a\left[(1+k)\beta\right]^{\frac{1}{b}}$ and shape parameter b. Thus, the OLLGIW density can be expressed as a double linear mixture of IW density. Then, several of its structural properties can be obtained from Equation (10) and those properties of the IW distribution.

2.2. **Moments.** The nth ordinary moment of X is given by

$$\mu_n' = \mathbf{E}(X^r) = \sum_{k=0}^{\infty} \zeta_{1+k} \int_{-\infty}^{\infty} x^n \, \pi_{((1+k)\beta,a,b)}(x) dx,$$

then we obtain

$$\mu_n' = \sum_{k=0}^{\infty} \zeta_{1+k} a^n \left[(1+k)\beta\right]^{\frac{n}{b}} \Gamma\left(1-\frac{n}{b}\right)|_{(n<b)}, \tag{11}$$

where

$$\Gamma\left(1+\boldsymbol{\xi}\right)|_{(\boldsymbol{\xi}\in\mathbb{R}^+)} = \boldsymbol{\xi}! = \prod_{w=0}^{\boldsymbol{\xi}-1} \left(\boldsymbol{\xi}-w\right).$$

by setting $r = 1$ in (11), we get the mean of X. The skewness (Ske(X)) and kurtosis (Kur(X)) measures can be calculated using (11) using and the well-known relationships. The mean ($E\left(X\right)$), variance (Var(X)), skewness and kurtosis of the new distribution are computed numerically for some selected values of parameter β, θ, a and b using the R software:

1-The skewness of the OLLGIW distribution can range in the interval $(-671.68, 2.9)$.

2-The kurtosis of the OLLGIW distribution varies in the interval $(-86, 56.38)$.

Table 2: Mean, variance, skewness and kurtosis of the OLLGIW distribution.

β	θ	a	b	$E(X)$	$\mathrm{Var}(X)$	$\mathrm{Ske}(X)$	$\mathrm{Kur}(X)$
100	1	1	1	1.442856	0.000357	0.0821314	4.21690
80				1.442947	0.0005573	0.1027041	4.22650
60				1.443143	0.0009916	0.1370542	4.247268
50				1.443341	0.0014290	0.1646038	4.26833
40				1.443704	0.0022363	0.2060811	4.30716
20				1.446741	0.0090596	0.4176729	4.644949
10				1.459027	0.0381571	0.8833694	6.293185
5				1.510446	0.1899747	2.347414	27.1085
4.5				1.527314	0.2523226	**2.935822**	**56.3756**
5	0.01	1.5	1.25	0.05209	0.000136	1.730015	**−86.00897**
	0.1			0.32871	0.005427	1.730015	13.64197
	0.5			1.191212	0.071275	1.730015	13.64197
	1			2.07402	0.2160653	1.730015	13.64197
	5			7.516038	2.837508	1.730015	13.64197
	10			13.08618	8.601714	1.730015	13.64196
	20			22.78437	26.07552	1.730015	13.64197
	40			39.66989	79.0462	1.730015	13.64197
	50			47.42299	112.9632	1.730015	13.64196
	100			82.56823	342.4404	1.730015	13.64197
	200			143.7596	1038.085	1.730015	13.64197
	500			299.2189	4497.146	1.730015	13.64197
	1000			520.9703	13632.8	1.730015	13.64197
10	1.25	0.001	1.25	$2.76692\times e^{-5}$	$1.61537\times e^{-6}$	1.619803	2.694399
		0.01		0.01615	$2.95362\times e^{-6}$	**−671.681**	18323.93
		0.1		0.1615	0.0002954	0.7300936	5.589085
		1		1.615	0.0295353	0.7300817	5.589259
		5		8.075002	0.7383819	0.730082	5.589256
		10		16.15	2.953528	0.730082	5.589256
		20		32.30001	11.81411	0.730082	5.589255
		50		80.75002	73.83819	0.730082	5.589256
		100		161.5	295.3528	0.730082	5.589256
		500		807.5002	7383.819	0.730082	5.589256
4.5	2.25	2	1	6.872914	5.109533	2.93582	56.37563
			2	3.665635	0.308949	1.244699	8.321656
			5	2.541821	0.022395	0.7009007	5.260133
			10	2.253732	0.004336	0.5477181	4.752984
			20	2.122854	0.00095544	0.4745802	4.556885

2.3. **Moment generating function.** Here, we provide two formulae for the moment generating function (mgf) $M_X(t) = \mathbf{E}\left(e^{t\,X}\right)$ of X. Clearly, the first one can be derived from using (11) as

$$\begin{aligned} M_X(t) &= \sum_{k=0}^{\infty} \zeta_{1+k}\, M_{1+k}(t) \\ &= \sum_{k,n=0}^{\infty} \zeta_{1+k} \frac{t^n a^n}{n!} \left[(1+k)\,\beta\right]^{\frac{n}{b}} \Gamma\left(1-\frac{n}{b}\right)|_{(n<b)}, \end{aligned}$$

As for the second formula for $M_X(t)$, setting $y = x^{-1}$ in (1), we can write this mgf as

$$M(t;a,b) = ba^b \int_0^{\infty} \mathrm{e}^{\frac{t}{y}}\, y^{(b-1)}\, \mathrm{e}^{-(ay)^b}.$$

By expanding the first exponential and calculating the integral, we have

$$M(t;a,b) = ba^b \int_0^{\infty} \sum_{m=0}^{\infty} \frac{t^m}{m!} \mathrm{e}^{\frac{t}{y}}\, y^{b-m-1}\, \mathrm{e}^{-(ay)^b} = \sum_{m=0}^{\infty} \frac{a^m\, t^m}{m!} \Gamma\left(\frac{b-m}{b}\right),$$

where the gamma function is well-defined for any non-integer b. Consider the Wright generalized hypergeometric function (Wright (1935)) defined by

$${}_p\mathbf{\Psi}_q\left[\begin{array}{c} (\alpha_1, A_1), \ldots, (\alpha_p, A_p) \\ (\beta_1, B_1), \ldots, (\beta_q, B_q) \end{array}; x\right] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} \Gamma\left(\alpha_j + A_j\, n\right)}{\prod_{j=1}^{q} \Gamma\left(\beta_j + B_j\, n\right)} \frac{x^n}{n!}.$$

Then, we have

$$M(t;a,b) = {}_1\mathbf{\Psi}_0\left[\begin{array}{c} \left(1, -\frac{1}{b}\right) \\ - \end{array}; a\,t\right]. \tag{12}$$

Combining expressions (10) and (12), we obtain the mgf of X, say $M(t)$, as

$$M(t) = \sum_{k=0}^{\infty} \zeta_{1+k}\, {}_1\mathbf{\Psi}_0\left[\begin{array}{c} \left(1, -\frac{1}{b}\right) \\ - \end{array}; at\left[(1+k)\,\beta\right]^{\frac{1}{b}}\right].$$

2.4. **Incomplete moment.** The sth incomplete moment, say $I_s(t)$, of X can be expressed from (10), for $n < b$, as

$$\begin{aligned} I_n(t) &= \sum_{k=0}^{\infty} \zeta_{1+k} \int_{-\infty}^{t} x^n\, \pi_{1+k}(x)\, dx \\ &= \sum_{k=0}^{\infty} \zeta_{1+k} a^n \left[(1+k)\,\beta\right]^{\frac{n}{b}} \gamma\left(1-\frac{n}{b}, (1+k)\left(\frac{a}{t}\right)^b\right)|_{(n<b)}. \end{aligned}$$

where

$$\begin{aligned}\gamma(\xi,p) &= \int_0^p t^{\xi-1}\mathrm{e}^{-t}dt = \left\{{}_{[1]}\mathbf{F}_{[1]}\left[\xi;\xi+1;-p\right]\right\}\frac{p^{\xi}}{\xi}\\ &= \sum_{k=0}^{\infty}\frac{(-1)^k p^{\xi+k}}{k!\,(\xi+k)}|_{(\xi\neq 0.-1,-2,\ldots)},\end{aligned}$$

denotes the complementary lower incomplete gamma function and ${}_{[1]}\mathbf{F}_{[1]}[\cdot]$ is a confluent hypergeometric function which can be evaluated by statistical software like R.

The mean deviations about the mean

$$m_1 = \mathbf{E}(|X-\mathbf{E}(X)|) = 2\mu_1^{'}F(\mathbf{E}(X)) - 2I_1(\mathbf{E}(X))$$

and about the median

$$m_2 = \mathbf{E}\left(\left|X - Q\left(\frac{1}{2}\right)\right|\right) = \mathbf{E}(X) - 2I_1\left(Q\left(\frac{1}{2}\right)\right)$$

where $Q\left(\frac{1}{2}\right)$ =Median(X) is the median, $F(\mathbf{E}(X))$ is easily calculated from (5) and $I_1(t)$ is the first incomplete moment given by the last Equation with $n=1$. The general formula for $I_1(t)$ can be obtained from $I_s(t)$ as

$$I_1(t) = \sum_{k=0}^{\infty}\zeta_{1+k}a\left[(1+k)\beta\right]^{\frac{1}{b}}\gamma\left(1-\frac{1}{b},\left[(1+k)\beta\right]\left(\frac{a}{t}\right)^b\right).$$

2.5. **The nth moment of the residual life.** The nth moment of the residual life is given by

$$m_n(t)\ \ |_{(X>t)}^{(n=1,2,\ldots)} = \mathbf{E}[(X-t)^n]\ ,$$

the nth moment of the residual life of X can be expressed as

$$m_n(t)\ |_{(X>t)}^{(n=1,2,\ldots)} = \frac{1}{1-F(t)}\int_t^{\infty}(x-t)^n dF(x),$$

therefore

$$m_n(t) = \frac{a^n}{1-F(t)}\sum_{k=0}^{\infty}\zeta_{1+k}^{(m_n)}\left[(1+k)\beta\right]^{\frac{n}{b}}\Gamma\left(1-\frac{n}{b},\left[(1+k)\beta\right]\left(\frac{a}{t}\right)^b\right)|_{(\ n<b)},$$

where

$$\zeta_{1+k}^{(m_n)} = \zeta_{1+k}\sum_{r=0}^{n}\binom{n}{r}(-t)^{n-r},$$

and

$$\begin{aligned}\Gamma(\xi,p)|_{(p>0)} &= \int_0^p t^{\xi-1}\mathrm{e}^{-t}dt = \Gamma(\xi) - \gamma(\xi,p)\\ &\sim \frac{p^{\xi-1}}{\mathrm{e}^p}\left[1+\frac{\xi-1}{p}+\frac{(\xi-1)(\xi-2)}{p^2}+\ldots\right],\end{aligned}$$

denotes the complementary upper incomplete gamma function.

2.6. **The nth moment of the reversed residual life.** The nth moment of the reversed residual life can be expressed as

$$M_n(t)\mid_{(X\leq t,\ t>0)}^{(n=1,2,\ldots)}=\mathbf{E}\left[(t-X)^n\right].$$

We obtain

$$M_n(t)\mid_{(X\leq t,\ t>0)}^{(n=1,2,\ldots)}=\frac{1}{F(t)}\int_0^t (t-x)^n dF(x).$$

Then, the nth moment of the reversed residual life of X becomes

$$M_n(t)=\frac{a^n}{F(t)}\sum_{k=0}^{\infty}\zeta_{1+k}^{(M_n)}\left[(1+k)\beta\right]^{\frac{n}{b}}\gamma\left(1-\frac{n}{b},\left[(1+k)\beta\right]\left(\frac{a}{t}\right)^b\right)\mid_{(n<b)},$$

where

$$\zeta_{1+k}^{(M_n)}=\zeta_{1+k}\sum_{r=0}^{n}(-1)^r\binom{n}{r}t^{n-r}.$$

3. Estimation

If X follows the OLLGIW distribution with vector of parameters $\mathbf{\Phi}=(\theta,\beta,a,b)^T$, the log-likelihood for $\mathbf{\Phi}$ from a single observation x of X is given by

$$\begin{aligned}\ell(\mathbf{\Phi}) &= \log(\theta)+\log(\beta)+\log(b)+b\log(a)-(b+1)\log(x)+\log s\\ &\quad+(-1+\theta)\log\left[s(-s+1)\right]-2\log\left[s^\theta+(-s+1)^\theta\right],\end{aligned}$$

where $s=\mathrm{e}^{-\beta\left(\frac{a}{x}\right)^b}$. The components of the unit score vector

$$\mathbf{I}=\mathbf{I}(\mathbf{\Phi})=(\partial\theta/\partial\ell,\partial\beta/\partial\ell,\partial a/\partial\ell,\partial b/\partial\ell)^T=\left(\mathbf{I}_{(\theta)},\mathbf{I}_{(\beta)},\mathbf{I}_{(a)},\mathbf{I}_{(b)}\right)^T$$

are given by

$$\mathbf{I}_{(\theta)}=\frac{1}{\theta}+\log\left[s(-s+1)\right]-2\frac{s^\theta\log(s)+(-s+1)^\theta\log(-s+1)}{s^\theta+(-s+1)^\theta},$$

$$\mathbf{I}_{(\beta)}=\frac{1}{\beta}+\frac{p}{s}-(-1+\theta)\frac{sp}{s(-s+1)}-2\frac{\theta ps^{-1+\theta}-\theta p(-s+1)^{-1+\theta}}{s^\theta+(-s+1)^\theta}$$

$$\mathbf{I}_{(a)}=\frac{b}{a}+\frac{w}{s}+(-1+\theta)\frac{w-2ws}{s(-s+1)}-2\frac{\theta ws^{-1+\theta}-\theta w(-s+1)^{-1+\theta}}{s^\theta+(-s+1)^\theta}$$

and

$$\mathbf{I}_{(b)}=\frac{1}{b}+\log(a)-\log(x)+\frac{q}{s}+(-1+\theta)\frac{q-2qs}{s(-s+1)}-2\frac{\theta qs^{-1+\theta}-\theta q(-s+1)^{-1+\theta}}{s^\theta+(-s+1)^\theta},$$

where

$$w=-\beta ba^{b-1}x^{-b}\mathrm{e}^{-\beta\left(\frac{a}{x}\right)^b},$$

$$q=-\left(\frac{a}{x}\right)^b\mathrm{e}^{-\left(\frac{a}{x}\right)^b}\log\left(\frac{a}{x}\right),$$

and

$$p=-\left(\frac{a}{x}\right)^b\mathrm{e}^{-\beta\left(\frac{a}{x}\right)^b}.$$

For a random sample $x = (x_1, ..., x_n)^T$ of size n from X, the total log-likelihood is

$$\ell_n(\mathbf{\Phi}) = \sum_{i=0}^{n} \ell^{(i)}(\mathbf{\Phi}),$$

where $\ell^{(i)}(\mathbf{\Phi})$ is the log-likelihood for the ith observation. The total score function is

$$\mathbf{I}_n = \sum_{i=0}^{n} \mathbf{I}^{(i)},$$

where $\mathbf{I}^{(i)}$ has the form given before. Maximization of $\ell(\mathbf{\Phi})$ (or $\ell_n(\mathbf{\Phi})$) can be easily performed using well-established routines such as the nlm or optimize in the R statistical package. Setting these equations to zero, $U(\mathbf{\Phi}) = 0$, and solving them simultaneously gives the MLE $\widehat{\mathbf{\Phi}}$ b of $\mathbf{\Phi}$. These equations cannot be solved analytically and statistical software can be used to evaluate them numerically using iterative techniques such as the Newton-Raphson algorithm.

4. Applications

In this section we provide two applications of the OLLGIW distribution using two real data sets. For the 1st application we shall compare the OLLGIW distribution with related models namely: the Marshall-Olkin IW (MOIW), Kumaraswamy IW (KwIW), beta IW (BIW), Kumaraswamy Marshall-Olkin Inverse exponential (KwMOIE), Kumaraswamy Marshall-Olkin Inverse Rayleigh (KwMOIR) and IW distributions. For the 2nd application we shall compare the OLLGIW distribution with related models namely: the MOIW, BIW, KwMOIR and IW distributions.

The first data set from [4] which consists of 72 observations of survival times guinea pigs injected with different doses of tubercle bacilli: 12, 15, 22, 24, 24, 32, 32, 33, 34, 38, 38, 43, 44, 48, 52, 53, 54, 54, 55, 56, 57, 58, 58, 59, 60, 60, 60, 60, 61, 62, 63, 65, 65, 67, 68, 70, 70, 72, 73, 75, 76, 76, 81, 83, 84, 85, 87, 91, 95, 96, 98, 99, 109, 110, 121, 127, 129, 131, 143, 146, 146, 175, 175, 211, 233, 258, 258, 263, 297, 341, 341, 376. The total time test (TTT) plot (see [1]) for the1^{st} real data sets is presented in Figure 3. This plot indicates that the empirical HRF of 1st data sets

is upside down then increasing.

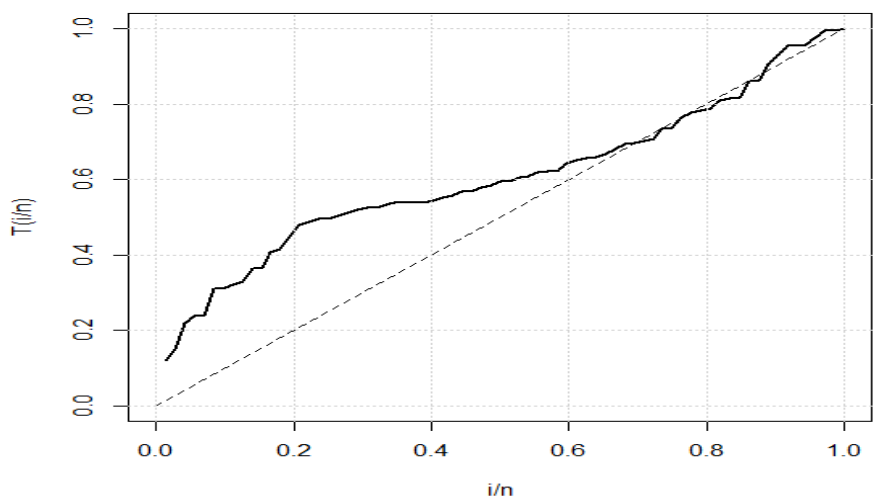

Figure 3: TTT plot for the 1st real data.

The second data set is obtained from [17]. The data are the strengths of 1.5 cm glass fibres, measured at the National Physical Laboratory, England. Unfortunately, the units of measurement are not given in the paper. The data set consisting of 63 observations are: 0.55, 0.93,1.25, 1.36, 1.49, 1.52, 1.58, 1.61, 1.64, 1.68, 1.73, 1.81, 2 , 0.74, 1.04, 1.27, 1.39, 1.49, 1.53,1.59, 1. 61, 1.66, 1.68, 1.76, 1.82, 2.01, 0.77, 1.11, 1.28, 1.42, 1.5, 1.54, 1.6, 1.62, 1.66, 1.69,1.76, 1.84, 2.24, 0.81, 1.13, 1.29, 1.48, 1.5, 1.55, 1.61, 1.62, 1.66, 1.7, 1.77, 1.84, 0.84, 1.24, 1.3, 1.48, 1.51, 1.55, 1.61, 1.63, 1.67, 1.7, 1.78, 1.89. TheTTT plot for the 2nd real data sets is presented in Figure

4. This plot indicates that the empirical HRF of 2nd data is increasing.

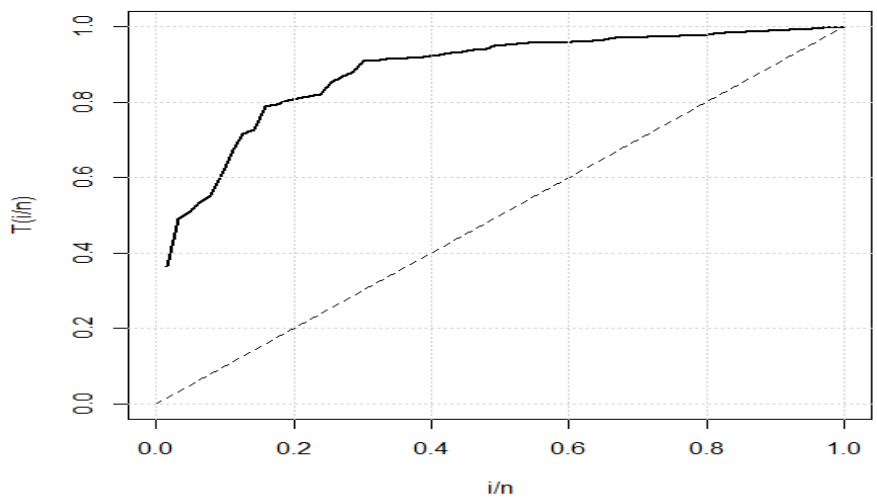

Figure 4: TTT plot for the 2nd real data.

The 1st data were previously studied by Krishna et al. (2013) for MOIW, BIW, and IW distributions. The 2nd data were previously studied by Barreto-Souza et. al. (2011) for BIW and IW distributions. In order to compare the distributions, we consider some criteria like $-2\widehat{\ell}$ (Maximized Log-likelihood), AI_C (Akaike Information Criterion), CAI_C (the consistent Akaike Information Criterion), BI_C (Bayesian information criterion) and HQI_C (Hannan-Quinn information) criterion for the real data set. The model with minimum AI_C or BI_C or HQI_C or CAI_C value is chosen as the best model to fit the data, where

$$AI_C = 2\left(-\widehat{\ell} + k\right),$$

$$BI_C = 2\left[-\widehat{\ell} + \frac{1}{2}k\log(n)\right],$$

$$HQI_C = 2\left\{-\widehat{\ell} + k\log\left[\log(n)\right]\right\},$$

and

$$CAI_C = 2\left[-\widehat{\ell} + kn/(n-k-1)\right].$$

where $\widehat{\ell}$ denotes the log-likelihood function evaluated at the maximum likelihood estimates, k is the number of parameters and n is the sample size. Tables 3 and 5 list the MLEs and their corresponding standard errors (in parentheses) of the model parameters, whilst the numerical values of $-2\widehat{\ell}, AI_C, BI_C, HQI_C$ and CAI_C are listed in Tables 4 and 6, respectively. These numerical results are obtained using R software. Figure 5 and 6 gives the fitted pdf, cdf, P-P plot and estimated hrf and for the two data sets respectively. These Figures indicates that the new model gives the adequate fit to the used data sets.

Table 3: MLEs and their standard errors (in parentheses) for the 1st data.

Model	Estimates				
	$\widehat{\theta}$	$\widehat{\beta}$	$\widehat{\delta}$	$\widehat{a}$	$\widehat{b}$
OLLGIW	4.7989	1.3108		13.990	0.38
	(5.1585)	(1.889)		(55.23)	(0.404)
KwIW		0.6207	0.7111	45.7326	8.2723
		(0.003)	(0.013)	(0.092)	(0.979)
BIW		0.322	24.5032	19.9786	20.1331
		(0.00115)	(0.087)	(7.246)	(7.26)
KwMOIE	8.8727		0.1758	68.1393	2.6258
	(1.174)		(0.000)	(0.020)	(0.512)
IW		1.4148	54.1888		
		(0.00271)	(0.111)		
MOIW	14.9816	1.7855	13.991		
	(4.6305)	(0.193)	(2.964)		
KwMOIR	9.993		1.6788	58.4697	0.6389
	(1.972)		(0.001)	(0.105)	(0.098)

Table 4: $-2\widehat{\ell}, AI_C, BI_C, HQI_C$ and CAI_C for the 1st data.

Model	$-2\widehat{\ell}$	AI_C	BI_C	HQI_C	CAI_C
OLLGIW	**779.2**	**787.4**	**796.5**	**791**	**788**
Kw-IW	780.5	788.5	797.6	792.1	789.1
BIW	780.6	788.6	797.7	792.3	789.2
KwMOIE	782.7	790.7	799.8	794.3	791.3
IW	791.3	795.3	799.9	797.1	795.5
MOIW	790.1	796.1	802.9	798.8	796.5
KwMOIR	800.2	808.2	817.3	811.8	808.8

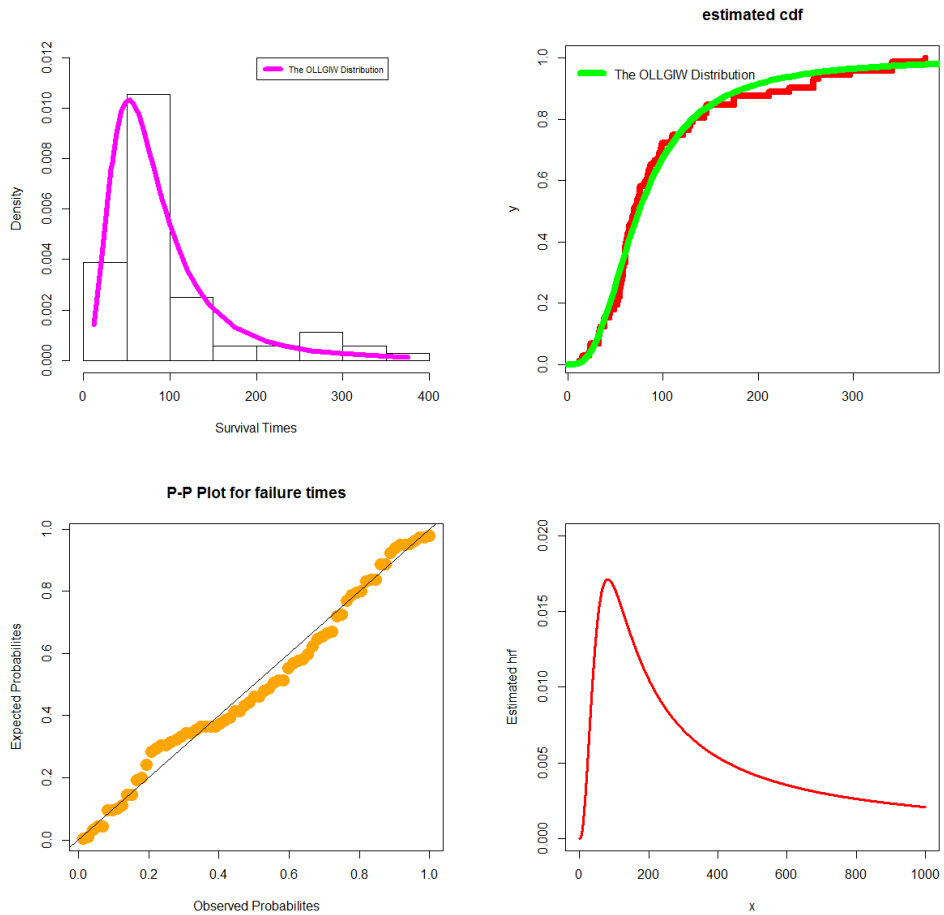

Figure 5: The fitted pdf, cdf, P-P plot and estimated hrf and for the first data set.

Table 5: MLEs and their standard errors (in parentheses) for the 2nd data.

Model	Estimates				
	$\widehat{\theta}$	$\widehat{\beta}$	$\widehat{\delta}$	$\widehat{a}$	$\widehat{b}$
OLLGIW	28.31	0.604		3.068	0.197
	(17.17)	(0.201)		(4.689)	(0.118)
BF		0.685	1.331	19.591	30.411
		(0.181)	(1.085)	(18.115)	(18.238)
Kw-MOIR	1		2.7498	0.5971	5.7974
	(0.192)		(0.079)	(0.034)	(0.0082)
F		2.888	1.264		
		(0.234)	(0.059)		
MOF	0.4816	2.3876	1.5441		
	(0.252)	(0.253)	(0.226)		

Table 6: $-2\widehat{\ell}, AI_C, BI_C, HQI_C$ and CAI_C for 2nd data.

Model	$-2\widehat{\ell}$	AI_C	BI_C	HQI_C	CAI_C
OLLGIW	**46.8**	**54.7**	**63.3**	**58.1**	**55.4**
BIW	61.7	69.7	78.3	73.1	70.4
Kw-MOIR	67.3	75.3	83.9	78.7	76
IW	93.7	97.7	102	99.4	97.9
MOIW	95.7	101.7	108.2	104.2	102.1

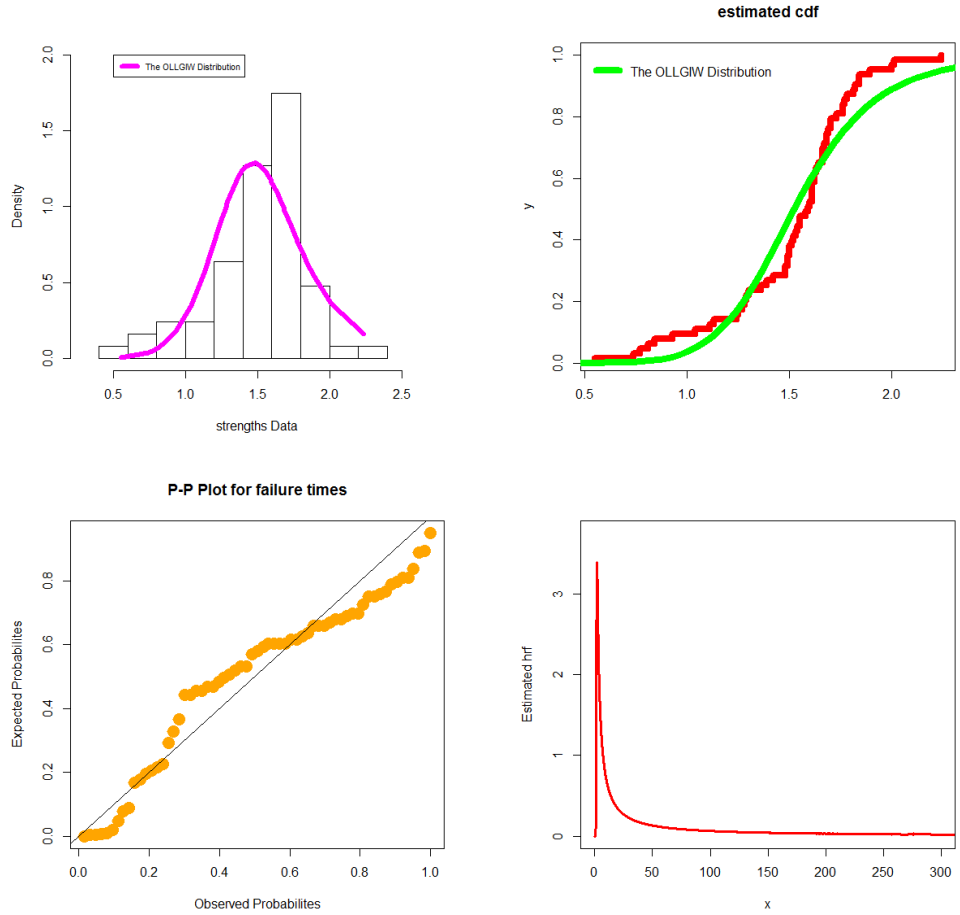

Figure 6: The fitted pdf, cdf, P-P plot and estimated hrf and for the second data set.

Tables 4 and 6 compares the OLLGIW model with other extensions of IW distribution. We note that the OLLGIW model gives the lowest values for the AI_C, BI_C, HQI_C and CAI_C statistics among all fitted models. So, the OLLGIW model could be chosen as the best model.

5. Concluding remarks

We propose a new model called the odd log-logistic generalized Inverse Weibull distribution which generalizes the generalized Inverse Weibull distribution and other ten known and unknown lifetime models. Various properties of the new model are explored. The maximum likelihood method is used to estimate the model parameters. We compare the flexibility of the proposed model with other related distributions by means of two real data sets. The skewness of the OLLGIW distribution can range in the interval $(-671.68, 2.9)$ and the kurtosis of the OLLGIW distribution varies in the interval $(-86, 56.38)$ which indicates to the wide flexibility of the new model.

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