ON EQUIVALENCE RESULTS FOR WELL-POSEDNESS OF MIXED HEMIVARIATIONAL-LIKE INEQUALITIES IN BANACH SPACES

LU-CHUAN CENG, JEN-CHIH YAO, YONGHONG YAO*

Abstract. In this paper, we devoted to explore conditions of well-posedness for mixed hemivariational-like inequalities in reflexive Banach spaces. By using some equivalent formulations of the mixed hemivariational-like inequality under different $\eta$-monotonicity assumptions, we establish two kinds of conditions under which the strong well-posedness and the weak well-posedness for the mixed hemivariational-like inequality are equivalent to the existence and uniqueness of its solution, respectively.

1. Introduction

Let $X$ be a real reflexive Banach space with the dual space $X^*$. Let $A : X \to X^*$ and $\eta : X \times X \to X$ be two mappings and $G : X \to \mathbb{R} \cup \{+\infty\}$ be a proper functional. Let $J : X \to \mathbb{R}$ be a locally Lipschitz functional and $J(\cdot, \cdot)$ stands for its Clarke's generalized directional derivative. Let $f \in X^*$ be some given element. Now, we consider the following mixed hemivariational-like inequality (in short, MHVLI($A, f, J, \eta, G$)): Find $x \in X$ such that

$$\langle Ax - f, \eta(y, x) \rangle + J^0(x, \eta(y, x)) + G(y) - G(x) \geq 0, \quad \forall y \in X.$$  \hspace{1cm} (1.1)

As an important subject in the theorem of optimization problems and their related problems such as variational inequalities, fixed point problems and equilibrium problems, well-posedness has been drawing more and more researchers’ attention. The classical concept of well-posedness for a global minimization problem was first introduced by Tykhonov [27]. For more literature, we refer the readers to [6, 8, 10, 11, 14]-[19], [21], [33]-[45] and the references therein.

Hemivariational inequality was introduced by Panagiotopoulos [25] in 1983. In 1995, Goeleven and Mentagui [13] first introduced the well-posedness for a hemivariational inequality and presented some basic results concerning the well-posed hemivariational inequality. Later, using the concept of approximating sequence,
Xiao et al. [29] [30] defined a concept of well-posedness for a hemivariational inequality and a variational-hemivariational inequality. Very recently, Xiao, Yang and Huang [31] studied the conditions of well-posedness for the hemivariational inequality considered in [30]. By using some equivalent formulations of the hemivariational inequality considered under different monotonicity assumptions, they established two kinds of conditions under which the strong well-posedness and the weak well-posedness for the hemivariational inequality considered are equivalent to the existence and uniqueness of its solution, respectively.

This article aims to explore some conditions of well-posedness for the mixed hemivariational-like inequality in reflexive Banach spaces. The paper is structured as follows. In Sect. 2, we recall briefly some preliminary material and introduce the definitions of strong (resp. weak) well-posedness for the mixed hemivariational-like inequality considered. Section 3 introduces a definition of strongly relaxed $\eta$-monotonicity for a class of multivalued operators and presents some equivalent formulations of the mixed hemivariational-like inequality considered under the assumptions of strongly relaxed $\eta$-monotonicity and relaxed $\eta$-monotonicity for the nonconvex and nonsmooth operator involved, respectively. In Sect. 4, we give some conditions under which the strong well-posedness and the weak well-posedness for the mixed hemivariational-like inequality are equivalent to the existence and uniqueness of its solution, respectively. Finally, some concluding remarks are provided in Sect. 5.

2. Preliminaries

Let $x$ be a given point and $y$ be a vector in $X$. The Clarke’s generalized directional derivative of $J$ at $x$ in the direction $y$, denoted by $J^o(x, y)$, is defined by

$$J^o(x, y) = \limsup_{z \to x} \frac{J(z + \lambda y) - J(z)}{\lambda}.$$ 

Let $G : X \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex and lower semicontinuous functional. We denote by $\partial G(x) : X \to 2^{X^*} \setminus \{\emptyset\}$ and $\overline{\partial} J(x) : X \to 2^{X^*} \setminus \{\emptyset\}$ the subgradient of convex functional $G$ and the Clarke’s generalized gradient of a locally Lipschitz functional $J$, respectively. That is,

$$\partial G(x) = \{\varrho \in X^* : G(y) - G(x) \geq \langle \varrho, y - x \rangle, \forall y \in X\}$$

and

$$\overline{\partial} J(x) = \{\xi \in X^* : J^o(x, y) \geq \langle \xi, y \rangle, \forall y \in X\}.$$ 

Remark. ([1]). The Clarke’s generalized gradient of a locally Lipschitz functional $J : X \to \mathbb{R}$ at a point $x$ is given by

$$\overline{\partial} J(x) = \partial (J^o(x, \cdot))(0).$$

Concerning the subgradient in the sense of convex analysis, the Clarke’s generalized directional derivative and the Clarke’s generalized gradient, we have the following basic properties (see e.g., [1] [9] [22] [24] [26]).

Proposition 2.1. Let $X$ be a Banach space and $G : X \to \mathbb{R} \cup \{+\infty\}$ be a convex and proper functional. Then we have

(i) $\partial G(x)$ is convex and weak* closed;
(ii) If \( G \) is continuous at \( x \in \text{dom} G \), then \( \partial G(x) \) is nonempty, convex, bounded, and weak*-compact;

(iii) If \( G \) is Gateaux differentiable at \( x \in \text{dom} G \), then \( \partial G(x) = \{ DG(x) \} \), where \( DG(x) \) is the Gateaux derivative of \( G \) at \( x \).

**Proposition 2.2.** Let \( X \) be a Banach space and \( G_1, G_2 : X \to \mathbb{R} \cup \{ +\infty \} \) be two convex functionals. If there is a point \( x_0 \in \text{dom} G_1 \cap \text{dom} G_2 \) at which \( G_1 \) is continuous, then the following equation holds:

\[
\partial (G_1 + G_2)(x) = \partial G_1(x) + \partial G_2(x), \quad \forall x \in X.
\]

**Proposition 2.3.** Let \( X \) be a Banach space, \( x, y \in X \) and \( J \) be a locally Lipschitz functional defined on \( X \). Then

(i) The function \( y \mapsto J^\circ(x, y) \) is finite, positively homogeneous, subadditive and then convex on \( X \);

(ii) \( J^\circ(x, y) \) is upper semicontinuous as a function of \((x, y)\), as a function of \( y \) alone, is Lipschitz continuous on \( X \);

(iii) \( J^\circ(x, -y) = (-J)^\circ(x, y) \);

(iv) \( \partial J(x) \) is a nonempty, convex, bounded and weak* compact subset of \( X^* \);

(v) For every \( y \in X \), one has

\[
J^\circ(x, y) = \max\{ \langle \xi, y \rangle : \xi \in \partial J(x) \};
\]

(vi) The graph of the Clarke's generalized gradient \( \partial J(x) \) is closed in \( X \times (w^*-X^*) \) topology, where \((w^*-X^*)\) denotes the space \( X^* \) equipped with weak* topology, i.e., if \( \{ x_n \} \subset X \) and \( \{ x_n^* \} \subset X^* \) are sequences such that \( x_n^* \in \partial J(x_n), \ x_n \to x \) in \( X \) and \( x_n^* \to x^* \) weakly* in \( X^* \), then \( x^* \in \partial J(x) \).

Let \( \eta : X \times X \to X \) and \( G : X \to \mathbb{R} \cup \{ +\infty \} \). A vector \( z^* \in X^* \) is called an \( \eta \)-subgradient of \( G \) at \( x \in \text{dom} G \) if

\[
\langle z^*, \eta(y, x) \rangle \leq G(y) - G(x), \quad \forall y \in X.
\]

Each \( G \) can be associated with the following \( \eta \)-subdifferential map \( \partial_\eta G \) defined by

\[
\partial_\eta G(x) = \begin{cases} 
\{ z^* \in X^* : \langle z^*, \eta(y, x) \rangle \leq G(y) - G(x), \forall y \in X \}, & x \in \text{dom} G, \\
0, & x \notin \text{dom} G.
\end{cases}
\]

Let \( X \) be a real Banach space with its dual \( X^* \), \( \eta : X \times X \to X \) be a mapping and \( T : X \to X^* \) be a single-valued operator.

**Definition 2.4.** \( T \) is said to be

(i) \( \eta \)-monotone, if

\[
\langle Tx - Ty, \eta(x, y) \rangle \geq 0, \quad \forall x, y \in X;
\]

(ii) strongly \( \eta \)-monotone with constant \( m > 0 \), if

\[
\langle Tx - Ty, \eta(x, y) \rangle \geq m \| x - y \| \| \eta(x, y) \|, \quad \forall x, y \in X.
\]

**Definition 2.5.** Let \( F : X \to 2^{X^*} \) be a multi-valued operator. \( F \) is said to be

(i) \( \eta \)-monotone, if

\[
\langle u - v, \eta(x, y) \rangle \geq 0, \quad \forall x, y \in X, u \in F(x), v \in F(y);
\]

(ii) strongly \( \eta \)-monotone with constant \( k > 0 \), if

\[
\langle u - v, \eta(x, y) \rangle \geq k \| x - y \| \| \eta(x, y) \|, \quad \forall x, y \in X, u \in F(x), v \in F(y);
\]
(iii) relaxed $\eta$-monotone with constant $c > 0$, if
\[ \langle u - v, \eta(x, y) \rangle \geq -c\|x - y\|\|\eta(x, y)\|, \quad \forall x, y \in X, u \in F(x), v \in F(y). \]

**Definition 2.6.** $T$ is said to be $\eta$-hemicontinuous if, for any $x, y \in X$, the function $t \mapsto (T(x + t\eta(y, x)), \eta(y, x))$ from $[0, 1]$ into $\mathbb{R} = (-\infty, \infty)$ is continuous at $0^+$. 

**Remark.** Clearly, whenever $\eta(x, y) = x - y$ for all $x, y \in X$, then Definitions 2.4, 2.5 and 2.6 reduce to Definitions 2.1, 2.2 and 2.3 in Xiao, Yang and Huang [31], respectively. In addition, continuity implies $\eta$-hemicontinuity, but the converse is not true in general. For the usual concepts of monotonicity and hemicontinuity of single-valued operators, we refer the readers to [43].

**Definition 2.7.** ([28]). The function $G : X \to \mathbb{R}$ is said to be preinvex w.r.t. $\eta$ iff, for all $x, y \in X$ and $t \in [0, 1]$,
\[ G(x + t\eta(y, x)) \leq (1 - t)G(x) + tG(y). \]

In the sequel, we need to use the following condition introduced by Mohan and Neogy [23].

**Hypothesis (A).** Let $\eta(\cdot, \cdot) : X \times X \to X$ be a mapping. For all $x, y \in X$ and $t \in [0, 1]$, the following relations hold:
\[ \eta(x, x + t\eta(y, x)) = -t\eta(y, x) \text{ and } \eta(y, x + t\eta(y, x)) = (1 - t)\eta(y, x). \]

Clearly, for $t = 0$, we have $\eta(x, x) = 0$, $\forall x \in X$. Yang et al. [24] have shown that if $\eta : X \times X \to X$ satisfies Hypothesis (A), then
\[ \eta(y + t\eta(x, y), y) = t\eta(x, y). \]

**Theorem 2.8.** ([12]). Let $C \subset X$ be nonempty, closed and convex, $C^* \subset X^*$ be nonempty, closed, convex and bounded, $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ be proper, convex and lower semicontinuous and $y \in C$ be arbitrary. Assume that, for each $x \in C$, there exists $x^*(x) \in C^*$ such that
\[ \langle x^*(x), x - y \rangle \geq \varphi(y) - \varphi(x). \]

Then, there exists $y^* \in C^*$ such that
\[ \langle y^*, x - y \rangle \geq \varphi(y) - \varphi(x), \quad \forall x \in C. \]

According to the above Theorem 2.8, we naturally introduce the following condition, which will be used in the sequel.

**Hypothesis (B).** Let $\varphi : X \times X \to X$ be a mapping. Let $C \subset X$ be nonempty, closed and convex, $C^* \subset X^*$ be nonempty, closed, convex and bounded, $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ be proper, preinvex w.r.t. $\eta$ and lower semicontinuous and $y \in C$ be arbitrary. Assume that, for each $x \in C$, there exists $x^*(x) \in C^*$ such that
\[ \langle x^*(x), \eta(x, y) \rangle \geq \varphi(y) - \varphi(x). \]

Then, there exists $y^* \in C^*$ such that
\[ \langle y^*, \eta(x, y) \rangle \geq \varphi(y) - \varphi(x), \quad \forall x \in C. \]

**Remark.** If $\eta(x, y) = x - y$ for all $x, y \in X$, then Hypothesis (B) reduces to Theorem 2.8.

Based on some concepts of well-posedness in [2, 3, 4, 5, 7, 20, 30, 31], we now introduce some definitions of well-posedness for the mixed hemivariational-like inequality MHVLI$(A, f, J, \eta, G)$. 

Definition 2.9. A sequence \( \{x_n\} \subset X \) is said to be an approximating sequence for the mixed hemivariational-like inequality \( \text{MHVLI}(A,f,J,\eta,G) \) if there exists a nonnegative sequence \( \{\epsilon_n\} \) with \( \epsilon_n \to 0 \) as \( n \to \infty \) such that
\[
\langle Ax_n-f,\eta(y,x_n)\rangle + J^\circ(x_n,\eta(y,x_n)) + G(y) - G(x_n) \geq -\epsilon_n \|\eta(y,x_n)\|, \quad \forall y \in X, \; n \in \mathbb{N}.
\]

Definition 2.10. The mixed hemivariational-like inequality \( \text{MHVLI}(A,f,J,\eta,G) \) is said to be strongly (resp. weakly) well-posed if it has a unique solution in \( X \) and every approximating sequence converges strongly (resp. weakly) to the unique solution.

Definition 2.11. The mixed hemivariational-like inequality \( \text{MHVLI}(A,f,J,\eta,G) \) is said to be strongly (resp. weakly) in the generalized sense if it has a nonempty solution set \( S \) in \( X \) and every approximating sequence has a subsequence which converges strongly (resp. weakly) to some point of solution set \( S \).

3. STRONGLY RELAXED \( \eta \)-MONOTONICITY

In this section, we present equivalent formulations of the mixed hemivariational-like inequality \( \text{MHVLI}(A,f,J,\eta,G) \) under the assumptions of strongly relaxed \( \eta \)-monotonicity and relaxed \( \eta \)-monotonicity for the nonconvex and nonsmooth mapping involved, respectively.

Definition 3.1. Let \( X \) be a real Banach space with its dual \( X^* \), \( \eta : X \times X \to X \) be a mapping and \( F : X \to 2^{X^*} \) a nonempty multi-valued mapping. \( F \) is said to satisfy the strongly relaxed \( \eta \)-monotonicity condition with constant \( c > 0 \) if, for all \( x,y \in X \) and \( u \in F(x) \) (or \( v \in F(y) \)), there exists a \( v \in F(y) \) (or \( u \in F(x) \)) such that
\[
\langle u - v, \eta(x,y) \rangle \geq -c \|x - y\| \|\eta(x,y)\|.
\]

Lemma 3.2. Let \( A \) be a mapping from a real Banach space \( X \) to its dual \( X^* \), \( \eta : X \times X \to X \) be a mapping, \( J : X \to \mathbb{R} \) be a locally Lipschitz functional and \( G : X \to \mathbb{R} \cup \{+\infty\} \) be a proper, preinvex w.r.t. \( \eta \) and lower semicontinuous functional with the \( \eta \)-subdifferential map \( \partial_{\eta}G \). Assume that Hypothesis (B) holds. Then, \( x \in X \) is a solution to the mixed hemivariational-like inequality \( \text{MHVLI}(A,f,J,\eta,G) \) if and only if \( x \) is a solution to the following inclusion problem:
\[
\Pi(A,f,J,\eta,G) : \text{Find } x \in X \text{ such that } f \in Ax + \partial J(x) + \partial_{\eta}G(x).
\]  

Proof. The lemma is easily proved by the definitions of the Clarke’s generalized gradient for locally Lipschitz functional and the \( \eta \)-subgradient for preinvex functional \( G \) w.r.t. \( \eta \). To this end, let \( x \in X \) be a solution to the inclusion problem \( \Pi(A,f,J,\eta,G) \). Then, there exist \( \xi \in \partial J(x) \) and \( \varrho \in \partial_{\eta}G(x) \) such that
\[
f = Ax + \xi + \varrho.
\]

For any \( y \in X \), multiplying the above Eq. (3.2) by \( \langle \eta(y),x \rangle \), we can get by the definitions of the Clarke’s generalized gradient for locally Lipschitz functional and the \( \eta \)-subgradient for preinvex functional \( G \) w.r.t. \( \eta \), that
\[
\langle \eta(y),x \rangle \leq \langle Ax, \eta(y) \rangle + J^\circ(x, \eta(y)) + G(y) - G(x), \forall y \in X.
\]

Thus, \( x \) is a solution to the mixed hemivariational-like inequality \( \text{MHVLI}(A,f,J,\eta,G) \).
On the other hand, let $x$ be a solution to the mixed hemivariational-like inequality MHVLI $(A, f, J, \eta, G)$, which means
\[
\langle Ax-f, \eta(y, x) \rangle + J^\circ(x, \eta(y, x)) + G(y) - G(x) \geq 0, \quad \forall y \in X.
\] (3.3)
From the fact that
\[
J^\circ(x, \eta(y, x)) = \max \{\langle \xi, \eta(y, x) \rangle : \xi \in \partial J(x) \},
\]
we get that there exists a $\xi(x, y) \in \partial J(x)$ such that
\[
\langle Ax-f, \eta(y, x) \rangle + \langle \xi(x, y), \eta(y, x) \rangle + G(y) - G(x) \geq 0, \quad \forall y \in X.
\]
By virtue of Proposition 2.3(iv), $\partial J(x)$ is a nonempty, convex, bounded and weak*-compact subset in $X^*$, which implies that $\{Ax-f+\xi : \xi \in \partial J(x)\}$ is a nonempty, convex, bounded and weak*-compact subset in $X^*$, and hence a nonempty, convex, bounded and weakly closed subset in $X^*$ by virtue of the reflexivity of $X$. Consequently, it is a nonempty, convex, bounded and closed subset in $X^*$. Since $G : X \rightarrow \mathbb{R} \cup \{+\infty\}$ is a proper, preinvex w.r.t. $\eta$ and lower semicontinuous functional, it follows from Hypothesis (B) with $\varphi(\cdot) = G(\cdot)$ and the last inequality that there exists $\xi(x) \in \partial J(x)$ such that
\[
\langle Ax-f, \eta(y, x) \rangle + \langle \xi(x), \eta(y, x) \rangle + G(y) - G(x) \geq 0, \quad \forall y \in X.
\]
For the sake of simplicity, we denote $\xi = \xi(x)$. Then, by the last inequality we have
\[
G(y) - G(x) \geq \langle -(Ax + Tx - f + \xi), \eta(y, x) \rangle, \quad \forall y \in X,
\]
which together with the definition of the $\eta$-subdifferential map $\partial_\eta G$, implies that
\[
-(Ax - f + \xi) \in \partial_\eta G(x).
\]
Thus, it follows from $\xi \in \partial J(x)$ that
\[
0 \in Ax - f + \partial J(x) + \partial_\eta G(x),
\]
which implies that $x$ is a solution to the inclusion problem $\text{IP}(A, f, J, \eta, G)$. This completes the proof. □

Lemma 3.3. Let $\eta : X \times X \rightarrow X$ satisfy Hypothesis (A). Let $G : X \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper, preinvex w.r.t. $\eta$ and lower semicontinuous functional with the $\eta$-subdifferential map $\partial_\eta G$. Assume that operator $A : X \rightarrow X^*$ is $\eta$-hemicontinuous and strongly $\eta$-monotone with constant $m > 0$ on $X$ and $J : X \rightarrow \mathbb{R}$ is a locally Lipschitz functional on $X$ such that the Clarke’s generalized gradient $\partial J(\cdot)$ satisfies the strongly relaxed $\eta$-monotonicity condition with constant $c > 0$. Assume that Hypothesis (B) holds. If $m \geq c$, then the following three statements are equivalent:

(i) $x$ is a solution of the mixed hemivariational-like inequality MHVLI $(A, f, J, \eta, G)$, that is,
\[
\langle Ax-f, \eta(y, x) \rangle + J^\circ(x, \eta(y, x)) + G(y) - G(x) \geq 0, \quad \forall y \in X;
\]

(ii) $x$ is a solution of the following associated mixed hemivariational-like inequality AMHVLI $(A, f, J, \eta, G)$: Find $x \in X$ such that
\[
\langle Ay-f, \eta(y, x) \rangle + J^\circ(y, \eta(y, x)) + G(y) - G(x) \geq 0, \quad \forall y \in X;
\]

(iii) $x$ is a solution of the following multi-valued mixed variational-like inequality MMVLI $(A, f, J, \eta, G)$: Find $x \in X$ such that, for all $y \in X$, there exists a $\zeta \in \partial J(y)$ satisfying
\[
\langle Ay+f, \eta(y, x) \rangle + G(y) - G(x) \geq 0, \quad \forall y \in X.
\]
Proof. Firstly, we prove that (i) ⇔ (ii). To this end, let \( x \in X \) be a solution to the mixed hemivariational-like inequality MHVLI\((A, f, J, \eta, G)\), which means that
\[
\langle Ax - f, \eta(y, x) \rangle + J^\circ(x, \eta(y, x)) + G(y) - G(x) \geq 0, \ \forall y \in X.
\]
By Lemma 3.2, \( x \) be a solution to the inclusion problem IP\((A, f, J, \eta, G)\), and thus there exist \( \xi \in \partial J(x) \) and \( \varrho \in \partial \eta G(x) \) such that
\[
f = Ax + \xi + \varrho.
\] (3.4)
For any \( y \in X \), by the strongly relaxed \( \eta \)-monotonicity of \( \overline{\partial} J(\cdot) \) on \( X \), there exists an \( \zeta \in \partial J(y) \) such that
\[
\langle \zeta - \xi, \eta(y, x) \rangle \geq -c\|y - x\|\|\eta(y, x)\|.
\] (3.5)
Note that for \( \varrho \in \partial \eta G(x) \) the definition of the \( \eta \)-subgradient of \( G \) at \( x \) leads to
\[
G(y) - G(x) \geq \langle \varrho, \eta(y, x) \rangle,
\]
which yields
\[
G(y) - G(x) - \langle \varrho, \eta(y, x) \rangle \geq 0.
\]
Thus, it follows from the strong \( \eta \)-monotonicity of the operator \( A \), (3.4), (3.5) and the condition \( m \geq c \) that
\[
\langle Ay + \zeta - f, \eta(y, x) \rangle + G(y) - G(x)
= \langle Ay + \zeta - (Ax + \xi + \varrho), \eta(y, x) \rangle + G(y) - G(x)
= \langle Ay - Ax, \eta(y, x) \rangle + \langle \zeta - \xi, \eta(y, x) \rangle - \langle \varrho, \eta(y, x) \rangle + G(y) - G(x)
\geq (m - c)\|y - x\|\|\eta(y, x)\|\geq 0,
\]
which together with the definition of the Clarke's generalized gradient and \( \zeta \in \overline{\partial} J(y) \), implies that
\[
\langle f - Ay, \eta(y, x) \rangle + G(x) - G(y) \leq \langle \zeta, \eta(y, x) \rangle \leq J^\circ(y, \eta(y, x)), \ \forall y \in X,
\]
i.e., \( x \) is a solution to the associated mixed hemivariational-like inequality AMHVLI\((A, f, J, \eta, G)\). Therefore, (i) ⇒ (ii) holds.

On the other hand, utilizing Hypothesis (A), Yang et al. \[32\] have shown that
\[
\eta(x + t\eta(y, x), x) = t\eta(y, x)
\]
for all \( x, y \in X \) and \( t \in [0, 1) \). Let \( x \) be a solution to the associated mixed hemivariational-like inequality AMHVLI\((A, f, J, \eta, G)\), which means that
\[
\langle Ay - f, \eta(y, x) \rangle + J^\circ(y, \eta(y, x)) + G(y) - G(x) \geq 0, \ \forall y \in X.
\] (3.6)
Given any \( y \in X \) we define \( y_t = x + t\eta(y, x) \) for all \( t \in (0, 1) \). Replacing \( y \) by \( y_t \) in the above inequality (3.6), we deduce from the preinvexity of \( G \) w.r.t. \( \eta \) and the positive homogeneity of the function \( y \mapsto J^\circ(x, y) \) that
\[
0 \leq \langle Ay_t - f, \eta(y_t, x) \rangle + J^\circ(y_t, \eta(y_t, x)) + G(y_t) - G(x)
= \langle Ay_t - f, \eta(x + t\eta(y, x), x) \rangle + J^\circ(y_t, \eta(x + t\eta(y, x), x)) + G(x + t\eta(y, x)) - G(x)
\leq \langle Ay_t - f, t\eta(y_t, x) \rangle + J^\circ(y_t, t\eta(y, x)) + (1 - t)G(x) + tG(y) - G(x)
= t[\langle Ay_t - f, \eta(y, x) \rangle + J^\circ(y_t, \eta(y_t, x)) + G(y) - G(x)],
\]
which hence implies that for each \( t \in (0, 1) \),
\[
\langle Ay_t - f, \eta(y, x) \rangle + J^\circ(y_t, \eta(y_t, x)) + G(y) - G(x) \geq 0.
\] (3.7)
It is obvious that \( y_t = x + t\eta(y, x) \to x \) as \( t \to 0^+ \) and the \( \eta \)-hemicontinuity of the operator \( A \) on \( X \) implies that
\[
\lim_{t \to 0^+} \langle Ay_t - f, \eta(y, x) \rangle = \lim_{t \to 0^+} (A(x + t\eta(y, x)) - f, \eta(y, x)) = \langle Ax - f, \eta(y, x) \rangle. \tag{3.8}
\]
Moreover, by Proposition \( 2.3 \) (i), (ii), \( J^\circ(x, y) \) is positively homogeneous with respect to \( y \) and upper semicontinuous with respect to \( (x, y) \). Thus, taking the limsup as \( t \to 0^+ \) at both sides of inequality \( 3.7 \), we obtain from \( 3.8 \) that
\[
\langle Ax - f, \eta(y, x) \rangle + J^\circ(x, y, \eta(y, x)) + G(y) - G(x) \\
\geq \limsup_{t \to 0^+} \langle A(x + t\eta(y, x)) - f, \eta(y, x) \rangle + J^\circ(x + t\eta(y, x), \eta(y, x)) + G(y) - G(x) \\
= \limsup_{t \to 0^+} \langle Ay_t - f, \eta(y, x) \rangle + J^\circ(y_t, \eta(y, x)) + G(y) - G(x) \\
\geq 0.
\]
By the arbitrariness of \( y \in X \), we conclude that \( x \) is a solution of the mixed hemivariational-like inequality \( \text{MHVLI}(A, f, J, \eta, G) \). Therefore, (ii) \( \Rightarrow \) (i) holds.

Secondly, we prove that (i) \( \Leftrightarrow \) (iii). Indeed, let \( x \) be a solution to the mixed hemivariational-like inequality \( \text{MHVLI}(A, f, J, \eta, G) \). By the same arguments as in the proof of (i) \( \Rightarrow \) (ii), from the definition of the \( \eta \)-subgradient of \( G \) at \( x \), the strong \( \eta \)-monotonicity of the mapping \( A \), the strongly relaxed \( \eta \)-monotonicity of the Clarke’s generalized gradient \( \partial J(\cdot) \), and the condition \( m \geq c \), we know that, for any \( y \in X \) there exists a \( \zeta \in \partial J(y) \) such that
\[
\langle Ay + \zeta - f, \eta(y, x) \rangle + G(y) - G(x) \geq 0, \tag{3.9}
\]
which actually implies that \( x \) is a solution to the multi-valued mixed variational-like inequality \( \text{MMVLI}(A, f, J, \eta, G) \). Therefore, (i) \( \Rightarrow \) (iii) holds. For (iii) \( \Rightarrow \) (i), let \( x \) be a solution to the multi-valued mixed variational-like inequality \( \text{MMVLI}(A, f, J, \eta, G) \), which means that, for any \( y \in X \), there exists a \( \zeta \in \partial J(y) \) satisfying \( 3.9 \). Given any \( y \in X \) we define \( y_t = x + t\eta(y, x) \) for all \( t \in (0, 1) \). Replacing \( y \) by \( y_t \) in the left side of the above inequality \( 3.9 \), we deduce that there exists \( \zeta_t \in \partial J(y_t) \) such that
\[
\langle Ay_t + \zeta_t - f, \eta(y_t, x) \rangle + G(y_t) - G(x) \geq 0, \tag{3.10}
\]
which together with the definition of the Clarke’s generalized gradient and \( \zeta_t \in \partial J(y_t) \), implies that \( \langle \zeta_t, \eta(y_t, x) \rangle \leq J^\circ(y_t, \eta(y_t, x)) \) and hence
\[
\langle Ay_t - f, \eta(y_t, x) \rangle + J^\circ(y_t, \eta(y_t, x)) + G(y_t) - G(x) \geq 0.
\]
By the same arguments as in the proof of (ii) \( \Rightarrow \) (i), from the preinvexity of \( G \) w.r.t. \( \eta \), the \( \eta \)-hemicontinuity of \( A \) on \( X \), the positive homogeneous of \( J^\circ(x, y) \) w.r.t. \( y \) and the upper semicontinuity of \( J^\circ(x, y) \) w.r.t. \( (x, y) \), we can conclude that
\[
\langle Ax - f, \eta(y, x) \rangle + J^\circ(x, \eta(y, x)) + G(y) - G(x) \geq 0.
\]
By the arbitrariness of \( y \in X \), we know that \( x \) is a solution of the mixed hemivariational-like inequality \( \text{MHVLI}(A, f, J, \eta, G) \). This completes the proof.

4. Equivalence results for well-posedness

In this section, we give some conditions under which the strong well-posedness and the weak well-posedness for the mixed hemivariational-like inequality \( \text{MHVLI}(A, f, J, \eta, G) \) are equivalent to the existence and uniqueness of its solution, respectively.
**Theorem 4.1.** Let $\eta : X \times X \to X$ be skew, i.e., $\eta(x, y) + \eta(y, x) = 0$, $\forall x, y \in X$. Let $A : X \to X^*$ be strongly $\eta$-monotone with constant $m > 0$, and $J : X \to \mathbb{R}$ be a locally Lipschitz functional such that the Clarke’s generalized gradient $\partial J(\cdot) : X \to 2^{X^*}$ satisfies the relaxed $\eta$-monotonicity condition with constant $c > 0$. Let $G : X \to \mathbb{R} \cup \{+\infty\}$ be a proper, preinvex w.r.t. $\eta$ and lower semicontinuous functional with the $\eta$-subdifferential map $\partial_\eta G$. Assume that Hypothesis (B) holds. If $m > c$, then the mixed hemivariational-like inequality $\text{MHVLI}(A, f, J, \eta, G)$ is strongly well-posed if and only if it has a unique solution in $X$.

**Proof.** Obviously, the necessity follows immediately from Definition 2.10 of the strong $\alpha$-well-posedness for the mixed hemivariational-like inequality $\text{MHVLI}(A, f, J, \eta, G)$. It remains to prove the sufficiency. Assume that the mixed hemivariational-like inequality $\text{MHVLI}(A, f, J, \eta, G)$ has a unique solution $x^* \in X$. We claim that $x_n \to x^*$ in $X$ for any approximating sequence $\{x_n\} \subset X$ for the mixed hemivariational-like inequality $\text{MHVLI}(A, f, J, \eta, G)$. Since $x^*$ is the unique solution to the mixed hemivariational-like inequality $\text{MHVLI}(A, f, J, \eta, G)$, we have that

$$\langle Ax^* - f, \eta(y, x^*) \rangle + J^\circ(x^*, \eta(y, x^*)) + G(y) - G(x^*) \geq 0, \forall y \in X.$$  

By Lemma 3.2, $x^*$ is also a solution to the inclusion problem

$$f \in Ax + \partial J(x) + \partial_\eta G(x),$$

and thus there exist $\xi \in \partial J(x^*)$ and $\varrho \in \partial_\eta G(x^*)$ such that

$$f = Ax^* + \xi + \varrho \quad (4.1)$$

(see the argument process of (i) $\Rightarrow$ (ii) in the proof of Lemma 3.2). Moreover, $\{x_n\} \subset X$ is an approximating sequence for the mixed hemivariational-like inequality $\text{MHVLI}(A, f, J, \eta, G)$, which means that there exists a nonnegative sequence $\{\epsilon_n\}$ with $\epsilon_n \to 0$ as $n \to \infty$ such that

$$\langle Ax_n - f, \eta(y, x_n) \rangle + J^\circ(x_n, \eta(y, x_n)) + G(y) - G(x_n) \geq -\epsilon_n ||\eta(y, x_n)||, \forall y \in X. \quad (4.2)$$

From the fact that

$$J^\circ(x_n, \eta(y, x_n)) = \max \{\nu, \eta(y, x_n) \in \partial J(x_n)\},$$

we obtain by the inequality (4.2) that there exists a $\xi(x_n, y) \in \partial J(x_n)$ such that

$$\langle Ax_n - f, \eta(y, x_n) \rangle + \langle \xi(x_n, y), \eta(y, x_n) \rangle + G(y) - G(x_n) \geq -\epsilon_n ||\eta(y, x_n)||, \forall y \in X. \quad (4.3)$$

Define the functional $Q_n(\cdot) : X \to \mathbb{R}$ as below

$$Q_n(y) = ||\eta(y, x_n)||, \forall y \in X.$$

It is easy to calculate that

$$\partial Q_n(y) = \{y^* \in X^* : ||y^*|| = 1 \text{ and } \langle y^*, \eta(y, x_n) \rangle = ||\eta(y, x_n)||\},$$

and hence, for each $n \in \mathbb{N}$ there exists a $\zeta(x_n, y) \in \partial Q_n(y)$ with $||\zeta(x_n, y)|| = 1$ such that

$$\langle \zeta(x_n, y), \eta(y, x_n) \rangle = ||\eta(y, x_n)||, \forall n \in \mathbb{N}.$$  

Then (4.3) can be rewritten as

$$\langle Ax_n + \zeta(x_n, y) + \epsilon_n \xi(x_n, y), \eta(y, x_n) \rangle \geq G(x_n) - G(y), \forall y \in X. \quad (4.4)$$

On the other hand, by virtue of Proposition 2.3 (vi), $\partial J(x_n)$ is a nonempty, convex, bounded and weak*-compact subset of $X^*$. Since $X$ is reflexive, it can be
readily seen that the weak topology \( \sigma(X^*, X^{**}) \) coincides with the weak* topology \( \sigma(X^*, X) \). So, it follows that \( \overline{\partial J}(x_n) \) is a nonempty, convex, bounded and weakly closed subset of \( X^* \). Note that, for any subset in \( X \), its closed convexity coincides with its weakly closed convexity. Thus, \( \overline{\partial J}(x_n) \) is a nonempty, convex, bounded and closed subset of \( X^* \), which immediately implies that \( \{Ax_n - f + \xi : \xi \in \overline{\partial J}(x_n)\} \) is a nonempty, convex, bounded and closed subset of \( X^* \). Consequently, we know that

\[
\{Ax_n - f + \xi : \xi \in \overline{\partial J}(x_n) \text{ and } \xi \in B(0,1)\}
\]

is a nonempty, convex, bounded and closed subset of \( X^* \), where \( B(0,1) \) is the closed ball centered at 0 with radius 1. We now set \( C = X^* \) and

\[
C^* = \{Ax_n - f + \xi : \xi \in \overline{\partial J}(x_n) \text{ and } \xi \in B(0,1)\}.
\]

So, it follows from \([4.4]\) and Hypothesis (B), with \( \varphi(\cdot) = G(\cdot) \) which is proper, preinvex w.r.t. \( \eta \) and lower semicontinuous, that there exists \( \omega(x_n) \in C^* \) such that

\[
(\omega(x_n), \eta(y, x_n)) \geq G(x_n) - G(y), \ \forall y \in X. \tag{4.5}
\]

From \( \omega(x_n) \in C^* \), it follows that there exist \( \xi(x_n) \in \overline{\partial J}(x_n) \) and \( \zeta(x_n) \in B(0,1) \) such that \( \omega(x_n) = Ax_n - f + \xi(x_n) + \epsilon_n \zeta(x_n) \). Then \([4.5]\) can be rewritten as

\[
G(y) - G(x_n) \geq \langle -(Ax_n - f + \xi(x_n) + \epsilon_n \zeta(x_n)), \eta(y, x_n) \rangle, \ \forall y \in X. \tag{4.6}
\]

For the sake of simplicity, we denote \( \xi_n = \xi(x_n) \) and \( \zeta_n = \zeta(x_n) \). So, it follows from \([4.6]\) that

\[
G(y) - G(x_n) \geq \langle -(Ax_n - f + \xi_n + \epsilon_n \zeta_n), \eta(y, x_n) \rangle, \ \forall y \in X. \tag{4.7}
\]

Specially, taking \( y = x^* \) in the above inequality \([4.7]\) yields

\[
G(x^*) - G(x_n) \geq \langle -(Ax_n - f + \xi_n + \epsilon_n \zeta_n), \eta(x^*, x_n) \rangle,
\]

which hence leads to

\[
\epsilon_n \langle \zeta_n, \eta(x^*, x_n) \rangle \geq G(x_n) - G(x^*) + \langle f - (Ax_n + \xi_n), \eta(x^*, x_n) \rangle. \tag{4.8}
\]

It follows from the strong \( \eta \)-monotonicity of the operator \( A \), the relaxed \( \eta \)-monotonicity of the Clarke’s generalized gradient \( \overline{\partial J}(\cdot) \), the skew property of \( \eta \), and the Eqs. \([4.4]\) and \([4.8]\) that

\[
\epsilon_n \|\eta(x^*, x_n)\| \geq \epsilon_n \langle \zeta_n, \eta(x^*, x_n) \rangle
\]

\[
\geq G(x_n) - G(x^*) + \langle f - (Ax_n + \xi_n), \eta(x^*, x_n) \rangle
\]

\[
= G(x_n) - G(x^*) + \langle Ax^* + \xi + \varrho - (Ax_n + \xi_n), \eta(x^*, x_n) \rangle
\]

\[
= G(x_n) - G(x^*) - \langle \varrho, \eta(x_n, x^*) \rangle + \langle Ax^* + \xi - (Ax_n + \xi_n), \eta(x^*, x_n) \rangle
\]

\[
\geq (Ax^* - Ax_n + \xi - \xi_n, \eta(x^*, x_n))
\]

\[
\geq (m - c)\|x^* - x_n\| \|\eta(x^*, x_n)\|,
\]

which implies from the condition \( m > c \) that

\[
\|x^* - x_n\| \leq \frac{\epsilon_n}{m - c}. \tag{4.9}
\]

Taking the limit at both sides of the above inequality \([4.9]\) yields \( x_n \to x^* \) in \( X \). This completes the proof of Theorem \([4.1]\). \( \square \)
Remark. By the proof of Theorem 4.1, the condition \( m > c \) plays an important role in the proof of the strong convergence of the approximating sequence \( \{x_n\} \) for the mixed hemivariational-like inequality MHVLI(\( A, f, J, \eta, G \)). It is clear that we cannot obtain the conclusion in Theorem 4.1 when the condition \( m > c \) fails to hold. The following theorem gives the conditions under which the existence and uniqueness of solutions of the mixed hemivariational-like inequality MHVLI(\( A, f, J, \eta, G \)) is equivalent to its weak well-posedness when \( m = c \).

**Theorem 4.2.** Let \( \eta : X \times X \to X \) satisfy the conditions:

(i) \( \eta(x, z) = \eta(x, y) + \eta(y, z), \forall x, y, z \in X; \)

(ii) \( ||\eta(x, y)|| \geq \gamma_0||x - y||, \forall x, y \in X \) for some \( \gamma_0 > 0; \)

(iii) Hypothesis (A) holds; and

(iv) \( \eta \) is weakly continuous in the first variable.

Let operator \( A : X \to X^* \) be \( \eta \)-hemiconvex and strongly \( \eta \)-monotone with constant \( m > 0 \), and \( J : X \to \mathbb{R} \) be a locally Lipschitz functional such that the Clarke’s generalized gradient \( \partial J(y) : X \to 2^{X^*} \) satisfies the relaxed \( \eta \)-monotonicity condition with constant \( c > 0 \). Let \( G : X \to \mathbb{R} \cup \{+\infty\} \) be a proper, preinvex w.r.t. \( \eta \) and weakly lower semicontinuous functional with the \( \eta \)-subdifferential map \( \partial \eta G \). Assume that Hypothesis (B) holds. If \( m = c \), then the mixed hemivariational-like inequality MHVLI(\( A, f, J, \eta, G \)) is weakly well-posed if and only if it has a unique solution in \( X \).

**Proof.** It is easy to see that \( \eta : X \times X \to X \) is skew. By Definition 2.10 of weak well-posedness for the mixed hemivariational-like inequality MHVLI(\( A, f, J, \eta, G \)), the necessity is obvious. For the sufficiency, suppose that the mixed hemivariational-like inequality MHVLI(\( A, f, J, \eta, G \)) has a unique solution \( x^* \in X \). If the mixed hemivariational-like inequality MHVLI(\( A, f, J, \eta, G \)) is not weakly well-posed, then there exists at least an approximating sequence \( \{x_n\} \subset X \) for the mixed hemivariational-like inequality MHVLI(\( A, f, J, \eta, G \)) such that \( x_n \) doesn’t converge weakly to \( x^* \). We claim that the approximating sequence \( \{x_n\} \) is bounded in \( X \). In fact, if \( x_n \) is unbounded, we may assume, without loss of generality, that \( ||x_n|| \to +\infty \). Utilizing condition (ii) w.r.t. \( \eta \), we get \( ||\eta(x_n, x^*)|| \to +\infty \). Let

\[
\begin{align*}
t_n &= \frac{1}{||\eta(x_n, x^*)||} \quad \text{and} \quad z_n = x^* + t_n \eta(x_n, x^*). \\
(4.10)
\end{align*}
\]

Clearly, \( \{z_n\} \) is a bounded sequence in \( X \) since \( ||z_n|| \leq ||x^*|| + 1 \). Thus, without loss of generality, we may assume by the reflexivity of the Banach space \( X \) that \( \{z_n\} \) converges weakly to some point \( z \in X \), which obviously is not equal to \( x^* \) by (4.10). Also, since the approximating sequence \( \{x_n\} \) is unbounded, we can suppose that \( t_n \in (0, 1] \) by (4.10). Now, for any \( y \in X \) and \( \zeta \in \partial J(y) \), it follows from condition (i) and Hypothesis (A) that

\[
\begin{align*}
(A y + \zeta - f, \eta(y, z)) &= \langle A y + \zeta - f, \eta(y, x^*) \rangle + \langle A y + \zeta - f, \eta(x^*, z_n) \rangle \\
&+ \langle A y + \zeta - f, \eta(z_n, z) \rangle \\
&= \langle A y + \zeta - f, \eta(y, x^*) \rangle - t_n \langle A y + \zeta - f, \eta(x_n, x^*) \rangle \\
&+ \langle A y + \zeta - f, \eta(z_n, z) \rangle \\
&= (1 - t_n) \langle A y + \zeta - f, \eta(y, x^*) \rangle + t_n \langle A y + \zeta - f, \eta(y, x_n) \rangle \\
&+ \langle A y + \zeta - f, \eta(z_n, z) \rangle. \\
(4.11)
\end{align*}
\]
Keep in mind that \( x^* \) is the unique solution to the mixed hemivariational-like inequality \( \text{MHVLI}(A, f, J, \eta, G) \). By the same arguments as in the proof of Theorem 4.1, there exist \( \xi \in \partial J(x^*) \) and \( \varrho \in \partial \eta G(x^*) \) such that
\[
f = Ax^* + \xi + \varrho. \tag{4.12}
\]
Since the operator \( A \) is strongly \( \eta \)-monotone with constant \( m \) and the Clarke’s generalized gradient \( \overline{\partial} J(\cdot) \) of the locally Lipschitz functional \( J \) satisfies the relaxed \( \eta \)-monotonicity with constant \( c \), the condition \( m = c \) implies that \( A + \overline{\partial} J(\cdot) \) is monotone on \( X \). So, it follows from \( \zeta \in \overline{\partial} J(y) \), \( \xi \in \overline{\partial} J(x^*) \) and (4.12) that
\[
\langle Ay + \zeta - f, \eta(y, x^*) \rangle = \langle Ay + \zeta - (Ax^* + \xi), \eta(y, x^*) \rangle - \langle \varrho, \eta(y, x^*) \rangle \geq G(x^*) - G(y). \tag{4.13}
\]
Moreover, since \( \{x_n\} \) is an approximating sequence for the mixed hemivariational-like inequality \( \text{MHVLI}(A, f, J, \eta, G) \), there exists a nonnegative sequence \( \{\epsilon_n\} \) with \( \epsilon_n \to 0 \) such that
\[
\langle Ax_n - f, \eta(y, x_n) \rangle + J^\ast(x_n, \eta(y, x_n)) + G(y) - G(x_n) \geq -\epsilon_n||\eta(y, x_n)||, \quad \forall y \in X.
\]
Also, by the same argument as in the proof of Theorem 4.1, there exist \( \xi_n \in \overline{\partial} J(x_n) \) and \( \zeta_n \in B(0, 1) \), which both are independent on \( y \), such that
\[
G(y) - G(x_n) \geq -(Ax_n - f + \xi_n + \epsilon_n \zeta_n, \eta(y, x_n)), \quad \forall y \in X.
\]
which implies by the strong \( \eta \)-monotonicity of \( A \), the relaxed \( \eta \)-monotonicity of the Clarke’s generalized gradient \( \overline{\partial} J(\cdot) \), the condition \( m = c \) and the last inequality that
\[
\langle Ay + \zeta - f, \eta(y, x_n) \rangle \geq \langle Ax_n + \xi_n - f, \eta(y, x_n) \rangle \geq G(x_n) - G(y) - \epsilon_n \langle \zeta_n, \eta(y, x_n) \rangle. \tag{4.14}
\]
Therefore, it follows from (4.11), (4.13), (4.14), \( t_n = 1/||\eta(x_n, x^*)|| \) and the preinvexity w.r.t. \( \eta \) that
\[
\begin{align*}
\langle Ay + \zeta - f, \eta(y, z) \rangle &= (1 - t_n)\langle Ay + \zeta - f, \eta(y, x^*) \rangle + t_n\langle Ay + \zeta - f, \eta(y, x_n) \rangle \\
&\quad + \langle Ay + \zeta - f, \eta(z_n, z) \rangle \\
&\geq (1 - t_n)[G(x^*) - G(y)] + t_n[G(x_n) - G(y) - \epsilon_n \langle \zeta_n, \eta(y, x_n) \rangle] \\
&\quad + \langle Ay + \zeta - f, \eta(z_n, z) \rangle \\
&\geq G(z_n) - G(y) - \epsilon_n \langle \zeta_n, t_n \eta(y, x_n) \rangle \\
&\quad + \langle Ay + \zeta - f, \eta(z_n, z) \rangle. \tag{4.15}
\end{align*}
\]
Since \( \eta \) is weakly continuous in the first variable, \( G \) is weakly lower semicontinuous, \( z_n \to z \) and \( \epsilon_n \to 0 \) as \( n \to \infty \), we get by taking the limit at both sides of the above inequality (4.15) that
\[
\langle Ay + \zeta - f, \eta(y, z) \rangle + G(y) - G(z) \geq 0.
\]
By Lemma 3.3 the arbitrariness of \( y \in X \) and \( \zeta \in \overline{\partial} J(y) \) implies that \( z \neq x^* \) is a solution to the mixed hemivariational-like inequality \( \text{MHVLI}(A, f, J, \eta, G) \), which reaches a contradiction to the uniqueness of solutions to the mixed hemivariational-like inequality \( \text{MHVLI}(A, f, J, \eta, G) \). Thus, our claim that the approximating sequence \( \{x_n\} \) is bounded in \( X \) is valid.

Since \( \{x_n\} \) is bounded in \( X \) and Banach space \( X \) is reflexive, we let \( \{x_{n_k}\} \) be any subsequence of the approximating sequence \( \{x_n\} \) such that \( x_{n_k} \to \hat{x} \) as \( k \to \infty \).
Thus, it follows that
\[
\langle Ax_{n_k} - f, \eta(y, x_{n_k}) \rangle + J^\circ(x_{n_k}, \eta(y, x_{n_k})) + G(y) - G(x_{n_k}) \geq -\epsilon_{n_k} \|\eta(y, x_{n_k})\|, \forall y \in X.
\]
(4.16)

By the similar arguments to those of (4.7) in the proof of Theorem 4.1, there exist \(\xi_{n_k} \in \partial J(x_{n_k})\) and \(\zeta_{n_k} \in B(0, 1)\) such that
\[
\langle Ax_{n_k} + \xi_{n_k} - f, \eta(y, x_{n_k}) \rangle \leq G(x_{n_k}) - G(y) - \epsilon_{n_k} \langle \zeta_{n_k}, \eta(y, x_{n_k}) \rangle, \forall y \in X.
\]
which together with the strong \(\eta\)-monotonicity of \(A\), the relaxed \(\eta\)-monotonicity of the Clarke’s generalized gradient \(\partial J(\cdot)\), \(x_{n_k} \rightharpoonup \hat{x}\), the weakly lower semicontinuity of \(G\), the weak continuity of \(\eta\) in the first variable (\(\Rightarrow\) the boundedness of \(\{\eta(x_{n_k}, y)\}\)), and \(m = c\), implies that for any \(y \in X\) and \(\zeta \in \partial J(y)\),
\[
\langle Ay + \zeta - f, \eta(y, \hat{x}) \rangle = \liminf_{k \to \infty} \langle Ay + \zeta - f, \eta(y, x_{n_k}) \rangle \\
\geq \liminf_{k \to \infty} \langle Ax_{n_k} + \xi_{n_k} - f, \eta(y, x_{n_k}) \rangle \\
\geq \liminf_{k \to \infty} [G(x_{n_k}) - G(y) - \epsilon_{n_k} \langle \zeta_{n_k}, \eta(y, x_{n_k}) \rangle] \\
= \liminf_{k \to \infty} [G(x_{n_k}) - G(y)] \\
\geq G(\hat{x}) - G(y).
\]

By Lemma 3.3 \(\hat{x}\) also solves the mixed hemivariational-like inequality \(MHVLI(A, f, J, \eta, G)\) and so we have \(\hat{x} = x^*\) in terms of the uniqueness of solutions to the mixed hemivariational-like inequality \(MHVLI(A, f, J, \eta, G)\). Therefore, the whole approximating sequence \(\{x_n\}\) converges weakly to \(x^*\). This completes the proof. \(\square\)

References


LU-CHUAN CENG
DEPARTMENT OF MATHEMATICS, SHANGHAI NORMAL UNIVERSITY, SHANGHAI, CHINA
E-mail address: zenglc@hotmail.com

JEN-CHIH YAO
CENTER FOR GENERAL EDUCATION, CHINA MEDICAL UNIVERSITY, TAICHUNG, TAIWAN
E-mail address: yaojc@mail.cmu.edu.tw

YONGHONG YAO
DEPARTMENT OF MATHEMATICS, TIANJIN POLYTECHNIC UNIVERSITY, TIANJIN 300387, CHINA
E-mail address: yaoyonghong@aliyun.com