STATISTICAL EPI-CONVERGENCE IN SEQUENCES OF FUNCTIONS

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Abstract. In this paper, statistical epi-limit is defined by using epigraphs in order to increase sensitivity by eliminating outliers for mathematical problems. Various characterizations of statistical epi-convergence and their relations are given and it is compared with statistical convergence. Also, its connections with level sets and monotone increasing or decreasing cases are studied. Moreover, statistical equi-lower semicontinuity and its relation with statistical epi-limit is examined.

1. Introduction


Statistical convergence was first studied by Zygmund [28] in 1935 and then it was introduced by Steinhaus [23] and Fast [6] and also Schoenberg [22] independently. The definitions of pointwise and uniform statistical convergence of real-valued functions were given by Gökhan and Güngör [10, 11] and by Duman and Orhan [4] independently. Statistical limit superior and statistical limit inferior were introduced by Frýd [8] and also statistical limit points and cluster points were defined by Frýd [7, 9]. Furthermore statistical lower and upper limits of closed sets were defined and characterized by Talo et al. [24].

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In this part, fundamental definitions and theorems will be given. First of all, let 
\((X, d)\) be a metric space and \(f, (f_n)\) are functions defined on \(X\) with \(n \in \mathbb{N}\). If it is not mentioned explicitly the symbol \(d\) stands for the metric on \(X\).

Let \(K \subseteq \mathbb{N}\) and if the limit \(\delta(K) = \lim_{n \to \infty} \frac{1}{n} |\{k \leq n : k \in K\}|\) exists then it is called asymptotic density of \(K\) where \(|\{k \leq n : k \in K\}|\) denotes the number of elements of \(K\) not exceeding \(n\) (see [1], [7]).

If \(\delta(K_1) = \delta(K_2) = 1\), then \(\delta(K_1 \cap K_2) = \delta(K_1 \cup K_2) = 1\).

If \(\delta(K_1) = \delta(K_2) = 0\), then \(\delta(K_1 \cap K_2) = \delta(K_1 \cup K_2) = 0\).

Statistical convergence of a sequence of scalars was introduced by Fast [9]. Let \(x = (x_k)\) be a sequence of real or complex numbers. If for all \(\varepsilon > 0\), there exists \(L\) such that,

\[
\lim_{n \to \infty} \frac{1}{n} |\{k \leq n : |x_k - L| > \varepsilon\}| = 0,
\]

then the sequence \((x_k)\) is statistically convergent to \(L\).

The concepts of statistical limit superior and statistical limit inferior were introduced by Fridy and Orhan [8]. Let \(k\) be a positive integer and \(x\) be a real number sequence. Define the sets \(B_x\) and \(A_x\) as

\[
B_x := \{b \in \mathbb{R} : \delta(\{n : x_n > b\}) \neq 0\}, \quad A_x := \{a \in \mathbb{R} : \delta(\{n : x_n < a\}) \neq 0\}.
\]

Then statistical limit superior and statistical limit inferior of \(x\) is given by

\[
st-\limsup x := \left\{ \begin{array}{ll}
\sup B_x & \text{if } B_x \neq \emptyset, \\
-\infty & \text{if } B_x = \emptyset.
\end{array} \right.
\]

\[
st-\liminf x := \left\{ \begin{array}{ll}
\inf A_x & \text{if } A_x \neq \emptyset, \\
+\infty & \text{if } A_x = \emptyset.
\end{array} \right.
\]

**Lemma 1.1.** [8] If \(\beta = st-\limsup x\) is finite, then for every \(\varepsilon > 0\),

\[
\delta(\{k \in \mathbb{N} : x_k > \beta - \varepsilon\}) \neq 0 \text{ and } \delta(\{k \in \mathbb{N} : x_k > \beta + \varepsilon\}) = 0 \quad (1.1)
\]

Conversely, if (1.1) holds for every \(\varepsilon > 0\) then \(\beta = st-\limsup x\).

The dual statement for \(st-\liminf x\) is as follows:

**Lemma 1.2.** [8] If \(\alpha = st-\liminf x\) is finite, then for every \(\varepsilon > 0\),

\[
\delta(\{k \in \mathbb{N} : x_k < \alpha + \varepsilon\}) \neq 0 \text{ and } \delta(\{k \in \mathbb{N} : x_k < \alpha - \varepsilon\}) = 0 \quad (1.2)
\]

Conversely, if (1.2) holds for every \(\varepsilon > 0\) then \(\alpha = st-\liminf x\).

A point \(\xi \in X\) is called a statistical limit point of a sequence \(x = (x_k)\) if there is a set \(K = k_1 < k_2 < k_3 < \ldots\) with \(\delta(K) \neq 0\) such that \(x_{k_n} \to \xi\) as \(n \to \infty\). The set of all statistical limit points of a sequence \(x\) will be denoted by \(\Lambda_x\).

A point \(\xi \in X\) is called a statistical cluster point of \(x = (x_k)\) if for any \(\varepsilon > 0\),

\[
\delta(\{k \in \mathbb{N} : d(x_k, \xi) < \varepsilon\}) \neq 0.
\]

The set of all statistical cluster points of \(x\) will be denoted by \(\Gamma_x\).

Let \(L_x\) denote the set of all limit points \(\xi\) (accumulation points) of the sequence \(x\); i.e. \(\xi \in L_x\) if there exists an infinite set \(K = k_1 < k_2 < k_3 < \ldots\) such that \(x_{k_n} \to \xi\) as \(n \to \infty\).

Obviously we have \(\Lambda_x \subseteq \Gamma_x \subseteq L_x\).

In our study we will be interested much more on sequence of functions. Statistical convergence on sequence of functions is defined by Gökhan and Güngör [10].
Following definitions are statistical inner and outer limits on the concept of set convergence which is fundamental to define statistical epi-limit using sets. In this paper, we deal with Painlevé-Kuratowski \cite{13} convergence and actually its statistical version will be studied here which is defined by Sever and Talo \cite{24}. In set convergence, following collections of subsets of $\mathbb{N}$ play an important role for defining statistical inner and outer limits on sequence of sets.

$$S := \{N \subset \mathbb{N} : \delta(N) = 1\},$$

$$S^\# := \{N \subset \mathbb{N} : \delta(N) \neq 0\}.$$

Let $(X, d)$ be a metric space. The statistical inner limit and statistical outer limit of a sequence $(A_n)$ of closed subsets of $X$ are defined as follows:

$$\text{st-lim inf } A_n := \{x | \forall V \in \mathcal{N}(x), \exists N \in S, \forall n \in N : A_n \cap V \neq \emptyset\},$$

$$\text{st-lim sup } A_n := \{x | \forall V \in \mathcal{N}(x), \exists N \in S^\#, \forall n \in N : A_n \cap V \neq \emptyset\}.$$

**Proposition 1.3.** \cite{24} Let $(X, d)$ be a metric space and $(A_n)$ be a sequence of closed subsets of $X$. Then

$$\text{st-lim inf } A_n = \{x | \exists N \in S, \forall n \in N, \exists y_n \in A_n : \lim y_n = x\}.$$

**Proposition 1.4.** \cite{24} Let $(X, d)$ be a metric space and $(A_n)$ be a sequence of closed subsets of $X$. Then

$$\text{st-lim sup } A_n = \{x | \exists N \in S^\#, \forall n \in N, \exists y_n \in A_n : x \in \Gamma_y\}.$$

Let $f$ be a function defined on $X$, the epigraph of $f$ is the set $\text{epif } := \{(x, \alpha) \in X \times \mathbb{R} | \alpha \geq f(x)\}$ and its level set is defined by $\text{lev}_{\leq \alpha} f := \{x \in X | f(x) \leq \alpha\}$. Hence for functions $f$ and $g$ from $X$ to $\mathbb{R}$, if $f \leq g$ for all $x \in X$ it is obvious that $\text{epif } \supseteq \text{epig}$.  

\begin{equation}
\text{epif } \supseteq \text{epig}.
\end{equation}

For any sequence $(f_n)$ of functions on $X$, the lower epi-limit, $e^{-\text{lim inf}} f_n$, is the function having as its epigraph the outer limit of the sequence of sets $\text{epif } f_n$:

$$\text{epi}(e^{-\text{lim inf}} f_n) := \limsup(n)(\text{epif } f_n).$$

The upper epi-limit, $e^{-\text{lim sup}} f_n$, is the function having as its epigraph the inner limit of the sequence of sets $\text{epif } f_n$:

$$\text{epi}(e^{-\text{lim sup}} f_n) := \liminf(n)(\text{epif } f_n).$$

When these two functions equal to each other, we have $e^{-\text{lim}} f_n = e^{-\text{lim inf}} f_n = e^{-\text{lim sup}} f_n$. Hence the functions $f_n$ are said to epi convergent to the function $f$. It is symbolized by $f_n \rightarrow e f$. Moreover, the relation between set convergence and convergence of sequence of functions appears in the following equality.

$$f_n \rightarrow e f \iff \text{epif } f_n \rightarrow \text{epif} f.$$

Following definition is a sequential characterization of epi-convergence.

Given a sequence $(f_n)$ on a metric space $(X, d)$ is epi-convergent to $f$, provided at each $x \in X$, if the following two conditions both hold:

(i) for all $x_n \in X$ whenever $(x_n)$ is convergent to $x$, we have $f(x) \leq \liminf f_n f_n(x_n)$,
(ii) there exists a sequence \((x_n)\) convergent to \(x\) such that \(f(x) = \lim_n f_n(x_n)\).

For every function \(f : X \to \mathbb{R}\) the lower semicontinuous envelope \(sc^{-}f\) of \(f\) is defined for every \(x \in X\) by

\[
(sc^{-}f)(x) = \sup_{g \in \mathcal{G}(f)} g(x),
\]

where \(\mathcal{G}(f)\) is the set of all lower semicontinuous functions \(g\) on \(X\) such that \(g(y) \leq f(y)\) for every \(y \in X\).

**Proposition 1.5.** \([14]\) Let \(f : X \to \overline{\mathbb{R}}\) be a function. Then

\[
(sc^{-}f)(x) = \sup_{V \in \mathcal{N}(x)} \inf_{y \in V} f(y)
\]

for every \(x \in X\) where \(\mathcal{N}(x)\) is the neighbourhood of \(x\).

We also advise to look at \([3, 5, 18, 20]\) for detailed information about new types of convergence of sequences of real valued functions and statistical convergence.

## 2. Main Result

In this part, statistical epi-convergence is defined by the help of Kuratowski convergence on sets. The functions will be taken lower semicontinuous in order to use properties on closed sets since epigraphs of lower semicontinuous functions are closed. Set properties will give a new characterization of statistical epi-convergence by using neighbourhoods of the point \(x \in X\) in a metric space. After that neighbourhoods will give another characterizations of statistical epi-convergence by using sequences this time. Actually almost all definitions of statistical epi-convergence will be achieved in this paper. Level sets which are important instruments in set theory are also included in our calculations for lower and upper epi-limits. Moreover, statistical epi-convergence and statistical pointwise convergence will be discussed at the end.

**Definition 2.1.** Let \((X, d)\) be a metric space and \((f_n)\) a sequence of lower semicontinuous functions defined from \(X\) to \(\mathbb{R}\). The lower statistical epi-limit, \(e_{st^{-}}\liminf_n f_n\) is defined by the help of the sequence of sets:

\[
epi(e_{st^{-}}\liminf_n f_n) := \text{st}^{-}\limsup_n \text{epi} f_n.
\]

Similarly, the upper statistical epi-limit \(e_{st^{-}}\limsup_n f_n\) is defined:

\[
epi(e_{st^{-}}\limsup_n f_n) := \text{st}^{-}\liminf_n \text{epi} f_n.
\]

When these two functions are equal, we get statistical epi-limit function:

\[
f = e_{st^{-}}\lim f_n := e_{st^{-}}\limsup_n f_n = e_{st^{-}}\liminf_n f_n.
\]

As defined in above and by \([1, 3]\) it is obvious that \(e_{st^{-}}\liminf_n f_n \leq e_{st^{-}}\limsup_n f_n\).

Here we use statistical Painlevé-Kuratowski convergence. Whenever \((f_n)\) is epi convergent to \(f\) we can use the inclusion \(st^{-}\limsup_n \text{epi} f_n \subset \text{epi} f \subset st^{-}\liminf_n \text{epi} f_n\). Moreover, following comparisons with e-limit are valid for every function \(f : X \to \overline{\mathbb{R}}\).

\[
e^{-}\liminf_n f_n \leq e_{st^{-}}\liminf_n f_n, \quad e^{-}\limsup_n f_n \leq e_{st^{-}}\limsup_n f_n.
\]

In the following example, the function is not epi-convergent whereas it has statistical epi-limit.
**Example 2.2.** Given a sequence \( f_n : \mathbb{R} \to \mathbb{R} \) defined as

\[
  f_n(x) = \begin{cases} 
    nx & \text{if } n \text{ is square}, \\
    nxe^{nx} & \text{if } n \text{ is nonsquare}.
  \end{cases}
\]

\[
e^{-}\lim \inf f_n(x) = \begin{cases} 
  0 & \text{if } x < 0, \\
  -\frac{1}{e} & \text{if } x = 0, \\
  \infty & \text{if } x > 0.
  \end{cases}
\]

\[
e^{-}\lim \sup f_n(x) = \begin{cases} 
  0 & \text{if } x < 0, \\
  -\frac{1}{2e} & \text{if } x = 0, \\
  \infty & \text{if } x > 0.
  \end{cases}
\]

\[
est \lim f_n(x) = \begin{cases} 
  0 & \text{if } x < 0, \\
  -\frac{1}{2e} & \text{if } x = 0, \\
  \infty & \text{if } x > 0.
  \end{cases}
\]

In general, statistical epi-convergence is neither stronger nor weaker than statistical pointwise convergence. The obvious difference between these convergence types is obtaining minimums. Next example gives the difference between statistical epi limit and statistical pointwise limit.

**Example 2.3.** Given a sequence \( f_n : [-1, 1] \to \mathbb{R} \) with \( (k \in \mathbb{N}) \) defined as

\[
f_n(x) = \begin{cases} 
  \min\{1, 1 - \frac{x}{2}, 3n|x + \frac{1}{n}| - 2\} & \text{if } n = k^2, \\
  \min\{1, 1 - x, 2n|x + \frac{1}{n}| - 1\} & \text{if } n \neq k^2.
  \end{cases}
\]

**Figure 1.** the sequence \((f_n)\) when \(n \neq k^2\)

In Figure 1, we see the graph of the sequence of \((f_n)\) when \(n \neq k^2\). Obviously, the functions take their infimum at \(x_n = -\frac{1}{n}\) as \(-1\).

In Figure 2, it is the graph of the same sequence \((f_n)\) when \(n = k^2\) and the functions take their infimum at \(x_n = -\frac{1}{n}\) as \(-2\).

It can be seen clearly that, when \(n \to \infty\), the sequence \((f_n)\) has not a pointwise limit but it converges statistically to the function \(f(x) = \min\{1, 1 - x\}\) for \(x \in [-1, 1]\). It takes all its values as bigger than 0. Actually, infimum of the function \(f\) is \(f(1) = 0\) whereas the sequence \((f_n)\) takes its infimum at \(x_n = -\frac{1}{n} \neq 0\). We can see the statistical limit function in Figure 3 for \(x \in [-1, 1]\).

On the other hand, it has no epi-limit function. Since \(e^{-}\lim \inf f_n(0) = -2\) and \(e^{-}\lim \sup f_n(0) = -1\) are different on \(x = 0\).
Moreover, it has statistical epi-limit function. If we take $M \in \mathbb{N}$ as a dense set, let $n \in M$ and $-\frac{1}{n} \to 0$ we have $f_n(-\frac{1}{n}) \to -1$. In other words, st-$\lim_n f_n(-\frac{1}{n}) = -1$. So the statistical epi-limit function of the sequence $(f_n)$ is written as,

$$h(x) = \begin{cases} 
 1, & x \in [-1, 0), \\
-1, & x = 0, \\
1 - x, & x \in (0, 1]. 
\end{cases}$$

We say $f_n \overset{\text{st}}{\to} h$ and we can see it in the following figure.

**Lemma 2.4.** Let $(X,d)$ be a metric space and $(f_n)$ a sequence of lower semicontinuous functions defined from $X$ to $\mathbb{R}$, for every $x \in X$, define $g : X \to \mathbb{R}$ by

$$g(x) = \sup_{V \in \mathcal{N}(x)} \left( \liminf_n \inf_{y \in V} f_n(y) \right).$$
We should establish the epigraphical inclusions of the sets st-\(\lim\sup\) \((epif_n)\) \(\subset\) epig and epig \(\subset\) st-\(\lim\sup\) \((epif_n)\). For the first inclusion, let \((x,\alpha)\) \(\in\) st-\(\lim\sup\) \((epif_n)\) be arbitrary. Let \(V_0 \in \mathcal{N}(x)\) and \(\varepsilon > 0\) be fixed. By definition of the statistical upper limit, \(\exists N \in \mathcal{S}^\#\) such that \(\forall n \in N\) we have
\[
V_0 \times (-\infty, \alpha + \varepsilon) \bigcap epif_n \neq \emptyset.
\]
As a result,
\[
\delta\left(\{n \in \mathbb{N} : \inf_{y \in V_0} f_n(y) < \alpha + \varepsilon\}\right) \neq 0
\]
By Lemma 1.2 we have,
\[
st-\lim\inf_n \inf_{y \in V_0} f_n(y) \leq \alpha + \varepsilon.
\]
\(V_0\) and \(\varepsilon\) were arbitrary, we have \(g(x) \leq \alpha\) and hence \((x,\alpha)\) \(\in\) epig.
For the second inclusion let \((x,\alpha)\) \(\in\) epig, for all \(V_0 \in \mathcal{N}(x)\) and for all \(\varepsilon > 0\) we have,
\[
\alpha + \varepsilon > g(x) \geq st-\lim\inf_n \inf_{y \in V_0} f_n(y).
\]
Again by Lemma 1.2 we get \(\delta\left(\{n \in \mathbb{N} : \inf_{y \in V_0} f_n(y) < \alpha + \varepsilon\}\right) \neq 0\). It means, \(\exists N \in \mathcal{S}^\#\) such that \(\forall n \in N\)
\[
V_0 \times (-\infty, \alpha + \varepsilon) \bigcap epif_n \neq \emptyset.
\]
and as epigraphs lie in the vertical direction, we have
\[
V_0 \times (\alpha - \varepsilon, \alpha + \varepsilon) \bigcap epif_n \neq \emptyset.
\]
Hence \((x,\alpha)\) \(\in\) st-\(\lim\sup\) \((epif_n)\). \(\square\)

**Lemma 2.5.** Let \((X, d)\) be a metric space and \((f_n)\) a sequence of lower semicontinuous functions defined from \(X\) to \(\mathbb{R}\), for every \(x \in X\), define \(h : X \rightarrow \mathbb{R}\) by
\[
h(x) = \sup_{V \in \mathcal{N}(x)} st-\lim\sup_n \inf_{y \in V} f_n(y).
\]
Then st-\(\lim\inf_n\) \((epif_n)\) \(=\) epih.

**Proof.** We want to show st-\(\lim\inf_n\) \((epif_n)\) \(\subset\) epih and epih \(\subset\) st-\(\lim\inf_n\) \((epif_n)\). For the first inclusion, let \((x,\alpha)\) \(\in\) st-\(\lim\inf_n\) \((epif_n)\) be arbitrary. Let \(V_0 \in \mathcal{N}(x)\) and \(\varepsilon > 0\) be fixed. By definition of the statistical lower limit, \(\exists N \in \mathcal{S}\) such that \(\forall n \in N\) we have
\[
V_0 \times (-\infty, \alpha + \varepsilon) \bigcap epif_n \neq \emptyset.
\]
As a result,
\[
\delta\left(\{n \in \mathbb{N} : \inf_{y \in V_0} f_n(y) > \alpha + \varepsilon\}\right) = 0
\]
By Lemma 1.1 we have,
\[
st-\lim\inf_n \inf_{y \in V_0} f_n(y) \leq \alpha + \varepsilon.
\]
\(V_0\) and \(\varepsilon\) was arbitrary, we have \(h(x) \leq \alpha\) and hence \((x,\alpha)\) \(\in\) epih.
For the second inclusion, fix \((x,\alpha)\) \(\in\) epih. Given \(V_0 \in \mathcal{N}(x)\) and \(\varepsilon > 0\), \(\exists N \in \mathcal{S}\) such that \(\forall n \in N\) we have
\[
st-\lim\inf_n \inf_{y \in V_0} f_n(y) \leq h(x) < \alpha + \varepsilon
\]
and it equals to the following equality
\[ \delta(\{ n \in \mathbb{N} : \inf_{y \in V_0} f_n(y) < \alpha + \varepsilon \}) = 1. \]

Hence,
\[ \delta(\{ n \in \mathbb{N} : V_0 \times (-\infty, \alpha + \varepsilon) \cap \text{epi} f_n \neq \emptyset \}) = 1. \]

We conclude that
\[ \delta(\{ n \in \mathbb{N} : V_0 \times (\alpha - \varepsilon, \alpha + \varepsilon) \cap \text{epi} f_n \neq \emptyset \}) = 1. \]

It gives \((x, \alpha) \in \text{st-} \lim \inf_n (\text{epi} f_n)\) and concludes the proof. □

Next definition gives us a characterization of epi-limits with the help of Lemma 2.4 and Lemma 2.5.

**Definition 2.6.** Let \((X, d)\) be a metric space and \((f_n)\) a sequence of lower semi-continuous functions from \(X\) into \(\mathbb{R}\), for every \(x \in X\), lower and upper statistical epi-limit functions are defined by
\[
\left( e_{st} \lim \inf_n f_n \right)(x) := \sup_{V \in \mathcal{N}(x)} \text{st-} \lim \inf_n f_n(y),
\]
\[
\left( e_{st} \lim \sup_n f_n \right)(x) := \sup_{V \in \mathcal{N}(x)} \text{st-} \lim \sup_n f_n(y).
\]

If there exists a function \(f : X \to \mathbb{R}\) such that \(e_{st} \lim \inf_n f_n = e_{st} \lim \sup_n f_n = f\), then we write \(f = e_{st} \lim_n f_n\) and we say that \((f_n)\) is \(e_{st}\)-convergent to \(f\) on \(X\).

**Lemma 2.7.** Let \(x = (x_n)\) be a real sequence. Then
\[
\text{st-} \lim \inf_{n \to \infty} x_n = \inf_{N \in \mathcal{S}^*} \sup_{n \in N} x_n = \inf_{N \in \mathcal{S}^*} \sup_{n \in N} x_n
\]
\[
\text{st-} \lim \sup_{n \to \infty} x_n = \sup_{N \in \mathcal{S}^*} \inf_{n \in N} x_n = \inf_{N \in \mathcal{S}^*} \sup_{n \in N} x_n.
\]

By lemma 2.7, the statistical epi-limit infimum can be expressed as follows:
\[
(e_{st} \lim \inf_n f_n)(x) = \sup_{V \in \Omega(x)} \inf_{N \in \mathcal{S}^*} \sup_{n \in N} \inf_{y \in V} f_n(y) = \sup_{V \in \Omega(x)} \inf_{N \in \mathcal{S}^*} \sup_{n \in N} \inf_{y \in V} f_n(y).
\]

Similarly, the statistical epi-limit supremum can be expressed as follows:
\[
(e_{st} \lim \sup_n f_n)(x) = \sup_{V \in \Omega(x)} \inf_{N \in \mathcal{S}^*} \sup_{n \in N} \inf_{y \in V} f_n(y) = \sup_{V \in \Omega(x)} \inf_{N \in \mathcal{S}^*} \sup_{n \in N} \inf_{y \in V} f_n(y).
\]

**Remark.** If the functions \(f_n(x)\) are independent of \(x\), for every \(n \in \mathbb{N}\) there exists a constant \(\alpha_n \in \mathbb{R}\) such that \(f_n(x) = \alpha_n\) for every \(x \in X\),
\[
e_{st} \lim \inf_n f_n(x) = \text{st-} \lim \inf_n \alpha_n, \quad e_{st} \lim \sup_n f_n(x) = \text{st-} \lim \sup_n \alpha_n.
\]

If the functions \(f_n(x)\) are independent of \(n\), there exists \(f : X \to \mathbb{R}\) such that \(f_n(x) = f(x)\) for every \(x \in X\) and for every \(n \in \mathbb{N}\),
\[
e_{st} \lim \inf_n f_n = e_{st} \lim \sup_n f_n = \text{se}^{-} f.
\]

**Proposition 2.8.** In a metric space \((X, d)\) for every \(x \in X\), the following inequalities hold:
\[
(e_{st} \lim \inf_n f_n)(x) \leq \text{st-} \lim \inf_n f_n(x), \quad (e_{st} \lim \sup_n f_n)(x) \leq \text{st-} \lim \sup_n f_n(x).
\]
Proof. \( \forall x \in X \) and \( \forall V \in \mathcal{N}(x) \), \( \exists N \in \mathcal{S} \) such that \( \forall n \in N \) we have
\[
\inf_{y \in V} f_n(y) \leq f_n(x), \quad \text{inf}_{y \in V} f_n(y) \leq f_n(x).
\]
Since by the choice of our index set \( (n \in N) \), we get the following inequalities,
\[
st-\lim \inf_n \inf_{y \in V} f_n(y) \leq st-\lim \inf_n f_n(x), \quad st-\lim \sup_n \inf_{y \in V} f_n(y) \leq st-\lim \sup_n f_n(x).
\]
After taking the supremum over all \( V \in \mathcal{N}(x) \) we get the desired conclusion. \( \square \)

**Theorem 2.9.** Let \( (X,d) \) be a metric space and let \( (f_n) \) be a sequence of lower semicontinuous functions. Suppose that for each \( \alpha \in \mathbb{R} \), \( \exists (\alpha_n) \) of reals statistically convergent to \( \alpha \) with \( lev_{\leq \alpha} f = st-\lim_n (lev_{\leq \alpha_n} f_n) \), then \( f = e_{st-\lim_n} f_n \).

Proof. The condition \( lev_{\leq \alpha} f \subset st-\lim \inf_n (lev_{\leq \alpha_n} f_n) \) valid for each \( \alpha \in \mathbb{R} \) and for some sequence \( \alpha_n \xrightarrow{st} \alpha \). Let \( (x,\alpha) \in epi f \) there exists a sequence \( \alpha_n \) statistically convergent to \( \alpha \) such that \( lev_{\leq \alpha_n} f_n \subset st-\lim \inf_n (lev_{\leq \alpha_n} f_n) \). Hence \( x \in st-\lim \inf_n (lev_{\leq \alpha_n} f_n) \). It means there exists a sequence \( (x_n) \) statistically convergent to \( x \) such that \( x_n \in (lev_{\leq \alpha_n} f_n) \). Finally we get \( (x_n,\alpha_n) \xrightarrow{st} (x,\alpha) \) and \( (x,\alpha) \in st-\lim \inf epi f_n \).

In order to get \( st-\lim \sup epi f_n \subset epi f \), suppose to the contrary that \( (x,\beta) \in \text{st-lim sup} epi f_n \) but that \( (x,\beta) \notin epi f \). Then \( \beta < f(x) \). We can find \( N \in \mathcal{S} \) such that \( \forall n \in N (x_n,\beta_n) \in epi f_n \) such that \( (x,\beta) \in \Gamma(x_n,\beta_n) \). Choose a scalar \( \alpha \) between \( \beta \) and \( f(x) \) and let \( (\alpha_n) \) be a sequence statistically convergent to \( \alpha \) for which \( lev_{\leq \alpha} f \supset st-\lim \inf_n (lev_{\leq \alpha_n} f_n) \). We have \( \delta(n : \beta_n < \alpha_n) \neq 0 \) and \( (x_n,\beta_n) \in epi f_n \), \( \exists N \in \mathcal{S} \), \( \forall n \in N, x_n \in lev_{\leq \alpha_n} f_n \) which means \( x \in st-\lim \sup_n lev\alpha_n f_n \).

By the inclusion \( st-\lim \sup_n lev_{\leq \alpha_n} f_n \subset lev_{\leq \alpha} f \) we get \( x \in lev_{\leq \alpha} f \) and \( f(x) \leq \alpha \) which is a contradiction. \( \square \)

**Theorem 2.10.** The following properties hold for any sequence of lower semicontinuous functions \( (f_n) \) defined on \( X \).

(i) The functions \( e_{st-\lim_n} \inf_n f_n \) and \( e_{st-\lim_n} \sup_n f_n \) are lower semicontinuous and so too is \( e_{st-\lim_n} f_n \) when it exists.

(ii) If the sequence \( (f_n) \) is monotone statistically decreasing, then \( e_{st-\lim_n} f_n \) exists and equals \( sc^{-} \inf_n f_n \).

(iii) If the sequence \( (f_n) \) is monotone statistically increasing, then \( e_{st-\lim_n} f_n \) exists and equals \( sc^{+} \sup_n f_n \).

Proof. (i) Let \( U \) be a family of open subsets of \( X \), \( \alpha : U \rightarrow \mathbb{R} \) be an arbitrary function and \( f : X \rightarrow \mathbb{R} \) be defined by \( f(x) = \sup_{U \in \mathcal{N}(x)} \alpha(U) \). \( \forall U \subseteq X \), \( \forall y \in U \) and \( \forall U \in \mathcal{N}(y) \) it is clear that \( f(y) \geq \alpha(U) \). Since the inequality is satisfied by for all \( U \in \mathcal{N}(x) \) we have
\[
inf_{y \in U} f(y) \geq \alpha(U).
\]
Taking supremum of both sides
\[
f(x) = \sup_{U \in \mathcal{N}(x)} \alpha(U) \leq \sup_{U \in \mathcal{N}(x)} \inf_{y \in U} f(y)
\]
for every \( x \in X \). Since the opposite inequality trivial we get
\[
\sup_{U \in \mathcal{N}(x)} \alpha(U) = \sup_{U \in \mathcal{N}(x)} \inf_{y \in U} f(y)
\]
If we write $\alpha(U) = \text{st-lim inf}_n \inf_{y \in U} f_n(y)$ we get the desired conclusion. The proof is similar for functions $e_{st-\lim sup} f_n$ and $e_{st-\lim} f_n$.

Now we will prove $(ii)$, the proof of $(iii)$ is similar. Since the sequence $(f_n)$ is statistically decreasing, then there exists a subset $K = \{k_1 < k_2 < k_3 < \cdots\} \subseteq \mathbb{N}$ such that $\delta(K) = 1$ and $f_{k_n} \geq f_{k_{n+1}}$ for all $n \in \mathbb{N}$ and its epigraph $epi f_n$ will statistically increase that is $epi f_{k_n} \subseteq epi f_{k_{n+1}}$. In statistical set convergence theory, we have

$$epi(sc^{-}[\inf_n f_n]) = cl \bigcup_{n \in \mathbb{N}} epi f_{n_k}.$$  \hspace{1cm} (2.1)

Moreover, Theorem 2.13 in [24] makes clear the following equality for statistically increasing sequences

$$\text{st-\lim}_n (epi f_n) = cl \bigcup_{n \in \mathbb{N}} epi f_{n_k}.$$  \hspace{1cm} (2.2)

By using (2.1) and (2.2) combining with Definition 2.1

$\text{st-\lim}_n (epi f_n) = e_{st-\lim} f_n(x)$.

Finally we get the desired equation $sc^{-}[\inf_n f_n] = e_{st-\lim} f_n(x)$.

**Definition 2.11.** The sequence $(f_n)$ is called statistically equi-lower semicontinuous at a point $x$ if and only if for all $\varepsilon > 0$ there exists $\delta > 0$ and $N \subseteq \mathbb{N}$ such that for all $y \in B(x, \delta)$ we have,

$$f_n(x) - f_n(y) < \varepsilon$$

for each $n \in M$.

Next theorem gives the basic condition for which statistical convergence and statistical epi-convergence coincide.

**Theorem 2.12.** $(f_n)$ and $f$ are functions from $X$ to $\mathbb{R}$, let $(f_n)$ be statistically equi-lower semicontinuous at $x$. $(f_n)$ is statistically epi-convergent to $f$ at $x$ if and only if $(f_n)$ is statistically convergent to $f$ at $x$.

**Proof.** Assuming $(f_n)$ is statistically equi-lower semicontinuous at $x$, we have that for all $\varepsilon > 0$, there exists $V \in \mathcal{N}(x)$ and $N \subseteq \mathcal{S}$ such that

$$f_n(x) - \varepsilon < \inf_{y \in V} f_n(y)$$

for all $n \in N$. This implies

$$\text{st-lim inf}_n f_n(x) - \varepsilon \leq \sup_{V \in \mathcal{N}(x)} \text{st-lim inf}_n f_n(y)$$

for every $\varepsilon > 0$. Combining with Proposition 2.8 we get

$$\text{st-lim inf}_n f_n(x) = \sup_{V \in \mathcal{N}(x)} \text{st-lim inf}_n f_n(y)$$

which means,

$$\text{st-lim inf}_n f_n(x) = e_{st-\lim} \inf f_n(x).$$

In similar way, we get $\text{st-lim sup}_n f_n(x) = e_{st-\lim} \sup f_n(x)$ and finally we reach the desired equality as follows

$$\text{st-lim} f_n(x) = e_{st-\lim} f_n(x).$$

$\square$
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REFERENCES


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