COMMON FIXED POINT FOR KANNAN TYPE
CONTRACTIONS VIA INTERPOLATION

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Abstract. In this paper, we use interpolation to obtain a common fixed point result for a new type of Kannan contraction mappings.

1. Introduction and Preliminaries

In 1968 Kannan introduced an interesting type of contraction mapping which is not continuous and it possesses a fixed point [1]. Kannan’s theorem asserts that if \( M \) is a complete metric space and \( T : M \to M \) is a mapping satisfying the following condition

\[
d(Tp, Tq) \leq \lambda [d(p, Tp) + d(q, Tq)],
\]

for all \( p, q \in M \), where \( \lambda \in [0, \frac{1}{2}) \). Then \( T \) has a unique fixed point. Kannan’s theorem has been generalized in different ways by many authors; one of the latest generalizations was given by Karapinar in [2]. Karapinar introduced a Kannan type contraction mapping called interpolative Kannan type contraction and proved a fixed point result on it.

Definition 1.1. [2] Let \((M, d)\) be a metric space. A self mapping \( T : M \to M \) is said to be an interpolative Kannan type contraction if there exist a constant \( \lambda \in [0, 1) \) and \( \alpha \in (0, 1) \) such that

\[
d(Tp, Tq) \leq \lambda [d(p, Tp)]^\alpha [d(q, Tq)]^{1-\alpha}.
\]

Theorem 1.2. [2] Let \((M, d)\) be a complete metric space and \( T : M \to M \) be an interpolative Kannan type contraction mapping. Then, \( T \) has a unique fixed point.

2. Main Result

In this section we are following Karapinar’s result in [2] to obtain a common fixed point result.

Theorem 2.1. Let \( M \) be a complete metric space, \( S, T : M \to M \) be self mappings. Assume that there are some \( \lambda \in [0, 1) \), \( \alpha \in (0, 1) \) such that the condition

\[
d(Tp, S) \leq \lambda [d(p, Tp)]^\alpha [d(q, S)]^{1-\alpha}.
\]

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is satisfied for all $p,q \in \mathcal{M}$ such that $Tp \neq p$ whenever $Sq \neq q$. Then, $S$ and $T$ have a unique common fixed point.

Proof. Let $p_0 \in \mathcal{M}$, define the sequence $\{p_n\}$ by

$$p_{2n+1} = Tp_{2n}, \quad p_{2n+2} = Sp_{2n+1} \quad \forall n \in \{0, 1, 2, \cdots\}.$$ 

If there exists $n \in \{0, 1, 2, \cdots\}$ such that $p_n = p_{n+1} = p_{n+2}$, then $p_n$ is a common fixed point of $S$ and $T$; so suppose that there does not exist three consecutive identical terms in the sequence $\{p_n\}$ and that $p_0 \neq p_1$.

Now, using (2.1) we deduce that

$$d(p_{2n+1}, p_{2n+2}) = d(Tp_{2n}, Sp_{2n+1}) \leq \lambda [d(p_{2n}, p_{2n+1})]^\alpha [d(p_{2n+1}, p_{2n+2})]^{1-\alpha}.$$ 

Thus,

$$[d(p_{2n+1}, p_{2n+2})]^\alpha \leq \lambda [d(p_{2n}, p_{2n+1})]^\alpha;$$

or,

$$d(p_{2n+1}, p_{2n+2}) \leq \lambda^{\frac{1}{\alpha}} d(p_{2n}, p_{2n+1}) \leq \lambda d(p_{2n}, p_{2n+1}).$$

Hence,

$$d(p_{2n+1}, p_{2n+2}) \leq \lambda d(p_{2n}, p_{2n+1}) \leq \lambda^2 d(p_{2n-1}, p_{2n}) \leq \lambda^3 d(p_{2n-2}, p_{2n-1}) \cdots \leq \lambda^{n+1} d(p_0, p_1);$$

or,

$$d(p_{2n+1}, p_{2n+2}) \leq \lambda^{2n+1} d(p_0, p_1). \quad (2.2)$$

Similarly,

$$d(p_{2n+1}, p_{2n}) = d(Tp_{2n}, Sp_{2n-1}) \leq \lambda [d(p_{2n}, p_{2n+1})]^\alpha [d(p_{2n-1}, p_{2n})]^{1-\alpha}.$$ 

Thus,

$$[d(p_{2n+1}, p_{2n})]^{1-\alpha} \leq \lambda [d(p_{2n-1}, p_{2n})]^{1-\alpha};$$

or,

$$d(p_{2n+1}, p_{2n}) \leq \lambda^{\frac{1}{1-\alpha}} d(p_{2n-1}, p_{2n}) \leq \lambda d(p_{2n-1}, p_{2n}).$$

Hence,

$$d(p_{2n+1}, p_{2n}) \leq \lambda d(p_{2n-1}, p_{2n}) \leq \lambda^2 d(p_{2n-2}, p_{2n-1}) \leq \lambda^3 d(p_{2n-3}, p_{2n-2}) \cdots \leq \lambda^n d(p_0, p_1).$$

Thus,

$$d(p_{2n+1}, p_{2n}) \leq \lambda^n d(p_0, p_1). \quad (2.3)$$

From (2.2) and (2.3) we can deduce that

$$d(p_n, p_{n+1}) \leq \lambda^n d(p_0, p_1). \quad (2.4)$$

Now, using (2.4) we prove that the sequence $\{p_n\}$ is a Cauchy sequence. Let $m, r \in \{0, 1, 2, \cdots\}$

$$d(p_m, p_{m+r}) \leq d(p_m, p_{m+r}) + d(p_{m+r-1}, p_{m+r}) \leq \lambda^m \cdot \lambda^{m+1} \cdots \lambda^{m+r-1} d(p_0, p_1) \leq \lambda^m \cdot \lambda^{m+1} \cdots \lambda^{m+r-1} + \cdots d(p_0, p_1) = \frac{1}{1-\lambda} d(p_0, p_1).$$

Letting $m \to \infty$, we deduce that $\{p_n\}$ is a Cauchy sequence.
As $\mathcal{M}$ is complete, so there exists $u \in \mathcal{M}$ such that $\lim_{n \to \infty} p_n = u$. Using the continuity of the metric in its both variables we can prove that $u$ is a fixed point of $T$ as follows

$$d(Tu, p_{2n+2}) = d(Tu, Sp_{2n+1})$$
$$\leq \lambda [d(u, Tu)]^\alpha [d(p_{2n+1}, p_{2n+2})]^{1-\alpha}.$$ 

Letting $n \to \infty$ we get $d(Tu, u) = 0$, so $Tu = u$. Similarly,

$$d(p_{2n+1}, Su) = d(Tp_{2n}, Su)$$
$$\leq \lambda [d(p_{2n}, p_{2n+1})]^\alpha [d(u, Su)]^{1-\alpha}.$$ 

Letting $n \to \infty$ we get $u = Su$.

To prove that $u$ is the unique common fixed point of $S$ and $T$, suppose that $v$ is another common fixed point of $S$ and $T$, then

$$d(u, v) = d(Tu, Sv) \leq \lambda [d(u, Tu)]^\alpha [d(v, Sv)]^{1-\alpha} = 0.$$ 

Hence, $u = v$. \hfill \Box

Now, we give an example of the previous result using a metric defined in [2].

**Example 2.2.** Let $M = \{p, q, z, w\}$, define a metric $d$ on $M$ as follows

$$d(p, p) = d(q, q) = d(z, z) = d(w, w) = 0$$
$$d(p, q) = d(q, p) = 3$$
$$d(z, p) = d(p, z) = 4$$
$$d(q, z) = d(z, q) = \frac{3}{2}$$
$$d(w, p) = d(p, w) = \frac{5}{2}$$
$$d(w, q) = d(q, w) = 2$$
$$d(w, z) = d(z, w) = \frac{3}{2}$$

Define self maps $T, S$ as follows

$$T: \begin{pmatrix} p & q & z & w \\ p & w & z & w \end{pmatrix}, \quad S: \begin{pmatrix} p & q & z & w \\ p & q & w & z \end{pmatrix}$$

It is clear that $S, T$ satisfies \([2.1]\) with $\lambda = \frac{9}{10}$ and $\alpha = \frac{1}{2}$, and $S$ and $T$ has a unique common fixed point $p$.

**Conclusion**

We can obtain more common fixed point results in similar ways and use them in more applications.

**References**