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SENSITIVITY ANALYSIS FOR A PARAMETRIC GENERALIZED MIXED MULTI-VALUED IMPLICIT QUASI-VARIATIONAL INCLUSION

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Abstract. In this paper, by using a resolvent operator technique of maximal monotone mappings and the property of a fixed-point set of multi-valued contractive mapping, we study the behavior and sensitivity analysis of a solution set for a parametric generalized mixed multi-valued implicit quasi-variational inclusion in a real Hilbert space. Further, under some suitable conditions, we discuss the continuity and Lipschitz continuity of the solution set with respect to the parameter. Our approach and results extend, improve and unify the previously many known results in this field.

1. Introduction

Variational inequality theory has become very effective and powerful tool for studying a wide range of problems arising in mechanics, optimization, operation research, equilibrium problems and boundary valued problems, etc. Variational inequalities have been generalized and extended in different directions using novel and innovative techniques. A useful and important generalization of variational inequality is called the variational inclusion. Hassouni and Moudafi [9], Agarwal *et al.* [2], Ding [5, 6], Ding and Luo [7], Fang and Huang [8], Huang [10] and Noor [17, 18] have used the resolvent operator technique to obtain some important extensions and generalizations in existence results for some classes of variational inequalities (inclusions).

In recent years, much attention has been given to develop general techniques for the sensitivity analysis of solution set of various classes of variational inequalities (inclusions). From the mathematical and engineering point of view, sensitivity properties of various classes of variational inequalities can provide new insight concerning the problem being studied and stimulate ideas for solving problems. The

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sensitivity analysis of solution set for variational inequalities have been studied extensively by many authors using quite different techniques. By using the projection technique, Dafermos [4], Mukherjee and Verma [15], Ding and Luo [7] and Yen [23] studied the sensitivity analysis of solution for some classes of variational inequalities with single-valued mappings. By using the implicit function approach that makes use of so-called normal mappings, Robinson [22] studied the sensitivity analysis of solutions for variational inequalities in finite-dimensional spaces. By using resolvent operator technique, Adly [1], Agarwal *et al.* [2], Lim [13], Liu *et al.* [14] and Noor [17] studied the sensitivity analysis of solution for some classes of quasi-variational inclusions involving single-valued mappings.

Recently, by using projection and resolvent techniques, Agarwal *et al.* [3], Ding [5, 6], Kazmi and Alvi [11], Kazmi and Khan [12], Noor [18], Peng and Long [20] and Ram [21] studied the behavior and sensitivity analysis of solution set for some classes of parametric generalized variational inclusions involving multi-valued mappings.

Motivated by recent work going in this direction, in this paper, we introduce the notion of resolvent mapping of a maximal monotone mapping and discuss some of its properties. Further, we consider a parametric generalized mixed multi-valued implicit quasi-variational inclusion problem (PGMMIQVIP, for short) involving maximal monotone mapping in a real Hilbert space. Further, by using a resolvent operator technique and the property of a fixed-point set of multi-valued contractive mapping, we study the behavior and sensitivity analysis of a solution set for the PGMMIQVIP. Further, under some suitable conditions, the continuity and Lipschitz continuity of the solution set for the PGMMIQVIP with respect to the parameter are proved. The results presented in this paper generalize and improve the results given by many authors, see for example [3, 6, 11, 12, 13, 18, 20, 21].

2. Preliminaries

We assume that H is a real Hilbert space equipped with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$; 2^H is the power set of H; $C(H)$ is the family of all nonempty compact subsets of H; $\mathcal{H}(\cdot, \cdot)$ is the Hausdorff metric on $C(H)$ defined by

$$\mathcal{H}(A,B) \ = \ \max\Big\{ \sup_{x\in A} \inf_{y\in B} d(x,y), \sup_{y\in B} \inf_{x\in A} d(x,y) \Big\}, \ A, B \in C(H).$$

First, we review the following concepts and known results.

Definition 2.1[19]**.** Let $W : H \to 2^H$ be a maximal monotone mapping. For any fixed $\rho > 0$, the mapping $J_\rho^W : H \to H$, defined by

$$J_\rho^W(x) \ = \ (I + \rho W)^{-1}(x), \ \forall\, x \in H,$$

is said to be the resolvent operator of W where I is the identity mapping on H.

Lemma 2.1[19]**.** Let $W : H \to 2^H$ be a maximal monotone mapping. Then the resolvent operator $J_\rho^W : H \to H$ of W is nonexpansive, i.e.,

$$\|J_\rho^W(x) - J_\rho^W(y)\| \ \le \ \|x - y\|, \ \forall\, x, y \in H.$$

Lemma 2.2[16]**.** Let (X,d) be a complete metric space. Suppose that $T : X \to C(X)$ satisfies

$$\mathcal{H}(T(x),\, T(y)) \;\leq\; \nu\; d(x,\, y),\; \forall\, x, y \in X,$$

where $\nu \in (0,1)$ is a constant. Then the mapping T has fixed point in X.

Lemma 2.3[13]**.** Let (X,d) be a complete metric space and let $T_1, T_2 : X \to C(X)$ be θ-$\mathcal{H}$-contraction mappings, then

$$\mathcal{H}(F(T_1),\, F(T_2)) \;\leq\; (1-\theta)^{-1} \sup_{x\in X} \mathcal{H}(T_1(x),\, T_2(x)),$$

where $F(T_1)$ and $F(T_2)$ are the sets of fixed points of T_1 and T_2, respectively.

Definition 2.2[5, 11, 12] A multi-valued mapping $R : H \times \Omega \to C(H)$ is said to be:

(i) δ-strongly monotone if there exists a constant $\delta > 0$ such that

$$\langle s_1 - s_2,\, x-y\rangle \;\geq\; \delta\|x-y\|^2,\; \forall\,(x,y,\lambda) \in H\times H\times\Omega,\; s_1 \in R(x,\lambda),\; s_2 \in R(y,\lambda);$$

(ii) L_R-Lipschitz continuous if there exists a constant $L_R > 0$ such that

$$\mathcal{H}(R(x,\lambda),\, R(y,\lambda)) \;\leq\; L_R\; \|x-y\|,\; \forall\,(x,y,\lambda) \in H\times H\times\Omega.$$

Definition 2.3[11, 12]**.** A multi-valued mapping $A : H \times \Omega \to C(H)$ is said to be (L_A, l_A)-$\mathcal{H}$-mixed Lipschitz continuous if there exist constants $L_A, l_A > 0$ such that

$$\mathcal{H}(A(x_1,\lambda_1), A(x_2,\lambda_2)) \;\leq\; L_A\|x_1-x_2\| + l_A\|\lambda_1-\lambda_2\|,\; \forall\,(x_1,\lambda_1),(x_2,\lambda_2) \in H\times\Omega.$$

Definition 2.4[11, 12, 20]**.** Let $A, B : H\times\Omega \to C(H)$ be multi-valued mappings. A single-valued mapping $N : H\times H\times\Omega \to H$ is said to be:

(i) α-strongly mixed monotone with respect to A and B if there exists a constant $\alpha > 0$ such that

$$\langle N(u_1,v_1,\lambda) - N(u_2,v_2,\lambda),\, x-y\rangle \;\geq\; \alpha\|x-y\|^2,$$

$$\forall\;(x,y,\lambda) \in H\times H\times\Omega,\; u_1 \in A(x,\lambda),\; u_2 \in A(y,\lambda),\; v_1 \in B(x,\lambda),\; v_2 \in B(y,\lambda);$$

(ii) σ-generalized mixed pseudocontractive with respect to A and B if there exists a constant $\sigma > 0$ such that

$$\langle N(u_1,v_1,\lambda) - N(u_2,v_2,\lambda),\, x-y\rangle \;\leq\; \sigma\|x-y\|^2,$$

$$\forall\;(x,y,\lambda) \in H\times H\times\Omega,\; u_1 \in A(x,\lambda),\; u_2 \in A(y,\lambda),\; v_1 \in B(x,\lambda),\; v_2 \in B(y,\lambda);$$

(iii) $(L_{(N,1)}, L_{(N,2)}, l_N)$-mixed Lipschitz continuous if there exist constants $L_{(N,1)}, L_{(N,2)}$, $l_N > 0$ such that

$$\|N(x_1,y_1,\lambda_1) - N(x_2,y_2,\lambda_2)\| \;\leq\; L_{(N,1)}\|x_1-x_2\| + L_{(N,2)}\|y_1-y_2\| + l_N\|\lambda_1-\lambda_2\|,$$

$$\forall\,(x_1,y_1,\lambda_1),(x_2,y_2,\lambda_2) \in H\times H\times\Omega.$$

3. Formulation of Problem

Let Ω be a nonempty open subset of H in which the parameter λ takes values. Let $N, M : H \times H \times \Omega \to H$ and $m, f : H \times \Omega \to H$ be single-valued mappings, and let $A, B, C, D, G, P, Q, R : H \times \Omega \to C(H)$ be multi-valued mappings. Suppose that $W : H \times H \times \Omega \to 2^H$ is a multi-valued mapping such that for each given $(z, \lambda) \in H \times \Omega$, $W(\cdot, z, \lambda) : H \to 2^H$ is a maximal monotone mapping with $(R(H, \lambda) - m(H, \lambda)) \cap \operatorname{dom} W(\cdot, z, \lambda) \neq \emptyset$. Throughout this paper, unless otherwise stated, we will consider the following parametric generalized mixed multi-valued implicit quasi-variational inclusion problem (PGMMIQVIP):

For each fixed $\lambda \in \Omega$, find $x(\lambda) \in H$, $u(\lambda) \in A(x(\lambda), \lambda)$, $v(\lambda) \in B(x(\lambda), \lambda)$, $w(\lambda) \in C(x(\lambda), \lambda)$, $y(\lambda) \in D(x(\lambda), \lambda)$, $z(\lambda) \in G(x(\lambda), \lambda)$, $n(\lambda) \in P(x(\lambda), \lambda)$, $t(\lambda) \in Q(x(\lambda), \lambda)$ and $s(\lambda) \in R(x(\lambda), \lambda)$ such that

$$0 \in N(u(\lambda), v(\lambda), \lambda) - M(w(\lambda), y(\lambda), \lambda) + f(t(\lambda), \lambda) + W(s(\lambda) - m(n(\lambda), \lambda), z(\lambda), \lambda). \tag{3.1}$$

Some Special Cases:

(1) If $R = g : H \times \Omega \to H$ is a single-valued mapping and $P(x, \lambda) = x$, for all $(x, \lambda) \in H \times \Omega$, then the PGMMIQVIP (3.1) reduces to the following parametric generalized quasi-variational inclusion problem: for each fixed $\lambda \in \Omega$, find $x(\lambda) \in H$, $u(\lambda) \in A(x(\lambda), \lambda)$, $v(\lambda) \in B(x(\lambda), \lambda)$, $w(\lambda) \in C(x(\lambda), \lambda)$, $y(\lambda) \in D(x(\lambda), \lambda)$, $z(\lambda) \in G(x(\lambda), \lambda)$, $t(\lambda) \in Q(x(\lambda), \lambda)$ such that

$$0 \in N(u(\lambda), v(\lambda), \lambda) - M(w(\lambda), y(\lambda), \lambda) + f(t(\lambda), \lambda) + W(g(x(\lambda), \lambda) - m(x(\lambda), \lambda), z(\lambda), \lambda). \tag{3.2}$$

Similar type problems have been studied by many authors given in [5, 11, 12, 18, 20, 21].

(2) If $m(x(\lambda), \lambda) = 0$, for all $(x, \lambda) \in H \times \Omega$, then the PGMMIQVIP (3.2) reduces to the following parametric generalized quasi-variational inclusion problem: for each fixed $\lambda \in \Omega$, find $x(\lambda) \in H$, $u(\lambda) \in A(x(\lambda), \lambda)$, $v(\lambda) \in B(x(\lambda), \lambda)$, $w(\lambda) \in C(x(\lambda), \lambda)$, $y(\lambda) \in D(x(\lambda), \lambda)$, $z(\lambda) \in G(x(\lambda), \lambda)$, $t(\lambda) \in Q(x(\lambda), \lambda)$ such that

$$0 \in N(u(\lambda), v(\lambda), \lambda) - M(w(\lambda), y(\lambda), \lambda) + f(t(\lambda), \lambda) + W(g(x(\lambda), \lambda), z(\lambda), \lambda). \tag{3.3}$$

Similar type problems have been studied by many authors given in [5, 11, 12, 18, 20, 21].

(3) If $M(w(\lambda), y(\lambda), \lambda) = 0$, then the PGMMIQVIP (3.1) reduces to the following parametric generalized quasi-variational inclusion problem: for each fixed $\lambda \in \Omega$, find $x(\lambda) \in H$, $u(\lambda) \in A(x(\lambda), \lambda)$, $v(\lambda) \in B(x(\lambda), \lambda)$, $z(\lambda) \in G(x(\lambda), \lambda)$, $n(\lambda) \in P(x(\lambda), \lambda)$, $t(\lambda) \in Q(x(\lambda), \lambda)$, $s(\lambda) \in R(x(\lambda), \lambda)$ such that

$$0 \in N(u(\lambda), v(\lambda), \lambda) + f(t(\lambda), \lambda) + W(s(\lambda) - m(n(\lambda), \lambda), z(\lambda), \lambda) \tag{3.4}$$

which has been introduced and studied by Ram [21].

(4) If $f(t(\lambda), \lambda) = 0$, then the PGMMIQVIP (3.4) reduces to the following parametric generalized quasi-variational inclusion problem: for each fixed $\lambda \in \Omega$, find $x(\lambda) \in H$, $u(\lambda) \in A(x(\lambda), \lambda)$, $v(\lambda) \in B(x(\lambda), \lambda)$, $z(\lambda) \in$

$G(x(\lambda),\lambda)$, $n(\lambda)\in P(x(\lambda),\lambda)$, $s(\lambda)\in R(x(\lambda),\lambda)$ such that

$$0\in N(u(\lambda),v(\lambda),\lambda)+W(s(\lambda)-m(n(\lambda),\lambda),z(\lambda),\lambda), \tag{3.5}$$

which has been introduced and studied by Ding [5].

(5) If $R=g:H\times\Omega\to H$ is a single-valued mapping; $P(x,\lambda)=x$ and $m(x,\lambda)=0$, for all $(x,\lambda)\in H\times\Omega$, then the PGMMIQVIP (3.5) reduces to the following parametric generalized quasi-variational inclusion problem: for each fixed $\lambda\in\Omega$, find $x(\lambda)\in H$, $u(\lambda)\in A(x(\lambda),\lambda)$, $v(\lambda)\in B(x(\lambda),\lambda)$, $z(\lambda)\in G(x(\lambda),\lambda)$ such that

$$0\in N(u(\lambda),v(\lambda),\lambda)+W(g(x(\lambda),\lambda),z(\lambda),\lambda), \tag{3.6}$$

which has been introduced and studied by Noor [17, 18].

In brief, for appropriate and suitable choices of the mappings A,B,C,D,G,P,Q,R,W,m,f and the space H, it is easy to see that PGMMIQVIP (3.1) includes a number of known classes of parametric generalized variational inclusions studied by many authors as special cases, see for example [3, 5, 6, 11, 12, 13, 18, 20, 21] and the references therein.

Now, for each fixed $\lambda\in\Omega$, the solution set $S(\lambda)$ of the PGMMIQVIP (3.1) is denoted as

$$S(\lambda):=\Big\{x(\lambda)\in H:\exists\ u(\lambda)\in A(x(\lambda),\lambda),v(\lambda)\in B(x(\lambda),\lambda),w(\lambda)\in C(x(\lambda),\lambda),y(\lambda)\in D(x(\lambda),\lambda),z(\lambda)\in G(x(\lambda),\lambda),n(\lambda)\in P(x(\lambda),\lambda),t(\lambda)\in Q(x(\lambda),\lambda),s(\lambda)\in R(x(\lambda),\lambda)\text{ such that } 0\in N(u(\lambda),v(\lambda),\lambda)-M(w(\lambda),y(\lambda),\lambda)+f(t(\lambda),\lambda)+W(s(\lambda)-m(n(\lambda),\lambda),z(\lambda),\lambda)\Big\}. \tag{3.7}$$

In this paper, our main aim is to study the behavior and sensitivity analysis of the solution set $S(\lambda)$, and the conditions on these mappings A,B,C,D,G,P,Q,R,W,m,f under which the solution set $S(\lambda)$ of the PGMMIQVIP (3.1) is nonempty and Lipschitz continuous (or continuous) with respect to the parameter $\lambda\in\Omega$.

4. Sensitivity Analysis of Solution Set $S(\lambda)$

First, we transfer the PGMMIQVIP (3.1) into a parametric fixed point problem.

Theorem 4.1. For each fixed $\lambda\in\Omega$, $x(\lambda)\in S(\lambda)$ is a solution of the PGMMIQVIP (3.1) if and only if there exist $u(\lambda)\in A(x(\lambda),\lambda)$, $v(\lambda)\in B(x(\lambda),\lambda)$, $w(\lambda)\in C(x(\lambda),\lambda)$, $y(\lambda)\in D(x(\lambda),\lambda)$, $z(\lambda)\in G(x(\lambda),\lambda)$, $n(\lambda)\in P(x(\lambda),\lambda)$, $t(\lambda)\in Q(x(\lambda),\lambda)$, $s(\lambda)\in R(x(\lambda),\lambda)$ such that the following relation holds:

$$\begin{aligned}s(\lambda)=m(n(\lambda),\lambda)+J_\rho^{W(\cdot,z(\lambda),\lambda)}(s(\lambda)&-m(n(\lambda),\lambda)-\rho N(u(\lambda),v(\lambda),\lambda)\\&+\rho M(w(\lambda),y(\lambda),\lambda)+f(t(\lambda),\lambda)),\end{aligned} \tag{4.1}$$

where $\rho>0$ is a constant.

Proof. For each fixed $\lambda\in\Omega$, by the definition of the resolvent operator $J_\rho^{W(\cdot,z(\lambda),\lambda)}$ of $W(\cdot,z(\lambda),\lambda)$, we have that there exist $x(\lambda)\in H$, $u(\lambda)\in A(x(\lambda),\lambda)$, $v(\lambda)\in B(x(\lambda),\lambda)$, $w(\lambda)\in C(x(\lambda),\lambda)$, $y(\lambda)\in D(x(\lambda),\lambda)$, $z(\lambda)\in G(x(\lambda),\lambda)$, $n(\lambda)\in P(x(\lambda),\lambda)$, $t(\lambda)\in Q(x(\lambda),\lambda)$ and $s(\lambda)\in R(x(\lambda),\lambda)$ such that (4.1) holds if and only if

$$s(\lambda)-m(n(\lambda),\lambda)-\rho N(u(\lambda),v(\lambda),\lambda)+\rho M(w(\lambda),y(\lambda),\lambda)+f(t(\lambda),\lambda)$$

$$\in s(\lambda)-m(n(\lambda),\lambda)+\rho W(s(\lambda)-m(n(\lambda),\lambda),z(\lambda),\lambda). \tag{4.2}$$

The above relation holds if and only if

$$0 \in N(u(\lambda),v(\lambda),\lambda) - M(w(\lambda),y(\lambda),\lambda) + f(t(\lambda),\lambda)$$
$$+W(s(\lambda) - m(n(\lambda),\lambda),z(\lambda),\lambda).$$

By the definition of $S(\lambda)$, we obtain that $x(\lambda) \in S(\lambda)$ is a solution of the PGMMIQVIP (3.1) if and only if there exist $x(\lambda) \in H$, $u(\lambda) \in A(x(\lambda),\lambda)$, $v(\lambda) \in B(x(\lambda),\lambda)$, $w(\lambda) \in C(x(\lambda),\lambda)$, $y(\lambda) \in D(x(\lambda),\lambda)$, $z(\lambda) \in G(x(\lambda),\lambda)$, $n(\lambda) \in P(x(\lambda),\lambda)$, $t(\lambda) \in Q(x(\lambda),\lambda)$ and $s(\lambda) \in R(x(\lambda),\lambda)$ such that (4.1) holds.

Remark 4.1. Theorem 4.1 is a generalized variant of Lemma 3.1 of Adly [1], Lemma 2.1 of Agarwal *et al.* [2], Theorem 3.1 of Ding [5], Lemma 3.1 of Ding *et al.* [6], Lemma 4.1 of Kazmi and Alvi [11], Lemma 2.1 of Peng *et al.* [20], and Theorem 3.1 of Ram [21].

Theorem 4.2. Let $A,B,C,D,G,P,Q,R : H \times \Omega \to C(H)$ be multi-valued mappings such that A,B,C,D,G,P,Q and R are $\mathcal{H}$-Lipschitz continuous in the first arguments with constant $L_A, L_B, L_C, L_D, L_G, L_P, L_Q$ and L_R, respectively, and let $R : H \times \Omega \to C(H)$ be δ-strongly monotone. Let $m : H \times \Omega \to H$ be (L_m, l_m)-mixed Lipschitz continuous and $f : H \times \Omega \to H$ be (L_f, l_f)-mixed Lipschitz continuous. Let $N : H \times H \times \Omega \to H$ be α-strongly mixed monotone with respect to A and B and $(L_{(N,1)}, L_{(N,2)})$-mixed Lipschitz continuous in first two arguments, and let $M : H \times H \times \Omega \to H$ be σ-generalized mixed pseudocontractive with respect to C and D and $(L_{(M,1)}, L_{(M,2)})$-mixed Lipschitz continuous in first two arguments. Suppose that the multi-valued mapping $W : H \times H \times \Omega \to 2^H$ is such that for each fixed $(z,\lambda) \in H \times \Omega$, $W(\cdot,z,\lambda) : H \to 2^H$ is a maximal monotone mapping satisfying $R(H,\lambda) - m(H,\lambda) \cap \mathrm{dom}\, W(\cdot,z,\lambda) \neq \emptyset$. Suppose that there exist constants $k_1, k_2 > 0$ such that

$$\|J_\rho^{W(\cdot,x_1,\lambda_1)}(t) - J_\rho^{W(\cdot,x_2,\lambda_2)}(t)\| \ \leq\ k_1\|x_1 - x_2\| + k_2\|\lambda_1 - \lambda_2\|,\ \forall x_1,x_2,t \in H;\ \lambda_1,\lambda_2 \in \Omega, \tag{4.3}$$

and suppose for $\rho > 0$, the following condition holds:

$$\theta \ =\ k + t(\rho) < 1,\ \text{where} \tag{4.4}$$

$$k = 2\sqrt{1-2\delta+\lambda_R^2} + 2L_mL_P + L_fL_Q + k_1L_G;\ t(\rho) = \sqrt{1-2\rho(\alpha-\sigma)+2\rho^2(L_N^2+L_M^2)};$$

$$L_N = (L_AL_{(N,1)}+L_BL_{(N,2)}); L_M := (L_CL_{(M,1)}+L_DL_{(M,2)}).$$

Then, for each $\lambda \in \Omega$, the solution set $S(\lambda)$ of the PGMMIQVIP (3.1) is nonempty and closed set in H.

Proof. Define a multi-valued mapping $F : H \times \Omega \to 2^H$ by

$$F(x,\lambda) = \bigcup_{u\in A(x,\lambda),v\in B(x,\lambda),w\in C(x,\lambda),y\in D(x,\lambda),z\in G(x,\lambda),n\in P(x,\lambda),t\in Q(x,\lambda),s\in R(x,\lambda)} \Big[x-s+m(n,\lambda)$$
$$+J_\rho^{W(\cdot,z,\lambda)}(s-m(n,\lambda)-\rho N(u,v,\lambda)+\rho M(w,y,\lambda)+f(t,\lambda))\Big],\ \forall (x,\lambda) \in H\times\Omega. \tag{4.5}$$

For any $(x,\lambda)\in H\times\Omega$, since $A(x,\lambda)$, $B(x,\lambda)$, $C(x,\lambda)$, $D(x,\lambda)$, $G(x,\lambda)$, $P(x,\lambda)$, $Q(x,\lambda)$, $R(x,\lambda)\in C(H)$, and $m, f, J_\rho^{W(\cdot,z,\lambda)}$ are continuous, we have $F(x,\lambda)\in C(H)$. Now for each fixed $\lambda\in\Omega$, we prove that $F(x,\lambda)$ is a multi-valued contractive mapping. For any $(x,\lambda),(y,\lambda)\in H\times\Omega$ and any $a\in F(x,\lambda)$, there exist $u_1\in A(x,\lambda)$, $v_1\in B(x,\lambda)$, $w_1\in C(x,\lambda)$, $y_1\in D(x,\lambda)$, $z_1\in G(x,\lambda)$, $n_1\in P(x,\lambda)$, $t_1\in Q(x,\lambda)$ and $s_1\in R(x,\lambda)$ such that

$$a = x-s_1+m(n_1,\lambda)+J_\rho^{W(\cdot,z_1,\lambda)}(s_1-m(n_1,\lambda)-\rho N(u_1,v_1,\lambda)+\rho M(w_1,y_1,\lambda)+f(t_1,\lambda)). \tag{4.6}$$

Since $A(y,\lambda)$, $B(y,\lambda)$, $C(y,\lambda)$, $D(y,\lambda)$, $G(y,\lambda)$, $P(y,\lambda)$, $Q(y,\lambda)$, $R(y,\lambda)\in C(H)$, so there exist $u_2\in A(y,\lambda)$, $v_2\in B(y,\lambda)$, $w_2\in C(y,\lambda)$, $y_2\in D(y,\lambda)$, $z_2\in G(y,\lambda)$, $n_2\in P(y,\lambda)$, $t_2\in Q(y,\lambda)$ and $s_2\in R(y,\lambda)$ such that

$$\begin{aligned}
\|u_1-u_2\| &\le \mathcal{H}(A(x,\lambda),\,A(y,\lambda)) \le L_A\|x-y\|,\\
\|v_1-v_2\| &\le \mathcal{H}(B(x,\lambda),\,B(y,\lambda)) \le L_B\|x-y\|,\\
\|w_1-w_2\| &\le \mathcal{H}(C(x,\lambda),\,C(y,\lambda)) \le L_C\|x-y\|,\\
\|y_1-y_2\| &\le \mathcal{H}(D(x,\lambda),\,D(y,\lambda)) \le L_D\|x-y\|,\\
\|z_1-z_2\| &\le \mathcal{H}(G(x,\lambda),\,G(y,\lambda)) \le L_G\|x-y\|,\\
\|n_1-n_2\| &\le \mathcal{H}(P(x,\lambda),\,P(y,\lambda)) \le L_P\|x-y\|,\\
\|t_1-t_2\| &\le \mathcal{H}(Q(x,\lambda),\,Q(y,\lambda)) \le L_Q\|x-y\|,\\
\|s_1-s_2\| &\le \mathcal{H}(R(x,\lambda),\,R(y,\lambda)) \le L_R\|x-y\|.
\end{aligned} \tag{4.7}$$

Let

$$b = y-s_2+m(n_2,\lambda)+J_\rho^{W(\cdot,z_2,\lambda)}(s_2-m(n_2,\lambda)-\rho N(u_2,v_2,\lambda)+\rho M(w_2,y_2,\lambda)+f(t_2,\lambda)), \tag{4.8}$$

then we have $b\in F(y,\lambda)$. It follows that

$$\begin{aligned}
\|a-b\| \le{}& \|x-y-(s_1-s_2)\|+\|m(n_1,\lambda)-m(n_2,\lambda)\|\\
&+\|J_\rho^{W(\cdot,z_1,\lambda)}(s_1-m(n_1,\lambda)-\rho N(u_1,v_1,\lambda)+\rho M(w_1,y_1,\lambda)+f(t_1,\lambda))\\
&-J_\rho^{W(\cdot,z_2,\lambda)}(s_2-m(n_2,\lambda)-\rho N(u_2,v_2,\lambda)+\rho M(w_2,y_2,\lambda)+f(t_2,\lambda))\|.\\
\le{}& \|x-y-(s_1-s_2)\|+\|m(n_1,\lambda)-m(n_2,\lambda)\|\\
&+\|J_\rho^{W(\cdot,z_1,\lambda)}(s_1-m(n_1,\lambda)-\rho N(u_1,v_1,\lambda)+\rho M(w_1,y_1,\lambda)+f(t_1,\lambda))\\
&-[J_\rho^{W(\cdot,z_1,\lambda)}(s_2-m(n_2,\lambda)-\rho N(u_2,v_2,\lambda)+\rho M(w_2,y_2,\lambda)+f(t_2,\lambda))]\|\\
&+\|J_\rho^{W(\cdot,z_1,\lambda)}(s_2-m(n_2,\lambda)-\rho N(u_2,v_2,\lambda)+\rho M(w_2,y_2,\lambda)+f(t_2,\lambda))\\
&-[J_\rho^{W(\cdot,z_2,\lambda)}(s_2-m(n_2,\lambda)-\rho N(u_2,v_2,\lambda)+\rho M(w_2,y_2,\lambda)+f(t_2,\lambda))]\|\\
\le{}& \|x-y-(s_1-s_2)\|+\|m(n_1,\lambda)-m(n_2,\lambda)\|\\
&+\|s_1-m(n_1,\lambda)-\rho N(u_1,v_1,\lambda)+\rho M(w_1,y_1,\lambda)+f(t_1,\lambda)\\
&-[s_2-m(n_2,\lambda)-\rho N(u_2,v_2,\lambda)+\rho M(w_2,y_2,\lambda)+f(t_2,\lambda)]\|+k_1\|z_1-z_2\|\\
\le{}& 2\|x-y-(s_1-s_2)\|+2\|m(n_1,\lambda)-m(n_2,\lambda)\|\\
&+k_1\|z_1-z_2\|+\|f(t_1,\lambda)-f(t_2,\lambda)\|\\
&+\|x-y-\rho(N(u_1,v_1,\lambda)-N(u_2,v_2,\lambda)-M(w_1,y_1,\lambda)+M(w_2,y_2,\lambda))\|.
\end{aligned} \tag{4.9}$$

Since N is α-strongly mixed monotone with respect to A and B and $(L_{(N,1)},\ L_{(N,2)})$-mixed Lipschitz continuous; M is σ-generalized mixed pseudocontractive with respect to C and D and $(L_{(M,1)},\ L_{(M,2)})$-mixed Lipschitz continuous. Also, the multi-valued mappings A, B, C, D are $\mathcal{H}$-Lipschitz continuous, then using $\|a+b\|^2 \le 2(\|a\|^2+\|b\|^2)$, we have

$$
\begin{aligned}
&\|x-y-\rho(N(u_1,v_1,\lambda)-N(u_2,v_2,\lambda)-M(w_1,y_1,\lambda)+M(w_2,y_2,\lambda))\|^2\\
&\le \|x-y\|^2-2\rho\Big[\langle N(u_1,v_1,\lambda)-N(u_2,v_2,\lambda)-M(w_1,y_1,\lambda)+M(w_2,y_2,\lambda),x-y\rangle\Big]\\
&+2\rho^2\Big[\|N(u_1,v_1,\lambda)-N(u_2,v_2,\lambda)\|^2+\|M(w_1,y_1,\lambda)-M(w_2,y_2,\lambda)\|^2\Big]\\
&\le \|x-y\|^2-2\rho(\alpha-\sigma)\|x-y\|^2+2\rho^2\Big[(L_AL_{(N,1)}+L_BL_{(N,2)})^2+(L_CL_{(M,1)}+L_DL_{(M,2)})^2\Big]\|x-y\|^2\\
&\le \Big(1-2\rho(\alpha-\sigma)+2\rho^2[(L_AL_{(N,1)}+L_BL_{(N,2)})^2+(L_CL_{(M,1)}+L_DL_{(M,2)})^2]\Big)\|x-y\|^2.
\end{aligned}
\tag{4.10}
$$

Since R is δ-strongly monotone and L_R-Lipschitz continuous, we have

$$
\begin{aligned}
\|x-y-(s_1-s_2)\|^2 &= \|x-y\|^2-2\langle x-y,s_1-s_2\rangle+\|s_1-s_2\|^2\\
&\le \|x-y\|^2-2\delta\|x-y\|^2+[\mathcal{H}(R(x,\lambda),R(y,\lambda))]^2\\
&\le \|x-y\|^2-2\delta\|x-y\|^2+L_R^2\|x-y\|^2,
\end{aligned}
$$

and hence,

$$\|x-y-(s_1-s_2)\| \le \sqrt{1-2\delta+L_R^2}\ \|x-y\|. \tag{4.11}$$

By the mixed Lipschitz continuity of m and the Lipschitz continuity of P, we have

$$
\begin{aligned}
\|m(n_1,\lambda)-m(n_2,\lambda)\| &\le L_m\|n_1-n_2\| \le L_m\mathcal{H}(P(x,\lambda),P(y,\lambda))\\
&\le L_mL_P\|x-y\|.
\end{aligned}
\tag{4.12}
$$

By the Lipschitz continuity of G, we have

$$\|z_1-z_2\| \le \mathcal{H}(G(x,\lambda),G(y,\lambda)) \le L_G\|x-y\|. \tag{4.13}$$

By the mixed Lipschitz continuity of f and the Lipschitz continuity of Q, we have

$$
\begin{aligned}
\|f(t_1,\lambda)-f(t_2,\lambda)\| &\le L_f\|t_1-t_2\| \le L_f\ \mathcal{H}(Q(x,\lambda),Q(y,\lambda))\\
&\le L_fL_Q\|x-y\|.
\end{aligned}
\tag{4.14}
$$

Combining (4.9)-(4.14), we obtain

$$\|a-b\| \le \theta\,\|x-y\|, \tag{4.15}$$

where $\theta := k+t(\rho)$; $k := 2\sqrt{1-2\delta+L_R^2}+2L_mL_P+L_fL_Q+k_1L_G$;

$t(\rho) := \sqrt{1-2\rho(\alpha-\sigma)+2\rho^2(L_N^2+L_M^2)}$; $L_N := (L_AL_{(N,1)}+L_BL_{(N,2)})$;

$L_M := (L_CL_{(M,1)}+L_DL_{(M,2)})$.

It follows from condition (4.4) that $\theta<1$. Hence, we have

$$d(a,F(y,\lambda)) = \inf_{b\in F(y,\lambda)} \|a-b\| \le \theta\|x-y\|.$$

Since $a\in F(x,\lambda)$ is arbitrary, we obtain

$$\sup_{a\in F(x,\lambda)} d(a,F(y,\lambda)) \le \theta\|x-y\|.$$

By using same argument, we can prove

$$\sup_{b\in F(y,\lambda)} d(F(x,\lambda),b) \;\leq\; \theta\|x-y\|.$$

By the definition of the Hausdorff metric $\mathcal{H}$ on $C(H)$, and for all $(x,y,\lambda)\in H\times H\times\Omega$, we obtain that

$$\mathcal{H}\Big(F(x,\lambda),F(y,\lambda)\Big) \;\leq\; \theta\|x-y\|, \tag{4.16}$$

that is, $F(x,\lambda)$ is a uniform θ-$\mathcal{H}$-contraction mapping with respect to $\lambda\in\Omega$. Also, it follows from condition (4.4) that $\theta<1$ and hence $F(x,\lambda)$ is a multi-valued contraction mapping which is uniform with respect to $\lambda\in\Omega$. By Lemma 2.2, for each $\lambda\in\Omega$, $F(x,\lambda)$ has a fixed point $x(\lambda)\in H$, that is, $x(\lambda)\in F(x(\lambda),\lambda)$ and hence Theorem 4.1 ensure that $x(\lambda)\in S(\lambda)$ is a solution of the PGMMIQVIP (3.1) and so $S(\lambda)\neq\emptyset$. Further, for each $\lambda\in\Omega$, let $\{x_n\}\subset S(\lambda)$ with $\lim\limits_{n\to\infty} x_n=x_0$, we have $x_n\in F(x_n,\lambda)$ for all $n\geq 1$. By virtue of (4.16), we have

$$\begin{aligned} d(x_0,F(x_0,\lambda)) \;&\leq\; \|x_0-x_n\|+\mathcal{H}(F(x_n,\lambda),F(x_0,\lambda))\\ &\leq\; (1+\theta)\|x_n-x_0\|\to 0 \text{ as } n\to\infty, \end{aligned}$$

that is, $x_0\in F(x_0,\lambda)$ and hence $x_0\in S(\lambda)$. Thus $S(\lambda)$ is closed set in H.

Now, we prove that the solution set $S(\lambda)$ of the PGMMIQVIP (3.1) is $\mathcal{H}$-Lipschitz continuous (or continuous) for each $\lambda\in\Omega$.

Theorem 4.3. Let the multi-valued mappings A,B,C,D,G,P,Q and R be $\mathcal{H}$-mixed Lipschitz continuous with pairs of constants (L_A,l_A), (L_B,l_B), (L_C,l_C), (L_D,l_D), (L_G,l_G), (L_P,l_P), (L_Q,l_Q) and (L_R,l_R), respectively. Let the mappings m,f be same as in Theorem 4.2. Let N be α-strongly mixed monotone with respect to A and B and $(L_{(N,1)},L_{(N,2)},l_N)$-mixed Lipschitz continuous, and let M be σ-generalized mixed pseudocontractive with respect to C and D and $(L_{(M,1)},L_{(M,2)},l_M)$-mixed Lipschitz continuous. Suppose that the multi-valued mapping W is same as in Theorem 4.2 and condition (4.4) holds, then for each $\lambda\in\Omega$, the solution set $S(\lambda)$ of the PGMMIQVIP (3.1) is a $\mathcal{H}$-Lipschitz continuous (or continuous) mapping from Ω to H.

Proof. For each $\lambda,\bar\lambda\in\Omega$, it follows from Theorem 4.2 that $S(\lambda)$ and $S(\bar\lambda)$ are both nonempty and closed subsets of H. It also follows from Theorem 4.2 that $F(x,\lambda)$ and $F(x,\bar\lambda)$ are both multi-valued θ-$\mathcal{H}$-contraction mappings with same contractive constant $\theta\in(0,1)$. By Lemma 2.3, we obtain

$$\mathcal{H}(S(\lambda),\,S(\bar\lambda)) \;\leq\; \Big(\frac{1}{1-\theta}\Big)\sup_{x\in H}\;\mathcal{H}(F(x,\lambda),F(x,\bar\lambda)). \tag{4.17}$$

Taking any $a\in F(x,\lambda)$, there exist $u(\lambda)\in A(x,\lambda)$, $v(\lambda)\in B(x,\lambda)$, $w(\lambda)\in C(x,\lambda)$, $y(\lambda)\in D(x,\lambda)$, $z(\lambda)\in G(x,\lambda)$, $n(\lambda)\in P(x,\lambda)$, $t(\lambda)\in Q(x,\lambda)$, $s(\lambda)\in R(x,\lambda)$ such that

$$\begin{aligned} a \;=\; x-s(\lambda)+m(n(\lambda),\lambda)+J_\rho^{W(\cdot,z(\lambda),\lambda)}(&s(\lambda)-m(n(\lambda),\lambda)-\rho N(u(\lambda),v(\lambda),\lambda)\\ &+\rho M(w(\lambda),y(\lambda),\lambda)+f(t(\lambda),\lambda)). \end{aligned}\tag{4.18}$$

It is easy to see that there exist $u(\bar\lambda)\in A(x,\bar\lambda)$, $v(\bar\lambda)\in B(x,\bar\lambda)$, $w(\bar\lambda)\in C(x,\bar\lambda)$, $y(\bar\lambda)\in D(x,\bar\lambda)$, $z(\bar\lambda)\in G(x,\bar\lambda)$, $n(\bar\lambda)\in P(x,\bar\lambda)$, $t(\bar\lambda)\in Q(x,\bar\lambda)$ and $s(\bar\lambda)\in R(x,\bar\lambda)$

such that

$$
\begin{aligned}
\|u(\lambda)-u(\bar{\lambda})\| &\leq \mathcal{H}(A(x,\lambda),\, A(x,\bar{\lambda})) \leq l_A\|\lambda-\bar{\lambda}\|,\\
\|v(\lambda)-v(\bar{\lambda})\| &\leq \mathcal{H}(B(x,\lambda),\, B(x,\bar{\lambda})) \leq l_B\|\lambda-\bar{\lambda}\|,\\
\|w(\lambda)-w(\bar{\lambda})\| &\leq \mathcal{H}(C(x,\lambda),\, C(x,\bar{\lambda})) \leq l_C\|\lambda-\bar{\lambda}\|,\\
\|y(\lambda)-y(\bar{\lambda})\| &\leq \mathcal{H}(D(x,\lambda),\, D(x,\bar{\lambda})) \leq l_D\|\lambda-\bar{\lambda}\|,\\
\|z(\lambda)-z(\bar{\lambda})\| &\leq \mathcal{H}(G(x,\lambda),\, G(x,\bar{\lambda})) \leq l_G\|\lambda-\bar{\lambda}\|,\\
\|n(\lambda)-n(\bar{\lambda})\| &\leq \mathcal{H}(P(x,\lambda),\, P(x,\bar{\lambda})) \leq l_P\|\lambda-\bar{\lambda}\|,\\
\|t(\lambda)-t(\bar{\lambda})\| &\leq \mathcal{H}(Q(x,\lambda),\, Q(x,\bar{\lambda})) \leq l_Q\|\lambda-\bar{\lambda}\|,\\
\|s(\lambda)-s(\bar{\lambda})\| &\leq \mathcal{H}(R(x,\lambda),\, R(x,\bar{\lambda})) \leq l_R\|\lambda-\bar{\lambda}\|.
\end{aligned}
\tag{4.19}
$$

Let

$$
\begin{aligned}
b \;=\; & x-s(\bar{\lambda})+m(n(\bar{\lambda}),\bar{\lambda})+J_\rho^{W(\cdot,z(\bar{\lambda}),\bar{\lambda})}(s(\bar{\lambda})-m(n(\bar{\lambda}),\bar{\lambda})-\rho N(u(\bar{\lambda}),v(\bar{\lambda}),\bar{\lambda})\\
&+\rho M(w(\bar{\lambda}),y(\bar{\lambda}),\bar{\lambda})+f(t(\bar{\lambda}),\bar{\lambda})),
\end{aligned}
\tag{4.20}
$$

then $b \in F(x,\bar{\lambda})$.

Since N and M are mixed Lipschitz continuous and in view of (4.3), (4.18)-(4.20) and with $t = s(\bar{\lambda})-m(n(\bar{\lambda}),\bar{\lambda})-\rho N(u(\bar{\lambda}),v(\bar{\lambda}),\bar{\lambda})+\rho M(w(\bar{\lambda}),y(\bar{\lambda}),\bar{\lambda})+f(t(\bar{\lambda}),\bar{\lambda})$, we have

$$
\begin{aligned}
\|a-b\| \leq\; & \|s(\lambda)-s(\bar{\lambda})\|+\|m(n(\lambda),\lambda)-m(n(\bar{\lambda}),\bar{\lambda})\|+\|J_\rho^{W(\cdot,z(\lambda),\lambda)}(s(\lambda)-m(n(\lambda),\lambda)\\
&-\rho N(u(\lambda),v(\lambda),\lambda)+\rho M(w(\lambda),y(\lambda),\lambda)+f(t(\lambda),\lambda))-J_\rho^{W(\cdot,z(\lambda),\lambda)}(t)\|\\
&+\|J_\rho^{W(\cdot,z(\lambda),\lambda)}(t)-J_\rho^{W(\cdot,z(\bar{\lambda}),\lambda)}(t)\|+\|J_\rho^{W(\cdot,z(\bar{\lambda}),\lambda)}(t)-J_\rho^{W(\cdot,z(\bar{\lambda}),\bar{\lambda})}(t)\|\\
\leq\; & 2\|s(\lambda)-s(\bar{\lambda})\|+2\|m(n(\lambda),\lambda)-m(n(\bar{\lambda}),\bar{\lambda})\|\\
&+\|f(t(\lambda),\lambda)-f(t(\bar{\lambda}),\bar{\lambda})\|+\rho\|N(u(\lambda),v(\lambda),\lambda)-N(u(\bar{\lambda}),v(\bar{\lambda}),\bar{\lambda})\|\\
&+\rho\|M(w(\lambda),y(\lambda),\lambda)-M(w(\bar{\lambda}),y(\bar{\lambda}),\bar{\lambda})\|+k_1\|z(\lambda)-z(\bar{\lambda})\|+k_2\|\lambda-\bar{\lambda}\|.
\end{aligned}
\tag{4.21}
$$

By the Lipschitz continuity of R in $\lambda \in \Omega$, we have

$$
\|s(\lambda)-s(\bar{\lambda})\| \leq \mathcal{H}(R(x,\lambda),R(x,\bar{\lambda})) \leq l_R\|\lambda-\bar{\lambda}\|. \tag{4.22}
$$

By the mixed Lipschitz continuity of m and the Lispchitz continuity of P, we have

$$
\begin{aligned}
\|m(n(\lambda),\lambda)-m(n(\bar{\lambda}),\bar{\lambda})\| &\leq \|m(n(\lambda),\lambda)-m(n(\bar{\lambda}),\lambda)\|+\|m(n(\bar{\lambda}),\lambda)-m(n(\bar{\lambda}),\bar{\lambda})\|\\
&\leq L_m\|n(\lambda)-n(\bar{\lambda})\|+l_m\|\lambda-\bar{\lambda}\|\\
&\leq L_m\,\mathcal{H}(P(x,\lambda),P(x,\bar{\lambda}))+l_m\|\lambda-\bar{\lambda}\|\\
&\leq (L_m l_P+l_m)\,\|\lambda-\bar{\lambda}\|.
\end{aligned}
\tag{4.23}
$$

By the mixed Lipschitz continuity of f and the Lipschitz continuity of Q, we have

$$
\begin{aligned}
\|f(t(\lambda),\lambda)-f(t(\bar{\lambda}),\bar{\lambda})\| &\leq \|f(t(\lambda),\lambda)-f(t(\bar{\lambda}),\lambda)\|+\|f(t(\bar{\lambda}),\lambda)-f(t(\bar{\lambda}),\bar{\lambda})\|\\
&\leq L_f\|t(\lambda)-t(\bar{\lambda})\|+l_f\|\lambda-\bar{\lambda}\|\\
&\leq L_f\,\mathcal{H}(Q(x,\lambda),Q(x,\bar{\lambda}))+l_f\|\lambda-\bar{\lambda}\|\\
&\leq (L_f l_Q+l_f)\,\|\lambda-\bar{\lambda}\|.
\end{aligned}
\tag{4.24}
$$

By the mixed Lipschitz continuity of N, we have

$$
\begin{aligned}
&\|N(u(\lambda),v(\lambda),\lambda)-N(u(\bar{\lambda}),v(\bar{\lambda}),\bar{\lambda})\| \leq \|N(u(\lambda),v(\lambda),\lambda)-N(u(\bar{\lambda}),v(\lambda),\lambda)\|\\
&+\|N(u(\bar{\lambda}),v(\lambda),\lambda)-N(u(\bar{\lambda}),v(\bar{\lambda}),\lambda)\|+\|N(u(\bar{\lambda}),v(\bar{\lambda}),\lambda)-N(u(\bar{\lambda}),v(\bar{\lambda}),\bar{\lambda})\|\\
&\leq L_{(N,1)}\|u(\lambda)-u(\bar{\lambda})\|+L_{(N,2)}\|v(\lambda)-v(\bar{\lambda})\|+l_N\|\lambda-\bar{\lambda}\|
\end{aligned}
$$

$$\leq\ (l_A L_{(N,1)}+l_B L_{(N,2)}+l_N)\ \|\lambda-\bar{\lambda}\|. \tag{4.25}$$

By the mixed Lipschitz continuity of M, we have

$$\begin{aligned}
&\|M(w(\lambda),y(\lambda),\lambda)-N(w(\bar{\lambda}),y(\bar{\lambda}),\bar{\lambda})\|\ \leq\ \|M(w(\lambda),y(\lambda),\lambda)-M(w(\bar{\lambda}),y(\lambda),\lambda)\|\\
&\quad+\|M(w(\bar{\lambda}),w(\lambda),\lambda)-M(w(\bar{\lambda}),y(\bar{\lambda}),\lambda)\|+\|M(w(\bar{\lambda}),y(\bar{\lambda}),\lambda)-M(w(\bar{\lambda}),y(\bar{\lambda}),\bar{\lambda})\|\\
&\leq\ L_{(M,1)}\|w(\lambda)-w(\bar{\lambda})\|+L_{(M,2)}\|y(\lambda)-y(\bar{\lambda})\|+l_M\|\lambda-\bar{\lambda}\|\\
&\leq\ (l_C L_{(M,1)}+l_D L_{(M,2)}+l_M)\ \|\lambda-\bar{\lambda}\|.
\end{aligned} \tag{4.26}$$

By the Lipschitz continuity of G, we have

$$\|z(\lambda)-z(\bar{\lambda})\|\ \leq\ \mathcal{H}(G(x,\lambda),G(x,\bar{\lambda}))\ \leq\ l_G\ \|\lambda-\bar{\lambda}\|. \tag{4.27}$$

In view of (4.21)-(4.27), we obtain that

$$\|a-b\|\ \leq\ \theta_1\ \|\lambda-\bar{\lambda}\|, \tag{4.28}$$

where

$$\begin{aligned}
\theta_1 :=&\ 2(l_R+L_m l_P+l_m)+\rho(l_A L_{(N,1)}+l_B L_{(N,2)}+l_N+l_C L_{(M,1)}+l_D L_{(M,2)}+l_M)+L_f l_Q+\\
&\ l_f+k_1 l_G+k_2.
\end{aligned}$$

Hence, we obtain

$$\sup_{a\in F(x,\lambda)} d(a,F(x,\bar{\lambda}))\ \leq\ \theta_1\|\lambda-\bar{\lambda}\|.$$

By using a similar argument as above, we can obtain

$$\sup_{b\in F(x,\bar{\lambda})} d(F(x,\lambda),b)\ \leq\ \theta_1\|\lambda-\bar{\lambda}\|.$$

Hence, it follows that

$$\mathcal{H}(F(x,\lambda),F(x,\bar{\lambda}))\leq\theta_1\|\lambda-\bar{\lambda}\|.$$

By Lemma 2.3, we obtain

$$\mathcal{H}(S(\lambda),S(\bar{\lambda})\ \leq\ \Big(\frac{\theta_1}{1-\theta}\Big)\|\lambda-\bar{\lambda}\|.$$

This proves that $S(\lambda)$ is $\mathcal{H}$-Lipschitz continuous in $\lambda\in\Omega$. If, each mapping in this theorem is assumed to be continuous in $\lambda\in\Omega$, then by similar argument as above, we can show that $S(\lambda)$ is also continuous in $\lambda\in\Omega$. This completes the proof.

Remark 4.2. For $k_1,k_2,\rho>0$ and $k\in(0,1)$, it is clear that $\alpha>\sigma$; $L_R>\sqrt{2\delta-1}$; $2\rho(\alpha-\sigma)-1<2\rho^2(L_N^2+L_M^2)$. Further, $\theta\in(0,1)$ and condition (4.4) of Theorem 4.2 holds for some suitable values of constants, for example, $\alpha=3,\ \sigma=\delta=L_R=1.5,\ k_1=0.1,\ k_2=0.2$.

Remark 4.3. Since the PGMMIQVIP (3.1) includes many known classes of parametric generalized variational inclusions as special cases, Theorems 4.1-4.3 improve and generalize the known results given in [3, 5, 6, 11, 12, 13, 18, 20, 21].

Conflict of Interests. The authors declare that there is no conflict of interests.

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