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A NEW EXTENSION OF THE GENERALIZED EXPONENTIAL DISTRIBUTION WITH PROPERTIES AND APPLICATIONS TO FAILURE AND SURVIVAL TIMES

T. H. M. ABOUELMAGD

Abstract. This paper aims to introduce a new extension of the exponentiated exponential distribution called the odd Lindley exponentiated exponential which exhibits the monotonically increasing, bathtub, constant, the monotonically decreasing hazard rates. A range of mathematical properties of new model is derived. The proposed model can be viewed as a mixture of the exponentiated exponential density. The new model can also be considered as a convenient model for fitting the symmetric, the left skewed, the right skewed, and the unimodal data sets. The proposed lifetime model is much better than the exponential, moment exponential, log Butr Hatke exponential, the two parameter odd Lindley exponential, the one parameter odd Lindley exponential, two parameter odd Lindley exponentiated exponential and the Burr X exponential models so the new lifetime model is a good alternative to those models in modeling failure and survival times data sets.

1. Introduction

The most popular continuous distributions which used for modeling lifetime data are the gamma (G), the Weibull (W), exponentiated exponential (EE) which also called the generalized exponential (GE) and lognormal (Log-N) distributions. However, these four models suffer from some serious drawbacks. one of them, none these four models exhibit the bathtub shapes for their hazard rate functions (HRFs), those four models exhibit only monotonic decreasing, monotonic increasing or constant hazard rates and this is a major weakness point because most real-life systems exhibit bathtub shapes for their HRFs. Secondly, at least three of those four models exhibit the constant hazard rates, and this is a very unrealistic feature because there are hardly any real-life systems that have constant hazard rates. The aim of this work is to introduce a new three parameter substitutional to the EE distribution that overcomes these mentioned drawbacks and exhibits the monotonically increasing, bathtub and monotonically decreasing shapes for its HRF. The main goal of the work is to introduce a new EE model using the Odd Lindley (OL-G) family whish exhibits the monotonically increasing, bathtub, constant, the monotonically

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decreasing hazard rates.

A continuous random variable (RV) T is said to have the GE distribution (see [3]) if its probability density function (PDF) given by

$$g\left(t;\alpha,\beta\right)=\alpha\beta\ \left(1-\mathrm{e}^{-\beta t}\right)^{\alpha-1}\mathrm{e}^{-\beta t},t>0 \tag{1}$$

and cumulative distribution function (CDF)

$$G\left(t;\alpha,\beta\right)=\left(1-\mathrm{e}^{-\beta t}\right)^{\alpha}, \tag{2}$$

respectively, when $\alpha=1$, we have the standard E model. The PDF and CDF of the OL-G family of distribution (see Silva et al. (2017)) are given by

$$f(x;a,\boldsymbol{\psi})=a^{2}\left(1+a\right)^{-1}\exp\left[-aG\left(x;\boldsymbol{\psi}\right)/\overline{G}\left(x;\boldsymbol{\psi}\right)\right]g\left(x;\boldsymbol{\psi}\right)\overline{G}\left(x;\boldsymbol{\psi}\right)^{-3}, \tag{3}$$

and

$$F(x;\alpha,\boldsymbol{\psi})=1-\left(1+a\right)^{-1}\left[a+\overline{G}\left(x;\boldsymbol{\psi}\right)\right]\exp\left[-aG\left(x;\boldsymbol{\psi}\right)/\overline{G}\left(x;\boldsymbol{\psi}\right)\right]\overline{G}\left(x;\boldsymbol{\psi}\right)^{-1}, \tag{4}$$

respectively. To this end, we use equations (1), (2) and (3) to obtain the three-parameter OLGE density (5), a RV X is said to have the OLGE distribution if its PDF and CDF are given by

$$f(x)=a^{2}\alpha\beta\ \frac{\left(1-\mathrm{e}^{-\beta x}\right)^{\alpha-1}\mathrm{e}^{\left\{-a\left(1-\mathrm{e}^{-\beta x}\right)^{\alpha}/\left[-\left(1-\mathrm{e}^{-\beta x}\right)^{\alpha}+1\right]\right\}}}{\left(1+a\right)\mathrm{e}^{\beta x}\left[1-\left(1-\mathrm{e}^{-\beta x}\right)^{\alpha}\right]^{3}},x\geq 0, \tag{5}$$

and

$$F(x)=1-\frac{a+\left[1-\left(1-\mathrm{e}^{-\beta x}\right)^{\alpha}\right]}{\left[1-\left(1-\mathrm{e}^{-\beta x}\right)^{\alpha}\right]\left(1+a\right)}\mathrm{e}^{\left\{-a\left(-\mathrm{e}^{-\beta x}+1\right)^{\alpha}/\left[1-\left(1-\mathrm{e}^{-\beta x}\right)^{\alpha}\right]\right\}},x\geq 0, \tag{6}$$

respectively. The following Table provides some submodels of the OLGE model.

Table 1: Some submodels of the OLGE model

Model	a	α	β	Author
Two parameter OLGE	1	α	β	**New**
Another two parameter OLGE	a	α	1	**New**
One parameter OLGE	1	α	1	**New**
OLE	a	1	β	Silva et al. (2017)
OLE	1	1	β	Silva et al. (2017)

The graphical presentation and justification of the new model are presented in Section 2. Section 3 presents some of the mathematical properties of the new distribution. Method of Maximum likelihood estimation for the model parameters addressed in Section 5. In Section 6, the potentiality of the proposed model is illustrated by means of a two data sets. Section 7 provides some remarks.

2. Graphical presentation and JUSTIFICATION

From Figure 1 the PDF of the OLGE model exhibits various important unimodal shapes, from Figure 2 the HRF of the OLGE distribution exhibits the monotonically increasing, bathtub, constant and the monotonically decreasing hazard rates.

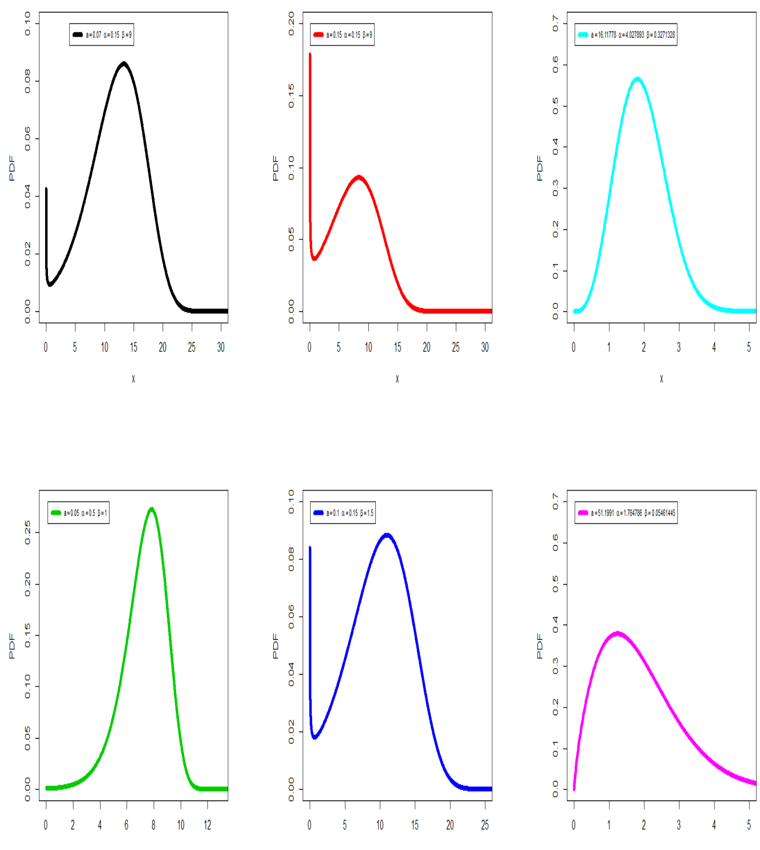

Figure 1: Plots of the OLGE PDF.

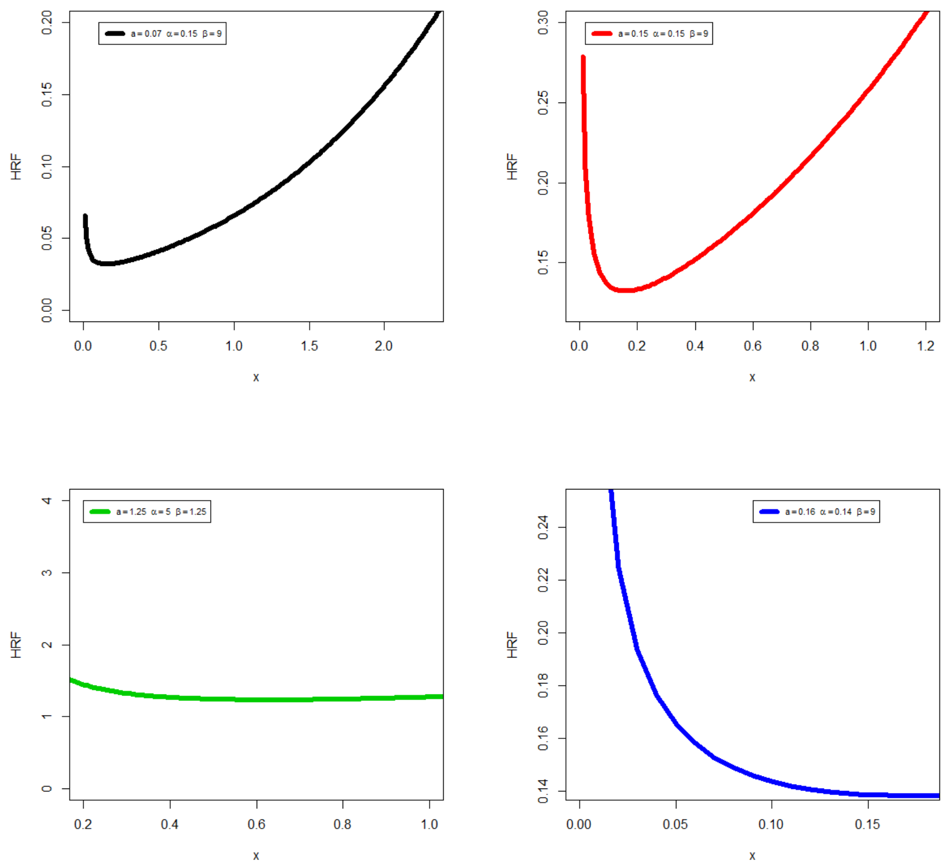

Figure 2: Plots of the OLGE HRF.

The major justification for the OLGE model is based on the enormous use of the E and GE lifetime models. Also we are motivated to introduce the OLGE lifetime model since it exhibits the monotonically increasing, bathtub, constant and the monotonically decreasing hazard rates (see Figure 2). The new model can be viewed as a mixture of the EE density. It can also be considered as a convenient model for fitting the symmetric, the left skewed, the right skewed, and the unimodal data (see Figure 1). The proposed lifetime model is better than the E, the Moment E (MomE), the LogBrHE, the **TI**OLE, the **TII**OLE and the Burr X exponential models so the new lifetime model is a good alternative to these models in modeling failure times and survival times data sets.

3. Properties

3.1. Critical points. The critical points of the OLGE PDF are the roots of the equation

$$0=\frac{\frac{d}{dx}\left\{\alpha\beta\ \left(1-\mathrm{e}^{-\beta x}\right)^{\alpha-1}\mathrm{e}^{-\beta x}\right\}}{\alpha\beta\ \left(1-\mathrm{e}^{-\beta x}\right)^{\alpha-1}\mathrm{e}^{-\beta x}}+3\frac{\alpha\beta\ \left(1-\mathrm{e}^{-\beta x}\right)^{\alpha-1}\mathrm{e}^{-\beta x}}{1-\left(1-\mathrm{e}^{-\beta x}\right)^{\alpha}}-a\frac{\alpha\beta\ \left(1-\mathrm{e}^{-\beta x}\right)^{\alpha-1}\mathrm{e}^{-\beta x}}{\left[1-\left(1-\mathrm{e}^{-\beta x}\right)^{\alpha}\right]^{2}}.$$

The critical points of the of the HRF of the OLGE are obtained from

$$0=\frac{\frac{d}{dx}\left\{\alpha\beta\ \left(1-\mathrm{e}^{-\beta x}\right)^{\alpha-1}\mathrm{e}^{-\beta x}\right\}}{\alpha\beta\ \left(1-\mathrm{e}^{-\beta x}\right)^{\alpha-1}\mathrm{e}^{-\beta x}}+\frac{\alpha\beta\ \left(1-\mathrm{e}^{-\beta x}\right)^{\alpha-1}\mathrm{e}^{-\beta x}}{a+\left[1-\left(1-\mathrm{e}^{-\beta x}\right)^{\alpha}\right]}+2\frac{\alpha\beta\ \left(1-\mathrm{e}^{-\beta x}\right)^{\alpha-1}\mathrm{e}^{-\beta x}}{\left[1-\left(1-\mathrm{e}^{-\beta x}\right)^{\alpha}\right]^{2}}.$$

We can examine the last two Equations to determine the local maximums and minimums and inflexion points via most computer algebra systems.

3.2. Useful representations. The PDF of X in (5) can be easily rewritten as

$$f(x)=\sum_{i,k=0}^{\infty}\omega_{i,k}\ g\left(x;\left(2+i+k\right)\alpha,\beta\right), \tag{7}$$

where

$$\omega_{i,k}=\frac{\left(-1\right)^{k}a^{2+k}\Gamma\left(i+k+3\right)}{i!\left(a+1\right)\left[\left[\left(2+i+k\right)\alpha\right]\right]\Gamma\left(k+3\right)},$$

and

$$g\left(x;\left(2+i+k\right)\alpha,\beta\right)=\left[\left(2+i+k\right)\alpha\right]\beta\ \mathrm{e}^{-\beta x}\left(1-\mathrm{e}^{-\beta x}\right)^{\left[\left(2+i+k\right)\alpha\right]-1},$$

is PDF of EE model with positive parameters $[(2+i+k)\alpha]$ and β. The CDF of X can be given by integrating (7) as

$$F(x)=\sum_{i,k=0}^{\infty}\omega_{i,k}\ G\left(x;\left(2+i+k\right)\alpha,\beta\right),$$

where

$$G\left(x;\left(2+i+k\right)\alpha,\beta\right)=\left(1-\mathrm{e}^{-\beta x}\right)^{\left[\left(2+i+k\right)\alpha\right]},$$

is PDF of EE model with positive parameters $[(2+i+k)\alpha]$ and β. For more details about the OL-G family see [10]. For more detail about properties of EE model see [4], [5] and [9]. Other useful works studied the E model such as the Burr X exponential (BrXE) model (see [11]), the one-parameter odd Lindley exponential model (see [7]), the two-parameter odd Lindley exponential model (see [10]) and the logarithmic Burr-Hatke exponential (LogBrHE) distribution (see [1]).

3.3. Moments. The r^{th} ordinary moment of X is given by $\mu_r'=\int_0^\infty x^r f(x)\,dx=\mathbf{E}(X^r)$. Using (7), we get

$$\mu_r'=\frac{\alpha}{\beta^r}\Gamma\left(1+r\right)\sum_{i,k,w=0}^{\infty}\omega_{i,k}v_w^{\{[(2+i+k)\alpha],r\}}|_{(r>-1)},$$

where

$$v_w^{(\alpha,r)}=\frac{\zeta\left(\alpha,w\right)}{\left(1+w\right)^{1+r}},$$

$$\zeta\left(\alpha,w\right)=\frac{\left(-1\right)^{w}\Gamma\left(\alpha\right)}{\Gamma\left(\alpha-w\right)},$$

$$\Gamma\left(1+v\right)|_{(v\in\mathbb{R}^{+})}=\prod_{m=0}^{v-1}\left(v-m\right)=v\left(v-1\right)\left(v-2\right)...1=v!,$$

and

$$\int_{0}^{\infty}t^{a-1}\,\mathrm{e}^{-t}dt=\Gamma\left(a\right),$$

is the well known complete gamma function. The r^{th} incomplete moment of X, say $I_r(t)$, is given by $I_r(t)=\int_0^t x^r f(x)\,dx$. Via (7), we obtain

$$I_r\left(t\right)=\frac{\alpha}{\beta^r}\left[\gamma\left(1+r,\frac{\beta}{t}\right)\right]\sum_{i,k,w=0}^{\infty}\omega_{i,k}v_w^{\{[(2+i+k)\alpha],r\}}|_{(r>-1)},$$

where $\gamma\left(\zeta,q\right)$ is the incomplete gamma function.

$$\begin{aligned}\gamma\left(\xi,q\right)|_{(\xi\neq0,-1,-2,\ldots)} &= \int_{0}^{q}t^{\xi-1}\exp\left(-t\right)dt\\ &= \frac{q\xi}{\xi}\left\{{}_{(1)}\mathbf{F}_{(1)}\left[\xi;\xi+1;-q\right]\right\}\\ &= \sum_{m=0}^{\infty}\frac{\left(-1\right)^{m}}{m!\left(a+m\right)}q^{a+m},\end{aligned}$$

where ${}_{(1)}\mathbf{F}_{(1)}\left[\cdot,\cdot,\cdot\right]$ is the confluent hypergeometric function.

3.4. **Order statistics.** Let $X_1,\ldots,X_n$ be a random sample (R.S.) from the OLGE model and let $X_{1:1},\ldots,X_{n:n}$ be the corresponding order statistics, so the PDF of the i^{th} order statistic, say $X_{i:n}$, can be expressed as

$$f_{i:n}\left(x\right)=\sum_{j=0}^{n-i}(-1)^{j}\left[\mathrm{B}^{-1}(i,n-i+1)\right]F(x)^{j+i-1}f(x)\binom{n-i}{j},$$

where $\mathrm{B}(\cdot,\cdot)$ is the well known beta function. By inserting (5) and (6) in $f_{i:n}(x)$, we obtain

$$f_{i:n}\left(x\right)=\sum_{m,p=0}^{\infty}\sum_{j=0}^{k+n-i}a_{i,m,p}\;g\left(x;\left(2+j+m+p\right)\alpha,\beta\right),$$

where

$$a_{i,m,p}=\sum_{k=0}^{i-1}\frac{\left(-1\right)^{k+m}\left(1+a\right)^{-(j+1)}a^{j+m+2}}{m!\mathrm{B}(i,n-i+1)\left[\left(2+j+m+p\right)\alpha\right]}\binom{-1+k+n}{j}\binom{j+m+p}{j+m}\binom{i-1}{k},$$

then, the z^{th} moment of $X_{i:n}$ is given by

$$\mathbf{E}\left(X_{i:n}^{z}\right)=\frac{\alpha}{\beta^{z}}\Gamma\left(1+z\right)\sum_{m,p,w=0}^{\infty}\sum_{j=0}^{k+n-i}a_{i,m,p}v_w^{\{[(2+j+m+p)\alpha],z\}}|_{(z>-1)}\;.$$

3.5. **Quantile spread ordering (QSO).** The QSO of the RV X ~OLGE(a, α, β) having CDF (6) is given by

$$\{QSO\}_X(\tau)|_{(\tau\in(0.5,1))} = \left[F^{-1}(\tau)\right] - \left[F^{-1}(1-\tau)\right],$$

which implies

$$\left[S^{-1}(1-\tau)\right] - \left[S^{-1}(\tau)\right] = \{QSO\}_X(\tau),$$

where

$$S(x) = 1 - F(x) \text{ and } F^{-1}(\tau) = S^{-1}(1-\tau)$$

is the survival function (SF). The QSO of a model describes how the probability mass (PM) is placed symmetrically about its median ($Median(X)$) and hence can be used to formalize concepts like peakedness ($Peak(X)$) and tail weight traditionally with kurtosis ($Kur(X)$). So, it allows to separate concepts of $Kur(X)$ and $Peak(X)$ for such a asymmetric models. Let X_1 and X_2 be two RVs following the OLGE model with $\{QSO\}_{X_1}$ and $\{QSO\}_{X_2}$. Then X_1 is called smaller than ($\leq$) X_2 in QSO, denoted as $X_1 \leq_{\{QSO\}} X_2$, if

$$\left(\{QSO\}_{X_1}(\tau) \leq \{QSO\}_{X_2}(\tau)\right)|_{(\tau\in(0.5,1))}.$$

Following properties of the QSO order can be obtained:

- The order $\leq_{\{QSO\}}$ is location-free

$$\left[X_1 \leq_{\{QSO\}} X_2 \text{ if } (X_1 + \mathbf{C}) \leq_{\{QSO\}} X_2\right]|_{(\mathbf{C}\geq 1)}.$$

- The order $\leq_{\{QSO\}}$ is dilative

$$\left[X_1 \leq_{\{QSO\}} \mathbf{C}X_1|_{(\mathbf{C}\geq 1)} \text{ and } X_2 \leq_{\{QSO\}} \mathbf{C}X_2\right]|_{(\mathbf{C}\geq 1)}.$$

- Let F_{X_1} and F_{X_2} be symmetric, then

$$\left[X_1 \leq_{\{QSO\}} X_2 \text{ if, and only if } F_{X_1}^{-1}(\tau) \leq F_{X_2}^{-1}(\tau)\right]|_{(\tau\in(0.5,1))}.$$

- The order $\leq_{\{QSO\}}$implies ordering of the mean absolute deviation around the median, say $\boldsymbol{\zeta}(X_i)|_{(i=1,2)}$,

$$\mathbf{E}\left[|-Median(X_1) + X_1|\right] = \boldsymbol{\zeta}(X_1),$$

and

$$\mathbf{E}\left[|-Median(X_2) + X_2|\right] = \boldsymbol{\zeta}(X_2),$$

where

$$X_1 \leq_{\{QSO\}} X_2,$$

then

$$\Rightarrow \boldsymbol{\zeta}(X_1) \leq_{\{QSO\}} \boldsymbol{\zeta}(X_2),$$

finally

$$X_1 \leq_{\{QSO\}} X_2 \text{ if, and only if } -X_1 \leq_{\{QSO\}} -X_2.$$

3.6. **Moment of residual and reversed residual life (MRL & MRRL).** The n^{th} MRL is given by

$$z_n(t) = \mathbf{E}[(X-t)^n \mid \binom{(n=1,2,\ldots)}{(X>t),\ (t>0)}].$$

So the n^{th} MRL of X can be given as

$$z_n(t) = [1-F(t)]^{-1} \int_t^\infty (x-t)^n dF(x),$$

subsequentlywe can write

$$\begin{aligned} z_n(t) &= [1-F(t)]^{-1} \sum_{i,k=0}^{\infty} \sum_{r=0}^{n} (-t)^{n-r} \ \ \omega_{i,k} \binom{n}{r} \int_t^\infty x^r \ g\left(x;(2+i+k)\alpha,\beta\right) dx \\ &= \frac{\alpha}{\beta^n} \frac{\Gamma\left(1+n,\frac{\beta}{t}\right)}{1-F(t)} \sum_{i,k,w=0}^{\infty} \sum_{r=0}^{n} (-t)^{n-r} \ \ \binom{n}{r} \omega_{i,k} v_w^{\{[(2+i+k)\alpha],n\}} \mid_{(n>-1)}, \end{aligned}$$

where

$$\Gamma\left(a,q\right)\mid_{(q>0)} = \int_q^\infty t^{a-1} \mathrm{e}^{-t} dt,$$

and

$$\Gamma\left(a,q\right) + \gamma\left(a,q\right) = \Gamma\left(a\right).$$

The n^{th} MRRL is given by

$$Z_n(t) = \mathbf{E}\left[(t-X)^n \mid \binom{(n=1,2,\ldots)}{(X\le t),\ t>0}\right] \ ,$$

uniquely determines $F(x)$, then we have

$$Z_n(t) = F(t)^{-1} \int_0^t (t-x)^n dF(x).$$

Then, the n^{th} moment of the reversed residual life of X becomes

$$\begin{aligned} Z_n(t) &= F(t)^{-1} \sum_{i,k=0}^{\infty} \sum_{r=0}^{n} (-1)^r \, t^{n-r} \binom{n}{r} \omega_{i,k} \int_0^t x^r \ g\left(x;(2+i+k)\alpha,\beta\right) dx \\ &= \frac{\alpha}{\beta^r} \frac{\gamma\left(1+r,\frac{\beta}{t}\right)}{F(t)} \sum_{i,k,w=0}^{\infty} \sum_{r=0}^{n} (-1)^r \, t^{n-r} \binom{n}{r} \omega_{i,k} v_w^{\{[(2+i+k)\alpha],n\}} \mid_{(n>-1)}. \end{aligned}$$

4. Maximum likelihood method

If $x_1, \ldots, x_n$ be a R.S. of the new distribution with parameter vector $\mathbf{\Psi} = (a, \alpha, \beta,)^\intercal$. The log-likelihood function for $\mathbf{\Psi}$, say $\ell = \ell(\mathbf{\Psi})$, is given by

$$\begin{aligned} \ell &= \ell(\mathbf{\Psi}) = 2n\log a - n\log(1+a) + n\log\alpha + n\log\beta + (\alpha-1)\sum_{i=1}^{n}\log\left(1-\mathrm{e}^{-\beta x_i}\right) \\ &\quad -\beta\sum_{i=1}^{n} x_i - 3\sum_{i=1}^{n}\log\left[1-\left(1-\mathrm{e}^{-\beta x_i}\right)^\alpha\right] - a\sum_{i=1}^{n} \frac{\left(1-\mathrm{e}^{-\beta x_i}\right)^\alpha}{\left[1-\left(1-\mathrm{e}^{-\beta x_i}\right)^\alpha\right]}. \end{aligned} \tag{8}$$

Equation (8) can be maximized either via the different programs like `R` (`optim` function), `SAS` (`PROC NLMIXED`) or via solving the nonlinear likelihood equations obtained by differentiating (8). The score vector elements, $\mathbf{U}(\mathbf{\Psi}) = \frac{\partial \ell}{\partial \mathbf{\Psi}} = \left(\frac{\partial \ell}{\partial a}, \frac{\partial \ell}{\partial \alpha}, \frac{\partial \ell}{\partial \beta}\right)^{\intercal}$, are exist and easily to be obtained.

5. Real data modeling

The first data set {1.3, 1.7, 1.8, 1.5, 1.2, 1.4, 2.2, 1.7, 3, 1.9, 1.8, 1.6, 1.1, 1.4, 2.7, 4.1, 1.7, 2.3, 1.6, 2} (see [6]) called the failure times data, this data represents the lifetime data relating to relief times (in minutes) of patients receiving an analgesic. The second data set {0.1, 0.330, 0.44, 0.560, 0.59, 0.72, 0.740, 0.77, 0.92, 0.930, 0.96, 1, 1, 1.020, 1.05, 1.07, 07, 1.080, 1.08, 1.08, 1.090, 1.12, 1.13, 1.150, 1.16, 1.2, 1.21, 1.220, 1.22, 1.24, 1.3, 1.34, 1.360, 1.39, 1.44, 1.460, 1.53, 1.59, 1.60, 1.63, 1.63, 1.680, 1.71, 1.720, 1.76, 1.83, 1.950, 1.960, 1.970, 2.020, 2.13, 2.150, 2.16, 2.220, 2.3, 2.310, 2.4, 2.450, 2.51, 2.530, 2.54, 2.54, 2.780, 2.93, 3.270, 3.42, 3.47, 3.61, 4.020, 4.32, 4.58, 5.55} called the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli (see [2]). We will consider some other goodness-of-fit measures including the consistent Akaike information criterion ($CAIC$), the Hannan-Quinn information criterion ($HQIC$), the Akaike information criterion (AIC), the Bayesian information criterion (BIC) and $-2\widehat{\ell}$, where $\widehat{\ell}$ is the maximized log-likelihood,

$$AIC = -2\widehat{\ell} + 2p,$$

$$BIC = -2\widehat{\ell} + p\,\log(n),$$

$$CAIC = -2\widehat{\ell} + 2pn/(n-p-1),$$

and

$$HQIC = -2\widehat{\ell} + 2p\log[\log(n)],$$

where p is the number of parameters and n is the sample size. Moreover, we consider the Cramér-Von Mises and the Anderson-Darling ($W^{\bigstar}$, $A^{\bigstar}$) and the Kolmogorov-Smirnov (KS) statistic. The $W^{\bigstar}$and $A^{\bigstar}$ statistics are given by

$$W^{\bigstar} = (1+1/2n)\left[[1/(12n)] + \sum\nolimits_{j=1}^{n} \omega_j\right],$$

and

$$A^{\bigstar} = a_{(n)}\left(n + n^{-1}\sum\nolimits_{j=1}^{n} a_j\right),$$

where

$$\omega_j = [z_i - (2j-1)/(2n)]^2,$$

$$a_{(n)} = 1 + \frac{9}{4}n^{-2} + \frac{3}{4}n^{-1},$$

and

$$a_j = (2j-1)\log[\tau_i(1-\tau_{n-j+1})],$$

where $F(y_j) = \tau_i$ and the y_j's values are the ordered observations.

5.1. **Modeling failure times data.** In this subsection, we shall compare the fits of the new distribution with those of other competitive models, namely: the exponential E(β), Moment exponential MomE(β), Log Butr Hatke exponential LogBrHE(β), the two parameter odd Lindley exponential OLE(a,β), one parameter odd Lindley exponential OLE(β) , the two parameter odd Lindley exponentiated exponential OLGE(α,β) and the Burr X exponential BrXE(θ,β) models. From Table 3 and Figure 1, the new lifetime model is much better than the exponential, Moment exponential, log Butr Hatke exponential, the two parameter odd Lindley exponential, the one parameter odd Lindley exponential, the two parameter odd Lindley exponentiated exponential and the Burr X exponential models so the new lifetime model is a good alternative to these models in modeling failure times data set.

Table 2: MLEs, SEs, C.I.s (in parentheses) values for the relief times data.

	β			α			a		
	MLE	S.E.	C.I.	MLE	S.E.	C.I.	MLE	S.E.	C.I.
E(β)	0.526	(0.117)	(0.29,0.75)						
MomE(β)	0.950	(0.150)	(0.66,1.24)						
LogBrHE(β)	0.5263	(0.118)	(0.43,0.63)						
BrXE(α,β)	0.3207	(0.031)	(0.26,0.38)	1.1635	(0.33)	(0.5, 1.82)			
OLE(β)	0.604	(0.054)	(0.5, 0.7)						
OLE(a,β)	0.68044	(0.164)	(0.36, 1)				0.78251	(0.391)	(0, 1.56)
OLGE(α,β)	0.749	(0.136)	(0.48, 1)	2.404	(0.979)	(0.6, 4.2)			
OLGE(a,α,β)	0.327	(0.258)	(0, 0.82)	4.027	(1.246)	(1.62, 6.42)	16.118	(26.679)	(0, 69.51)

Table 3: $AIC, BIC, CAIC, HQIC$, $A^\star$, $W^\star$, K.S. and (p-value) for the relief times data.

Models	$AIC, BIC, CAIC, HQIC$	$A^\star, W^\star$	K.S. and (p-value)
E(β)	67.67, 68.67, 67.89, 67.87	4.60,0.96	0.44, (0.004)
MomE(β)	54.32, 55.31, 54.54, 54.50	2.76, 0.53	0.32, (0.07)
LogBrHE(β)	67.67, 68.67, 67.89, 67.87	0.62,0.105	0.44, (0.0009)
BrXE(θ,β)	48.1, 50.1, 8.8, 48.5	1.39,0.24	0.248, (0.1705)
OLE(β)	49.1, 50.1, 49.3, 49.3	1.3,0.22	0.85, (6.231e^{-13})
OLE(a,β)	50.89, 52.88, 51.6, 51.3	1.39,0.24	0.86, (2.721e^{-13})
OLGE(α,β)	49.8, 51.8, 50.5, 50.2	1.19,0.20	0.54, (1.465e^{-5})
OLGE(a,α,β)	**45.5, 48.5, 47.02, 46.1**	**0.98,0.16**	**0.11, (0.96)**

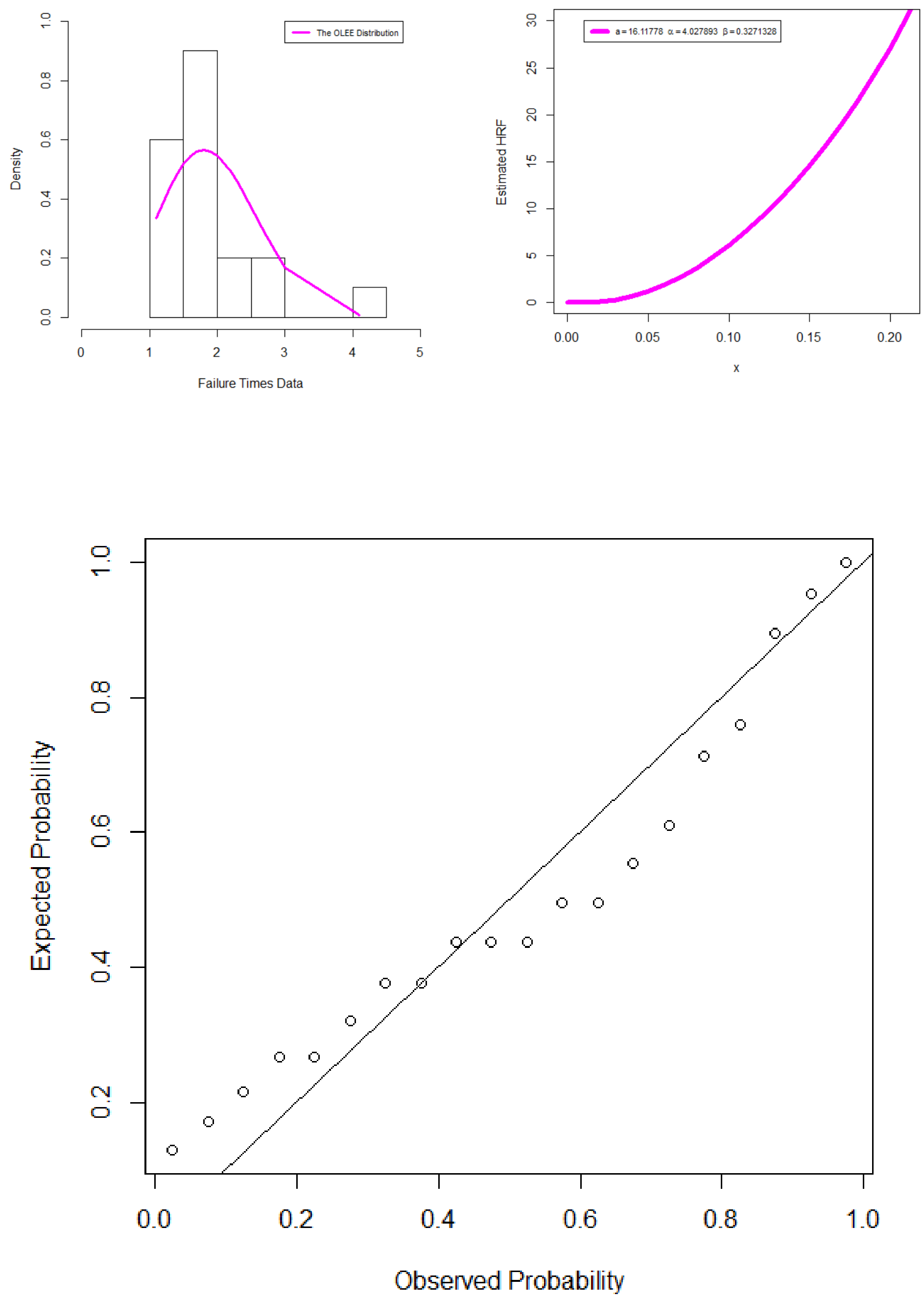

Figure 3: Estimated PDF, Estimated HRF, and P-P plot for the $1^{(st)}$ data.

5.2. **Modeling survival times data.** In this subsection, we shall compare the fits of the new model with those of same competitive models except the MomE model. From Table 5 and Figure 4, the proposed lifetime model is much better than the exponential, Moment exponential, log Butr Hatke exponential, the two parameter odd Lindley exponential, the one parameter odd Lindley exponential, the two parameter odd Lindley exponentiated exponential and the Burr X exponential models so the new lifetime model is a good alternative to these models in survival times data set.

Table 4: MLEs, SEs, C.I.s (in parentheses) values for the urvival times data.

	β			α			a		
	MLE	S.E.	C.I.	MLE	S.E.	C.I.	MLE	S.E.	C.I.
E(β)	0.540	(0.063)	(0.42,0.66)						
LogBrHE(β)	0.54	(0.064)	(0.412,0.668)						
BrXE(α, β)	0.2055	(0.012)	(0.18, 0.23)	0.475	(0.06)	(0.355, 0.494)			
OLE(β)	0.381	(0.021)	(0.34, 0.42)						
OLE(a, β)	0.247	(0.058)	(0.14, 0.34)				1.996	(0.607)	(0.9, 2.9)
OLGE(α, β)	0.148	(0.025)	(0.1, 0.18)	1.771	(0.202)	(1.3, 2.1)			
OLGE(a, α, β)	0.054	0.073	(0, 0.18)	1.764	0.186	(1.34, 2.16)	51.199	105.503	(0, 261)

Table 5: $AIC, BIC, CAIC, HQIC, A^\star$, $W^\star$, K.S. and (p-value) for survival times data.

Models	$AIC, BIC, CAIC, HQIC$	$A^\star$, $W^\star$	K.S. and (p-value)
Exp(β)	$234.63, 236.91, 234.68, 235.54$	6.53,1.25	0.27, (0.06)
LogBrHE(β)	$234.63, 236.9, 234.7, 235.5$	0.71,0.115	0.28, (2.38e^{-5})
BrXE(θ, β)	$235.3, 239.9, 235.5, 237.1$	2.9,0.52	0.22, (0.002)
OLE(β)	$229.1, 231.4, 229.2, 230$	1.94,0.33	0.49, (9.99e^{-16})
OLE(a, β)	$223.7, 228.3, 223.9, 225.6$	1.46,0.24	0.44, (9.45e^{-13})
OLGE(α, β)	$210.02, 214.5, 210.2, 211.8$	1.19,0.20	0.15, (0.090)
OLGE(a, α, β)	**210.6, 217.5, 211.05, 213.4**	**1.11,0.18**	**0.11, (0.29)**

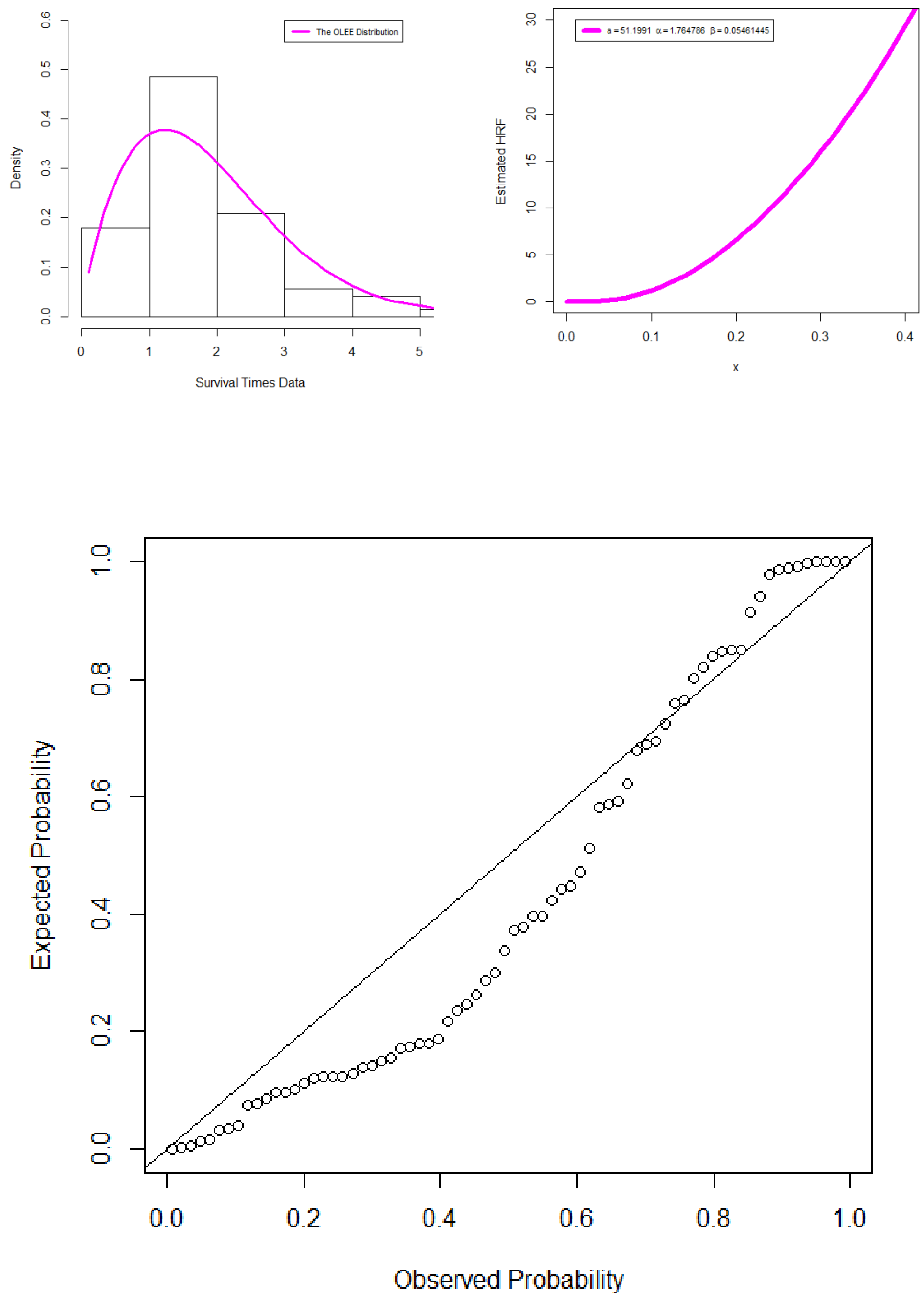

Figure 4: Estimated PDF, Estimated HRF, and P-P plot for the $2^{(nd)}$ data.

From Tables 3 and 5 we conclude that the proposed lifetime model is much better than the E(β), MomE(β), LogBrHE(β), the OLE(a, β), the OLE(β), the

OLGE(α, β) and the BrXE(θ, β) models. so the new lifetime model is a good alternative to these models in modeling failure times and survival times data sets.

6. Conclusions

This work aims to introduce a new extension of the exponentiated exponential distribution called the odd Lindley exponentiated exponential which exhibits the monotonically increasing, bathtub, constant, the monotonically decreasing hazard rates. A range of mathematical properties of new is derived. The proposed model can be viewed as a mixture of the exponentiated exponential density. The new model can also be considered as a convenient model for fitting the symmetric, the left skewed, the right skewed, and the unimodal data sets. The proposed lifetime model is much better than the exponential, Moment exponential, log Butr Hatke exponential, the two parameter odd Lindley exponential, the one parameter odd Lindley exponential, the two parameter odd Lindley exponentiated exponential and the Burr X exponential models so the new lifetime model is a good alternative to these models in modeling failure times and survival times data sets.

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Management Information System Department, Taibah University, Saudi Arabia., Department of Statistics, Mathematics and Insurance, Benha University, Egypt.

E-mail address: Tabouelmagd@taibahu.edu.sa