Journal of Mathematical Analysis
ISSN: 2217-3412, URL: http://ilirias.com/jma
Volume 9 Issue 4 (2018), Pages 61-74.

THE ORDER-CONVERGENCE OF THE THAKUR ITERATIVE PROCESS FOR HARDY-ROGERS CONTRACTIONS IN ORDER-BANACH SPACES

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Abstract. This paper proves several fixed point results for mappings satisfying the Hardy-Rogers inequality on order-metric spaces. Using ordered vector space valued norm-type mappings, the concept of completeness is redefined resulting order-Banach spaces. One of the main outcomes proves that each Dedekind σ-complete Riesz space is complete with respect to its natural absolute value. This statement leads to consistent examples of order-Banach structures. Ultimately, the Thakur iterative process is analyzed in the newly defined order-Banach framework; it results that the Thakur iteration order-converges faster than the Picard iterative process, for the class of Hardy-Rogers contractions.

1. Introduction

A short survey on fixed point theory reveals a very rich and complex domain, with a continuous flow of significant extensions and applicative outcomes. Important contributions have been made so far by defining generalized forms of metric structures (see, e.g. [2, 3, 8, 9, 11]), various types of contractive conditions (see, e.g. [1, 4, 5, 12, 13, 14]) or non-standard iterative processes [15, 16].

An interesting way to redefine metrics is by changing the image support space. In this direction, a very popular generalization of the metric structure was initiated by Huang and Zhang [8]; they have introduced the concept of cone metric spaces. The novelty of their definition consisted in taking distance functions with values in some cone ordered Banach space. Nevertheless, despite the huge interest their paper generated, it has been proved in [6] that the notion of a cone metric space is not more general than that of a standard metric space, via the norm function of the image Banach space. On the other hand, a more interesting approach, which eliminated the necessity of a norm was that of Cevik *et al*. [2, 3], who considered distance functions with Riesz space-type value set and called the resulted structures vector metric spaces. Nevertheless, from many points of view, the Riesz-structure requirement is unnecessary restrictive. Li *et al*. [9] have weakened this condition, resulting the concept of ordered metric space; the image space of their so called

2010 *Mathematics Subject Classification.* 47H10, 54H25.
Key words and phrases. Ordered vector spaces; order-metric spaces; order-convergent and order-Cauchy sequences; order-norm; order-Banach space.

Submitted July 9, 2018. Published August 20, 2018.
Communicated by M. Postolache.

ordered distance function was some arbitrary ordered vector space. Now, since there exists a one-to-one correspondence between the partial orders on a vector space that are compatible with the vector space structure and its proper convex cones, the ordered metrics of Li *et al*. are in fact cone metrics, without the interference of a norm function. From another perspective, they also are vector metrics in the sense of Cevik *et al*. [2, 3], without the necessity of a lattice structure.

This paper adopts the metric structure introduced by Li *et al*. in [9], and proves several fixed points results for mappings with more general contractive features. Nevertheless, some slight changes of names and notations will be operated. The terminology ordered metric space gets one thinking to a classical metric space endowed additionally with a partial order, which is not the case here. The term vector metric space is also deficient. To emphasize the fact that the distance function, and not its domain, is the element order-determined, we change the notions ordered metric space or vector metric space into order-metric space. In addition, the new concept of order-Banach space is defined. A major outcome of this paper is related to the order-Banach structure of some Dedekind σ-complete Riesz space, leading to an entire new class of (order-)complete spaces. The arithmetic vector space, the set of all real sequences or the set of all real functions with arbitrary domain are basic examples of order-Banach spaces. Moreover, based on a properly defined convergence comparison criterion, the Thakur iterative process [15, 16] is proved to converge faster than Picard iteration for Hardy-Rogers contractions.

2. Preliminaries

First, let recall some basic concepts related to ordered vector spaces and order-metric spaces from the literature, especially [9]. We adopt the notations in [2, 3, 8, 11].

A real vector space E endowed with a partial order $\succeq^E$ is called a partially ordered vector space or, simply, an *ordered vector space*, denoted $(E, \succeq^E)$, if the following properties hold:

(i) $a \succeq^E b$ implies that $a + c \succeq^E b + c$, for each $a, b, c \in E$;

(ii) $a \succeq^E b$ implies that $\alpha a \succeq^E \alpha b$, for each $a, b \in E$ and $\alpha \geq 0$.

A sequence $\{a_n\}$ in an ordered vector space $(E, \succeq^E)$ is said to be order-decreasing, whenever $m > n$ implies that $a_m \preceq^E a_n$. Such a sequence is denoted by $a_n \downarrow$. An order-decreasing sequence $\{a_n\}$ is said to be order-convergent to a (denoted $a_n \downarrow a$) if $\wedge\{a_n\} = \inf\{a_n : n \in \mathbb{N}\}$ exists with $\wedge\{a_n\} = a$.

Lemma 2.1 ([9]). *Let $\{a_n\}$, $\{b_n\}$ be two sequences in an ordered vector space $(E, \succeq^E)$. Then*

(i) *$a_n \downarrow a$ implies that $\alpha a_n \downarrow \alpha a$, for each $\alpha \geq 0$;*

(ii) *$a_n \downarrow a$ and $b_n \downarrow b$ implies that $(a_n + b_n) \downarrow (a + b)$.*

As consequence of (i) and (ii), the following implication holds: if $a_n \downarrow a$ and $b_n \downarrow b$, then $(\alpha a_n + \beta b_n) \downarrow (\alpha a + \beta b)$, for any $\alpha, \beta \geq 0$.

Definition 2.2 ([9]). *An ordered vector space $(E, \succeq^E)$ is said to be generalized Archimedean, if for any given element $a \succeq^E 0$ and any decreasing sequence of positive numbers $\{\alpha_n\}$ with limit 0, we have $\alpha_n a \downarrow 0$.*

Li *et al*. [9] introduced the concept of order-metric spaces in the following way:

Definition 2.3 ([9]). *Let X be a nonempty set and let $(E, \succeq^E)$ be an ordered vector space. A mapping $d_E\colon X \times X \to E$ is called an order-metric on X with respect to $(E, \succeq^E)$ (or E-metric) if, for every x, y, and z in X, it satisfies the following conditions:*

(i) $d_E(x, y) \succeq^E 0$ *with* $d_E(x, y) = 0$ *if and only if* $x = y$;
(ii) $d_E(x, y) = d_E(y, x)$;
(iii) $d_E(x, z) \preceq^E d_E(x, y) + d_E(y, z)$.

Then (X, d_E) is called an order-metric space, and $d_E(x, y)$ is called the order-distance between x and y, with respect to the ordered vector space $(E, \succeq^E)$.

Example 2.4 ([9]). Every metric space is an order-metric space. Consequently, every Banach space with the metric induced by its norm is also an order-metric space. Moreover, every cone metric space is an order-metric space.

Definition 2.5 ([9]). *A sequence $\{x_n\}$ in an order-metric space (X, d_E) is called*

(i) *order-convergent to $x \in X$ (or E-convergent), denoted as $x_n \to^o x$, if there exists a sequence $\{a_n\}$ in $(E, \succeq^E)$ with $a_n \downarrow 0$ such that $d_E(x_n, x) \preceq^E a_n$ holds, for each n.*
(ii) *an order-Cauchy sequence, if there exists a sequence $\{a_n\}$ in $(E, \succeq^E)$ with $a_n \downarrow 0$ such that $d_E(x_m, x_n) \preceq^E a_n$ holds for each n and for every $m \geq n$.*

Lemma 2.6 ([9]). *If a sequence $\{x_n\}$ in an order-metric space (X, d_E) is order-convergent, then its order-limit is unique.*

Definition 2.7 ([9]). *An order-metric space (X, d_E) is called order-complete if every order-Cauchy sequence in X is order-convergent.*

3. New fixed points results on order-metric spaces

The first outcome of this section is related to the existence of fixed points for a class of Hardy-Rogers contractions [7, 12], with admissible function in the sense of Samet *et al.* [13].

Theorem 3.1. *Let (X, d_E) be an order-complete order-metric space, with respect to a generalized Archimedean ordered vector space $(E, \succeq^E)$. Let $\alpha\colon X \times X \to [0, \infty)$ and $T\colon X \to X$ be two mappings such that for each $x, y \in X$ with $\alpha(x, y) \geq 1$, the following Hardy-Rogers inequality holds:*

$$d_E(Tx, Ty) \preceq^E \gamma_1 d_E(x, y) + \gamma_2 d_E(x, Tx) + \gamma_3 d_E(y, Ty) + \gamma_4 d_E(x, Ty) + \gamma_5 d_E(y, Tx) \tag{3.1}$$

where $\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5 \geq 0$ and $\eta = \min\left\{\frac{\gamma_1+\gamma_2+\gamma_4}{1-\gamma_3-\gamma_4}, \frac{\gamma_1+\gamma_3+\gamma_5}{1-\gamma_2-\gamma_5}\right\} \in [0, 1)$.

Further, assume that

(i) *there exists $x_0 \in X$ with $\alpha(x_0, Tx_0) \geq 1$;*
(ii) *$\alpha(Tx, Ty) \geq 1$, whenever $\alpha(x, y) \geq 1$;*
(iii) *for any sequence $\{x_n\} \subseteq X$ with $\alpha(x_n, x_{n+1}) \geq 1$, $\forall n \in \mathbb{N}$ and $x_n \to^o x$, we have $\alpha(x_n, x) \geq 1$, $\forall n \in \mathbb{N}$.*

Then T has a fixed point.

Proof. Using hypotheses (i) and (ii), one builds a sequence $x_n = Tx_{n-1} = T^n x_0$ satisfying $\alpha(x_n, x_{n+1}) \geq 1$, for each $n \in \mathbb{N}$. Relation (3.1) leads to

$$\begin{aligned} d_E(x_{n+1}, x_{n+2}) &= d_E(Tx_n, Tx_{n+1}) \\ &\preceq^E \gamma_1 d_E(x_n, x_{n+1}) + \gamma_2 d_E(x_n, Tx_n) + \gamma_3 d_E(x_{n+1}, Tx_{n+1}) \\ &\quad +\gamma_4 d_E(x_n, Tx_{n+1}) + \gamma_5 d_E(x_{n+1}, Tx_n) \\ &= \gamma_1 d_E(x_n, x_{n+1}) + \gamma_2 d_E(x_n, x_{n+1}) + \gamma_3 d_E(x_{n+1}, x_{n+2}) \\ &\quad +\gamma_4 d_E(x_n, x_{n+2}) + \gamma_5 d_E(x_{n+1}, x_{n+1}) \\ &\preceq^E \gamma_1 d_E(x_n, x_{n+1}) + \gamma_2 d_E(x_n, x_{n+1}) + \gamma_3 d_E(x_{n+1}, x_{n+2}) \\ &\quad +\gamma_4 d_E(x_n, x_{n+1}) + \gamma_4 d_E(x_{n+1}, x_{n+2}), \end{aligned}$$

and, ultimately to

$$d_E(x_{n+1}, x_{n+2}) \preceq^E \frac{\gamma_1 + \gamma_2 + \gamma_4}{1 - \gamma_3 - \gamma_4} d_E(x_n, x_{n+1}).$$

Similarly, using the symmetry of the order-distance function d_E, one finds that

$$d_E(x_{n+1}, x_{n+2}) \preceq^E \frac{\gamma_1 + \gamma_3 + \gamma_5}{1 - \gamma_2 - \gamma_5} d_E(x_n, x_{n+1}),$$

thus

$$d_E(x_{n+1}, x_{n+2}) \preceq^E \eta d_E(x_n, x_{n+1}), \tag{3.2}$$

where $\eta = \min\left\{\frac{\gamma_1+\gamma_2+\gamma_4}{1-\gamma_3-\gamma_4}, \frac{\gamma_1+\gamma_3+\gamma_5}{1-\gamma_2-\gamma_5}\right\} \in [0,1)$.

Inductively, inequality (3.2) leads to

$$d_E(x_{n+1}, x_{n+2}) \preceq^E \eta^{n+1} d_E(x_0, x_1) \text{ for each } n \in \mathbb{N}.$$

We prove next that $\{x_n\}$ is an order-Cauchy sequence in X. For an arbitrary natural number p, using the triangle inequality, we have

$$\begin{aligned} d_E(x_{n+p}, x_n) &\preceq^E d_E(x_n, x_{n+1}) + d_E(x_{n+1}, x_{n+2}) + \cdots + d_E(x_{n+p-1}, x_{n+p}) \\ &\preceq^E \eta^n d_E(x_0, x_1) + \eta^{n+1} d_E(x_0, x_1) + \cdots + \eta^{n+p-1} d_E(x_0, x_1) \\ &\preceq^E \frac{\eta^n}{1-\eta} d_E(x_0, x_1). \end{aligned}$$

Let $a_n = \frac{\eta^n}{1-\eta} d_E(x_0, x_1)$ for each $n \in \mathbb{N}$. Since $(E, \succeq^E)$ is generalized Archimedean, by Definition 2.2, one obtains $a_n \downarrow 0$. Hence, $\{x_n\}$ is an order-Cauchy sequence in (X, d_E). Since (X, d_E) is order-complete, there exists $x^* \in X$ such that $x_n \to^o x^*$ and, using (iii), one finds $\alpha(x_n, x^*) \geq 1$. Then, by applying again inequality (3.1), we have

$$\begin{aligned} d_E(x_{n+1}, Tx^*) &= d_E(Tx_n, Tx^*) \\ &\preceq^E \gamma_1 d_E(x_n, x^*) + \gamma_2 d_E(x_n, Tx_n) + \gamma_3 d_E(x^*, Tx^*) \\ &\quad +\gamma_4 d_E(x_n, Tx^*) + \gamma_5 d_E(x^*, Tx_n) \\ &= \gamma_1 d_E(x_n, x^*) + \gamma_2 d_E(x_n, x_{n+1}) + \gamma_3 d_E(x^*, Tx^*) \\ &\quad +\gamma_4 d_E(x_n, Tx^*) + \gamma_5 d_E(x^*, x_{n+1}). \end{aligned}$$

By using the triangle and the above inequalities, we obtain

$$\begin{aligned} d_E(x^*, Tx^*) &\preceq^E d_E(x^*, x_{n+1}) + d_E(x_{n+1}, Tx^*) \\ &\preceq^E d_E(x^*, x_{n+1}) + \gamma_1 d_E(x_n, x^*) + \gamma_2 d_E(x_n, x_{n+1}) \\ &\quad +\gamma_3 d_E(x^*, Tx^*) + \gamma_4 d_E(x_n, Tx^*) + \gamma_5 d_E(x^*, x_{n+1}). \end{aligned} \tag{3.3}$$

As $x_n \to^o x^*$, there is a sequence $\{b_n\} \subseteq E$ such that $b_n \downarrow 0$ and $d_E(x_n, x^*) \preceq^E b_n$ for each $n \in \mathbb{N}$. Thus, from Lemma 2.1 and (3.3), it follows that

$$d_E(x^*, Tx^*) \preceq^E 0.$$

Hence, $d_E(x^*, Tx^*) = 0$ and $x^* = Tx^*$. □

Corollary 3.2. *Let (X, d_E) be an order-complete order-metric space, with respect to a generalized Archimedean ordered vector space $(E, \succeq^E)$ and $C \subset X$ be a closed and non-empty subset. Let $T\colon C \to C$ be a mapping such that, for each $x, y \in C$, the Hardy-Rogers inequality* (3.1) *from Theorem 3.1 is satisfied. Then T admits a fixed point in C. Moreover, whenever $\gamma_1 + \gamma_4 + \gamma_5 < 1$, the fixed point is unique.*

Proof. Consider α to be the indicator function of $C \times C$, i.e.

$$\alpha\colon X \times X \to [0, \infty), \quad \alpha(x, y) = \begin{cases} 1 \text{ if } x, y \in C \\ 0 \text{ otherwise.} \end{cases}$$

Denote by $\bar{T}$ and arbitrary extension of T on X. Since C is closed, it follows that $\bar{T}$ and α satisfy all the hypotheses in 3.1, hence $\bar{T}$ admits fixed points which are obtained as order-limits for some Picard iterative processes. Moreover, by starting the construction of the approximation sequence $\{x_n\}$ with a point x_0 from C, the resulted fixed point also lies in C. Ultimately, let us prove the uniqueness of the fixed point in C. Suppose x^* and y^* are two fixed points of T. Then, according to (3.1),

$$\begin{aligned} d_E(Tx^*, Ty^*) \quad &\preceq^E \quad \gamma_1 d_E(x^*, y^*) + \gamma_2 d_E(x^*, Tx^*) + \gamma_3 d_E(y^*, Ty^*) \\ &+ \quad \gamma_4 d_E(x^*, Ty^*) + \gamma_5 d_E(y^*, Tx^*), \end{aligned}$$

resulting

$$(1 - \gamma_1 - \gamma_4 - \gamma_5) d_E(Tx^*, Ty^*) \preceq^E 0.$$

Therefore, if $\gamma_1 + \gamma_4 + \gamma_5 < 1$ it follows $d_E(Tx^*, Ty^*) \preceq^E 0$, hence $x^* = y^*$. □

Example 3.3. Let $E = l_1$ be the set of all real sequences which are absolutely convergent. Then E is a generalized Archimedean ordered vector space with respect to the pointwise order $\{a_n\} \succeq \{b_n\} \Leftrightarrow a_n \geq b_n,\ \forall n \in \mathbb{N}$. Let $X = l_\infty$ be the set of all bounded real sequences. According to [3], X is order-complete with respect to the order-metric

$$d : l_\infty \times l_\infty \to l_1,\ d(\{x_n\}, \{y_n\}) = \left\{ \frac{1}{2^n} |x_n - y_n| \right\}.$$

Consider also the mapping $T : l_\infty \to l_\infty,\ T\{x_n\} = \left\{ \frac{1}{2^n} x_n \right\}$. Then T fails to be an order-contraction. Indeed, supposing there exists a number $k \in [0, 1)$ such that T satisfies the contractive condition $d(T\{x_n\}, T\{y_n\}) \preceq kd(\{x_n\}, \{y_n\})$ with respect to the pointwise order relation in l_1, one finds $\left\{ \frac{1}{2^{2n}} |x_n - y_n| \right\} \preceq k \left\{ \frac{1}{2^n} |x_n - y_n| \right\}$, which leads to the inequality $\frac{1}{2^n} \leq k < 1,\ \forall n = 0, 1, 2....$ Absurd.

Instead, taking $\gamma_1 = \gamma_4 = \frac{1}{2}$ and $\gamma_2 = \gamma_3 = \gamma_5 = 0$, it can be easily checked that T satisfies the Hardy-Rogers inequality (3.1) and the additional assumption $\eta \in [0, 1)$. Therefore, according to the Corollary 3.2, T admits a (not necessarily unique) fixed point.

Throughout the paper, we assume that $G = (V, \mathcal{E})$ is a directed graph such that the set of its vertices $V = X$ and the set of its edges contains all loops but has no parallel edge.

Corollary 3.4. *Let (X, d_E) be an order-complete order-metric space, with respect to a generalized Archimedean ordered vector space $(E, \succeq^E)$ and suppose X is equipped with the graph G. Let $T\colon X \to X$ be a mapping such that, for each $x, y \in X$ with $(x, y) \in \mathcal{E}$, T satisfies the Hardy-Rogers inequality* (3.1) *from Theorem 3.1. Further, assume that the following conditions hold:*

(i) *there exists $x_0 \in X$ with $(x_0, Tx_0) \in \mathcal{E}$;*
(ii) *$(Tx, Ty) \in \mathcal{E}$, whenever $(x, y) \in \mathcal{E}$;*
(iii) *for any sequence $\{x_n\} \subseteq X$ with $(x_n, x_{n+1}) \in \mathcal{E}$, $\forall n \in \mathbb{N}$ and $x_n \to^o x$, we have $(x_n, x) \in \mathcal{E}$, for each $n \in \mathbb{N}$.*

Then T has a fixed point.

Proof. By defining the mapping α as

$$\alpha\colon X \times X \to [0, \infty), \quad \alpha(x, y) = \begin{cases} 1 \text{ if } (x, y) \in \mathcal{E} \\ 0 \text{ otherwise,} \end{cases}$$

this result is a direct consequence of Theorem 3.1. □

We now state and prove the second result of this section:

Theorem 3.5. *Let (X, d_E) be an order-complete order-metric space, with respect to a generalized Archimedean ordered vector space $(E, \succeq^E)$. Let $\alpha\colon X \times X \to [0, \infty)$ and $T\colon X \to X$ be two mappings such that for each $x \in X$ with $\alpha(x, Tx) \geq 1$, we have the following inequality:*

$$d_E(Tx, T^2x) \preceq^E \gamma d_E(x, Tx) \tag{3.4}$$

where $0 \leq \gamma < 1$. Further, assume that the following conditions hold:

(i) *there exists $x_0 \in X$ with $\alpha(x_0, Tx_0) \geq 1$;*
(ii) *$\alpha(Tx, T^2x) \geq 1$, whenever $\alpha(x, Tx) \geq 1$;*
(iii) *$\forall \{x_n\} \subseteq X$ such that $x_n \to^o x$, we have $d_E(x_n, Tx_n) \downarrow d_E(x, Tx)$.*

Then T has a fixed point.

Proof. Using hypotheses (i) and (ii), we build a sequence $x_n = Tx_{n-1} = T^n x_0$ such that $\alpha(x_n, Tx_n) \geq 1$ for each $n \in \mathbb{N}$. From (3.4), it follows that

$$\begin{aligned} d_E(x_{n+1}, x_{n+2}) &= d_E(Tx_n, T^2x_n) \\ &\preceq^E \gamma d_E(x_n, Tx_n) \\ &= \gamma d_E(x_n, x_{n+1}) \ \forall n \in \mathbb{N}. \end{aligned} \tag{3.5}$$

Due to the similarities between (3.2) and (3.5) and by following the ideas in the proof of Theorem 3.1, we conclude that $\{x_n\}$ is an order-Cauchy sequence in the order-complete order-metric space (X, d_E). Then, there exists $x^* \in X$ such that $x_n \to^o x^*$. Moreover, from combining this with (3.5) it follows $d_E(x_n, Tx_n) \downarrow 0$. Finally, using hypothesis (iii), one obtains $d_E(x^*, Tx^*) = 0$, which ultimately makes $x^* = Tx^*$. □

Corollary 3.6. *Let (X, d_E) be an order-complete order-metric space, with respect to a generalized Archimedean ordered vector space $(E, \succeq^E)$ and suppose that X is equipped with the graph $G = (V, \mathcal{E})$. Let $T\colon X \to X$ be a mapping satisfying inequality* (3.4) *from Theorem 3.5. Further, assume that the following conditions hold:*

(i) *there exists $x_0 \in X$ with $(x_0, Tx_0) \in \mathcal{E}$;*

(ii) $(Tx, T^2x) \in \mathcal{E}$, *whenever* $(x, Tx) \in \mathcal{E}$;
(iii) *for any sequence* $x_n \to^o x$, *we have* $d_E(x_n, Tx_n) \downarrow d_E(x, Tx)$.

Then T *has a fixed point.*

Proof. This result can be obtained from Theorem 3.5, by defining

$$\alpha\colon X \times X \to [0, \infty),\ \alpha(x, y) = \begin{cases} 1 \text{ if } (x, y) \in \mathcal{E} \\ 0 \text{ otherwise.} \end{cases}$$

□

Now, we state and prove the last result of the section.

Theorem 3.7. *Let* (X, d_E) *be an order-complete order-metric space, with respect to a generalized Archimedean ordered vector space* $(E, \succeq^E)$. *Let* $x_0 \in X$, $r \in E$ *and let* $T\colon X \to X$ *be a mapping which, for each* $x, y \in B(x_0, r) = \{\hat{x} \in X : d_E(x_0, \hat{x}) \preceq^E r\}$, *satisfies the Hardy-Rogers inequality* (3.1) *from Theorem 3.1. If, in addition,* $d_E(x_0, Tx_0) \preceq^E (1-\eta)r$, *where* $\eta = \min\left\{\frac{\gamma_1+\gamma_2+\gamma_4}{1-\gamma_3-\gamma_4}, \frac{\gamma_1+\gamma_3+\gamma_5}{1-\gamma_2-\gamma_5}\right\} \in [0, 1)$, *then* T *has a fixed point.*

Proof. Consider the sequence $x_n = T^n x_0,\ n \in \mathbb{N}$. Since $x_0, x_1 = Tx_0 \in B(x_0, r)$, based on the inequality (3.1) and using similar arguments as in the proof of Theorem 3.1, one finds

$$d_E(x_1, x_2) \preceq^E \eta d_E(x_0, x_1). \tag{3.6}$$

From the triangle inequality and the above inequality it follows

$$\begin{aligned} d_E(x_0, x_2) &\preceq^E d_E(x_0, x_1) + d_E(x_1, x_2) \\ &\preceq^E d_E(x_0, x_1) + \eta d_E(x_0, x_1) \\ &\preceq^E (1-\eta^2)r \\ &\preceq^E r, \end{aligned}$$

proving that $x_2 \in B(x_0, r)$. Applying once more (3.1) for $x_1, x_2 \in B(x_0, r)$, we obtain

$$d_E(x_2, x_3) \preceq^E \eta d_E(x_1, x_2),$$

which leads to

$$d_E(x_2, x_3) \preceq^E \eta^2 d_E(x_0, x_1). \tag{3.7}$$

Again, by using the triangle inequality, (3.6) and (3.7), it follows that

$$\begin{aligned} d_E(x_0, x_3) &\preceq^E d_E(x_0, x_1) + d_E(x_1, x_2) + d_E(x_2, x_3) \\ &\preceq^E d_E(x_0, x_1) + \eta d_E(x_0, x_1) + \preceq^E \eta^2 d_E(x_0, x_1) \\ &\preceq^E (1-\eta^3)r \\ &\preceq^E r. \end{aligned}$$

Continuing the same procedure we find that $x_n \in B(x_0, r),\ \forall n \in \mathbb{N}$ and

$$d_E(x_n, x_{n+1}) \preceq^E \eta^n d_E(x_0, x_1),\ \forall n \in \mathbb{N},$$

which leads to the conclusion that the sequence is order-Cauchy. Since the hypotheses state that (X, d_E) is order-complete, there exists $x^* \in X$ such that $x_n \to^o x^*$

(i.e. $d_E(x_n, x^*) \preceq^E a_n,\ \forall n \in \mathbb{N}$ and $a_n \downarrow 0$). Let us prove that $x^* \in B(x_0, r)$. By using the triangle inequality, we get

$$\begin{aligned} d_E(x_0, x^*) &\preceq^E d_E(x_0, x_n) + d_E(x_n, x^*) \\ &\preceq^E r + a_n. \end{aligned}$$

From Lemma 2.1, we have $(r + a_n) \downarrow r$. Thus, we obtain $d_E(x_0, x^*) \preceq^E r$. This implies that $x^* \in B(x_0, r)$. The proof ends by applying again the Hardy-Rogers inequality for $d_E(x_{n+1}, Tx^*)$, followed by the triangle inequality for $d_E(x^*, Tx^*)$, the same way as in the proof of Theorem 3.1. □

Example 3.8. Let $E = C[1, 2]$ be the set of all real valued continuous functions; this set is an ordered vector space with the pointwise partial order ($f \preceq^E g$ iff $f(t) \leq g(t)$, for each $t \in [1, 2]$). Let $X = \mathbb{R}$ and consider

$$d_E\colon X \times X \to E, \quad d_E(x, y) = |x - y|t \text{ for each } t \in [1, 2].$$

Then (X, d_E) is an order-complete order-metric space.

Define the mapping

$$T\colon X \to X, \quad Tx = \begin{cases} \frac{2x-1}{3} & \text{if } |x| \leq 2 \\ \frac{3x-1}{2} & \text{otherwise.} \end{cases}$$

For $r = 2t$, one can see that $B(0, 2t) = \{\hat{x} \in X : |\hat{x}|t \leq 2t,\ \forall t \in [1, 2]\} = [-2, 2]$. Moreover, T satisfies (3.1) for $\gamma_1 = \frac{2}{3}$ and $\gamma_i = 0$, $i = \overline{2, 5}$. Indeed, for each $x, y \in B(0, 2t)$, we have

$$d_E(Tx, Ty) = \left| \frac{2x-1}{3} - \frac{2y-1}{3} \right| t = \frac{2}{3}|x - y|t = \frac{2}{3} d_E(x, y).$$

Further, note that

$$d_E(0, T0) = \frac{1}{3}t \preceq^E \frac{1}{3}2t = (1 - \eta)r.$$

Therefore, by Theorem 3.7, it follows that T has a fixed point.

4. Order-Banach spaces

Inspired by the definition of standard Banach spaces, this section introduces new concepts as order-norm, order-normed space and order-Banach space as it follows.

Definition 4.1. *Let X be a real vector space. A mapping $||\cdot||_E : X \to E$ is called order-norm on X with respect to the ordered vector space $(E, \succeq^E)$ if satisfies the following conditions:*

(i) $||x||_E \succeq^E 0_E,\ \forall x \in X$ *and* $||x||_E = 0_E \Leftrightarrow x = 0_X$
(ii) $||\alpha x||_E = |\alpha| \cdot ||x||_E,\ \forall \alpha \in \mathbb{R},\ \forall x \in X$
(iii) $||x + y||_E \preceq^E ||x||_E + ||y||_E,\ \forall x, y \in X.$

The pair $(X, ||\cdot||_E)$ is called order-normed space with respect to the ordered vector space $(E, \succeq^E)$.

Example 4.2.

(1) Consider the vector space $E = C([0, 1]) = \{f : [0, 1] \to \mathbb{R} : \text{f continuous}\}$ endowed with the pointwise order $f \succeq^E g \Leftrightarrow f(t) \geq g(t),\ \forall t \in [0, 1]$. The mapping

$$||\cdot||_E : \mathbb{R} \to C([0, 1]),\ ||x||_E(t) = |x|t,\ \forall t \in [0, 1]$$

is an order-norm on $\mathbb{R}$ with respect to $C([0, 1])$.

(2) Similarly, if $E = \mathbb{R}^{\mathbb{N}}$ is the vector space of real sequences with the pointwise order, then

$$||\cdot||_E : \mathbb{R} \to \mathbb{R}^{\mathbb{N}},\ ||x||_E = \{|x|n\}_n,\ \forall n \in \mathbb{N}$$

is also an order-norm on $\mathbb{R}$.

(3) Let $\{\alpha_n\}$ be an arbitrary fixed sequence of positive real numbers. The mapping

$$||\cdot||_E : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}},\ ||\{x_n\}||_E = \{\alpha_n|x_n|\},\ \forall n \in \mathbb{N}$$

is an order-norm on $\mathbb{R}^{\mathbb{N}}$.

Proposition 4.3. *If $(X, ||\cdot||_E)$ is an order-normed space with respect to the ordered vector space $(E, \succeq^E)$, then (X, d_E) is an order-metric space, where the order-metric d_E is induced by the order-norm $||\cdot||_E$ in the usual way, i.e. $d_E(x, y) = ||x - y||_E$.*

Proof. The statement is a direct consequence of the properties of the order-norm mapping. $\square$

Due to the statement above, we can accordingly define convergence of sequences.

Definition 4.4. *Let $\{x_n\}$ be a sequence in an order-normed space $(X, ||\cdot||_E)$. Then, the sequence $\{x_n\}$ is:*

(i) *order-convergent to x in X, denoted as $x_n \to^o x$, whenever there exists another sequence $\{a_n\}$ in $(E, \succeq^E)$ with $a_n \downarrow 0$ such that $||x_n - x||_E \preceq^E a_n$ holds, for each n.*

(ii) *an order-Cauchy sequence, whenever there exists another sequence $\{a_n\}$ in $(E, \succeq^E)$ with $a_n \downarrow 0$ such that $||x_{n+p} - x_n||_E \preceq^E a_n$ holds for all non-negative integers n and p.*

One can easily check that every order-convergent sequence is also order-Cauchy.

Definition 4.5. *An order-normed space $(X, ||\cdot||_E)$ is said to be order-Banach space, whenever every order-Cauchy sequence in X is order-convergent (i.e (X, d_E) is order-metric complete).*

The most relevant examples of order-Banach spaces are related to Riesz spaces. In the following we recall some classic definitions. A Riesz space is an ordered vector space $(E, \succeq^E)$ which is also a lattice, i.e. any two points $x, y \in E$ have a supremum $x \vee y$ and an infinmum $x \wedge y$. In addition, one can define the *order-modulus* of some element $x \in E$ as $|x| = x \vee (-x)$. Moreover, a Riesz space is called *Dedekind σ-complete* if every countable non-empty subset which is bounded above has a supremum. Also, a Riesz space is called Archimedean if for any given element $x \succeq^E 0$ and any decreasing sequence of positive numbers $\{a_n\}$ with limit 0, we have $a_n x \downarrow 0$. An important statement is that any Dedekind σ-complete Riesz space is also Archimedean.

Theorem 4.6. *Every Riesz space $(E, \succeq^E)$ is an order-normed space with respect to the order-modulus norm*

$$||\cdot||_E : E \to E,\ ||x||_E = |x|,\ \forall x \in E.$$

Moreover, if $(E, \succeq^E)$ is Dedekind σ-complete, then $(E, ||\cdot||_E)$ is an order-Banach space.

Proof. One can easily check that the order-modulus on some Riesz space satisfies the properties of an order-norm. Suppose now that $(E, \succeq^E)$ is Dedekind σ-complete, therefore is also Archimedean. We prove the order-completeness in several steps.

Step1. We prove that each order-Cauchy sequence $\{x_n\}_n$ is order-bounded, i.e. there exists two comparable elements $a, b \in E$, $a \preceq^E b$, $a \neq b$ such that $a \preceq^E x_n \preceq^E b$, $\forall n \in \mathbb{N}$ (we write $x_n \in [a, b]$).

Suppose $\{x_n\}_n$ is an order-Cauchy sequence. According to Definition 4.4, there exists another sequence $\{\xi_n\} \subset E$, with $\xi_n \downarrow 0$, such that $|x_{n+p} - x_n| \preceq^E \xi_n,\ \forall n,\ \forall p$. Taking $n = 0$, one finds $|x_p - x_0| \preceq^E \xi_0,\ \forall p \in \mathbb{N}$ and then

$$|x_p| \preceq^E |x_p - x_0| + |x_0| \preceq^E \xi_0 + |x_0|,\ \forall p \in \mathbb{N}.$$

Denoting $M = \xi_0 + |x_0|$, it follows $M \succeq^E 0$ and $x_p \in [-M, M],\ \forall p$, hence $\{x_n\}$ is bounded.

Step2. Let us prove that every order-bounded sequence contains an order-convergent subsequence. Suppose $x_n \in [a_0, b_0],\ \forall n \in \mathbb{N}$, with $a_0, b_0 \in E$, $a_0 \preceq^E b_0$, $a_0 \neq b_0$. Consider $c_0 = \frac{1}{2}a_0 + \frac{1}{2}b_0$. Then c_0 is comparable with a_0 and b_0; more precisely, $a_0 \preceq^E c_0 \preceq^E b_0$, $c_0 \neq a_0, b_0$ and at least one of the order-intervals $[a_0, c_0]$ or $[c_0, b_0]$ contains an infinite number of sequence terms. Denote this order-interval $[a_1, b_1]$. Repeating the above arguments one finds a sequence of nested intervals $[a_0, b_0] \supset^E [a_1, b_1] \supset^E ... \supset^E [a_n, b_n] \supset^E ...$, all of them containing an infinite number of sequence elements. In addition, due to the properties of the order relation $\succeq^E$, the inequality $b_n - a_n \preceq^E \frac{1}{2^n}(b_0 - a_0)$ holds. Since $(E, \succeq^E)$ is Dedekind σ-complete, there exists $\xi = \sup\{a_n : n \in \mathbb{N}\}$. Moreover, $a_m \preceq^E b_n,\ \forall n, m$, therefore $a_n \preceq^E \xi \preceq^E b_n,\ \forall n$, hence $\xi \in \cap_{n \in \mathbb{N}}[a_n, b_n]$. Let $k_0 < k_1 < k_2 < ...$ be such that $x_{k_n} \in [a_n, b_n]$. This is possible since each order interval contains an infinite number of sequence elements. It follows $|x_{k_n} - \xi| \preceq^E \frac{1}{2^n}(b_0 - a_0)$. Since $(E, \succeq^E)$ is also Archimedean, the sequence $\eta_n = \frac{1}{2^n}(b_0 - a_0)$ is an order-decreasing sequence order-convergent to 0, therefore, according to Definition 4.4, the subsequence $\{x_{k_n}\}$ is order-convergent.

Step3. In conclusion, if $\{x_n\}$ is an order-Cauchy sequence then it contains an order-convergent subsequence. Let us prove next that the sequence $\{x_n\}$ entirely is order-convergent. Indeed,

$$|x_n - \xi| \preceq^E |x_{k_n} - x_n| + |x_{k_n} - \xi| \preceq^E \xi_n + \eta_n,\ \forall n \in \mathbb{N}$$

and $(\xi_n + \eta_n) \downarrow 0$, which ends the proof. □

Example 4.7. The set $\mathbb{R}$ of real numbers with classic order relation and the vector space $\mathbb{R}^S$ of real functions defined on some arbitrary nonempty set, with pointwise order relation are Dedekind σ-complete Riesz spaces, thus they are order-Banach spaces with respect to their natural order-modulus. More explicitly, the following list contains basic examples of order-Banach spaces:

(i) $(\mathbb{R}, |\cdot|)$
(ii) $\mathbb{R}^n$ with the order-modulus $|\cdot| : \mathbb{R}^n \to \mathbb{R}^n,\ |x| = (|x_1|, |x_2|, ..., |x_n|)$
(iii) the set of real sequences $\mathbb{R}^{\mathbb{N}}$ endowed with the order-modulus $|\cdot| : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$, $|\{x_n\}| = \{|x_n|\}$.
(iv) the space of real functions $\mathbb{R}^S$ in general and the space of linear functions on some closed interval $E = \{f : [a, b] \to \mathbb{R} : f(t) = \alpha t + \beta\}$ with the order-norm $|f|(t) = |f(t)|,\ \forall t$.

(v) the L^p spaces, with the same order-norm on functions as above, naturally induced by the pointwise order relation.

Other nontrivial examples of Dedekind σ-complete Riesz spaces can be extracted from [10].

Example 4.8. The set $\mathbb{R}^S$ of real functions defined on a non-empty set S, endowed with the norm

$$||\cdot||_\varphi : \mathbb{R}^S \to \mathbb{R}^S,\ ||f||_\varphi(t) = \varphi(t)|f(t)|,\ \forall t \in S,$$

where $\varphi : S \to (0, \infty)$ denotes an arbitrary positive function, is an order-Banach space. Indeed, suppose $\{f_n\}$ is an order-Cauchy sequence in $\mathbb{R}^S$. Then, according to Definition 4.4, there exists another function sequence $\{h_n\}$, order-decreasing (or pointwise decreasing) to the zero function, such that

$$||f_{n+p} - f_n||_\varphi \preceq h_n,\ \forall n, p \in \mathbb{N}.$$

Explicitly, this means $\varphi(t)|f_{n+p}(t) - f_n(t)| \leq h_n(t)$, $\forall n, p \in \mathbb{N}$, $\forall t \in S$. Since $\varphi(t) > 0$, $\forall t$, it follows that $|f_{n+p}(t) - f_n(t)| \leq \tilde{h}_n(t) = \frac{h_n(t)}{\varphi(t)}$, $\forall n, p \in \mathbb{N}$, $\forall t$ and $\{\tilde{h}_n\} \downarrow 0$. Hence, for each $t \in S$, the number sequence $\{f_n(t)\}$ is pointwise Cauchy in $\mathbb{R}$, therefore it is pointwise convergent. Let $f(t)$ denote its limit. Then, $|f_n(t) - f(t)| \leq g_n(t)$, for some decreasing to 0 sequence $\{g_n(t)\}$. Multiplying with the positive value $\varphi(t)$, and varying t, one may conclude that $\{f_n\}$ is order-convergent to f with respect to the order-norm $||\cdot||_\varphi$.

In particular, taking $S = \mathbb{N}$, we may state that the order-norm defined in Example 4.2 (iii),

$$||\cdot|| : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}},\ ||\{x_n\}|| = \{\alpha_n |x_n|\},\ \forall n \in \mathbb{N},$$

for some arbitrary fixed sequence of positive real numbers $\{\alpha_n\}$, generates an order-Banach structure on $\mathbb{R}^{\mathbb{N}}$.

Moreover, if $\alpha_n = \frac{1}{2^n}$, the corresponding order-norm determines the order-metric used in Example 3.3.

5. An order-convergence theorem for the Thakur iterative process in order-Banach spaces

The main purpose of defining order-Banach spaces in the section above is to create a framework for more general iterative processes (for example Mann, Ishikawa, Agarwal, Noor, Abbas, Thakur, S_n iterative schemes and so on). Just to make some choice, we consider the Thakur iteration [15], although any of the other enumerated processes may be a valid candidate for our arguments.

Let $(X, ||\cdot||_E)$ be an order-Banach space with respect to the ordered vector space $(E, \succeq^E)$ and $C \subset X$ be a closed convex subset. A Thakur iterative process [15] is a sequence $\{x_n\} \subset C$ generated from an arbitrary $x_0 \in C$ in the following way:

$$\begin{cases} x_{n+1} = (1 - \alpha_n)Tx_n + \alpha_n T y_n \\ y_n = (1 - \beta_n) z_n + \beta_n T z_n \\ z_n = (1 - \gamma_n) x_n + \gamma_n T x_n, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in the interval $(0, 1)$.

We also define a comparison criterion for the order-convergence of sequences as follows.

Definition 5.1. *Let $\{a_n\}$ and $\{b_n\}$ be two sequences in the ordered vector space $(E, \succeq^E)$ such that $a_n \downarrow 0$ and $b_n \downarrow 0$. We say that $\{a_n\}$ is faster descending than $\{b_n\}$ if there exists $\theta \in [0,1)$ such that $a_n \preceq^E \theta^n b_n$, for all n.*

Definition 5.2. *Let $\{x_n\}$ and $\{\bar{x}_n\}$ be two order-convergent sequences of an order-Banach space $(X, ||\cdot||_E)$, with order-limits x and $\bar{x}$, respectively, for which the following error estimates are available*

$$||x_n - x||_E \preceq^E a_n \text{ and } ||\bar{x}_n - \bar{x}||_E \preceq^E b_n, \ \forall n,$$

where $\{a_n\} \downarrow 0$ and $\{b_n\} \downarrow 0$. If $\{a_n\}$ is faster descending than $\{b_n\}$, then $\{x_n\}$ is faster order-convergent than $\{\bar{x}_n\}$.

Theorem 5.3. *Let $(X, ||\cdot||_E)$ be an order-Banach space with respect to an Archimedean ordered vector space $(E, \succeq^E)$ and $C \subset X$ be a closed convex subset. Let $T : C \to C$ be a Hardy-Rogers contraction, i.e.*

$$||Tx-Ty||_E \preceq^E \gamma_1||x-y||_E+\gamma_2||x-Tx||_E+\gamma_3||y-Ty||_E+\gamma_4||x-Ty||_E+\gamma_5||y-Tx||_E \tag{5.1}$$

where $\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5 \geq 0$, $\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 + \gamma_5 < 1$. If $\{x_n\}$ is a Thakur iterative process with $0 < \alpha \leq \alpha_n \leq 1$, $0 < \beta \leq \beta_n \leq 1$ and $0 < \gamma \leq \gamma_n \leq 1$, then $\{x_n\}$ order-converges to the unique fixed point of T faster than the Picard iterative process.

Proof. Let $\eta = \min\left\{\frac{\gamma_1+\gamma_2+\gamma_4}{1-\gamma_3-\gamma_4}, \frac{\gamma_1+\gamma_3+\gamma_5}{1-\gamma_2-\gamma_5}\right\}$ as in Theorem 3.1 and its corollaries. Suppose $\eta \geq 1$; it follows $\gamma_1 + \gamma_2 + \gamma_3 + 2\gamma_4 \geq 1$ and $\gamma_1 + \gamma_2 + \gamma_3 + 2\gamma_5 \geq 1$, leading to $\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 + \gamma_5 \geq 1$, which contradicts the hypotheses. Therefore, $\eta < 1$. Moreover, $\eta \geq 0$ and $\gamma_1 + \gamma_4 + \gamma_5 < 1$. Then, Corollary 3.2 guarantees the existence and uniqueness of a fixed point $p \in C$. We will prove that $x_n \to^o p$. For this, we start with a general inequality. From relation (5.1), for each element $x \in C$,

$$\begin{array}{rcl} ||Tx - Tp||_E & \preceq^E & \gamma_1||x - p||_E + \gamma_2||x - Tx||_E + \gamma_3||p - Tp||_E \\ & + & \gamma_4||x - Tp||_E + \gamma_5||p - Tx||_E \\ & \preceq^E & \gamma_1||x - p||_E + \gamma_2||x - p||_E + \gamma_2||Tx - Tp||_E \\ & + & \gamma_4||x - p||_E + \gamma_5||Tp - Tx||_E \end{array}$$

which leads to

$$||Tx - Tp||_E \preceq^E \frac{\gamma_1 + \gamma_2 + \gamma_4}{1 - \gamma_2 - \gamma_5}||x - p||_E$$

Similarly, since $||Tx - Tp||_E = ||Tp - Tx||_E$, we also find

$$||Tx - Tp||_E \preceq^E \frac{\gamma_1 + \gamma_3 + \gamma_5}{1 - \gamma_3 - \gamma_4}||x - p||_E,$$

thus

$$||Tx - Tp||_E \preceq^E k||x - p||_E, \ \forall x \in C,$$

where $k = \min\left\{\frac{\gamma_1+\gamma_2+\gamma_4}{1-\gamma_2-\gamma_5}, \frac{\gamma_1+\gamma_3+\gamma_5}{1-\gamma_3-\gamma_4}\right\} \in [0,1)$.

Then

$$\begin{aligned}
||z_n - p||_E &= ||(1-\gamma_n)(x_n - p) + \gamma_n(Tx_n - Tp)||_E \\
&\preceq^E (1-\gamma_n)||x_n - p||_E + \gamma_n||Tx_n - Tp||_E \\
&\preceq^E (1-(1-k)\gamma_n)\,||x_n - p||_E
\end{aligned}$$

$$\begin{aligned}
||y_n - p||_E &= ||(1-\beta_n)(z_n - p) + \beta_n(Tz_n - Tp)||_E \\
&\preceq^E (1-(1-k)\beta_n)\,||z_n - p||_E \\
&\preceq^E (1-(1-k)\beta_n)(1-(1-k)\gamma_n)\,||x_n - p||_E \\
&= \left(1-(1-k)(\beta_n+\gamma_n)+(1-k)^2\beta_n\gamma_n\right)||x_n - p||_E \\
&\preceq^E \left(1-2(1-k)\beta_n\gamma_n+(1-k)^2\beta_n\gamma_n\right)||x_n - p||_E \\
&= \left(1-(1-k^2)\beta_n\gamma_n\right)||x_n - p||_E
\end{aligned}$$

$$\begin{aligned}
||x_{n+1} - p||_E &= ||(1-\alpha_n)(Tx_n - Tp) + \alpha_n(Ty_n - Tp)||_E \\
&\preceq^X k\left[(1-\alpha_n)||x_n - p||_E + \alpha_n||y_n - p||_E\right] \\
&\preceq^E k\left[1-\alpha_n+\alpha_n\left(1-(1-k^2)\beta_n\gamma_n\right)\right]||x_n - p||_E \\
&= k\left(1-(1-k^2)\alpha_n\beta_n\gamma_n\right)||x_n - p||_E \\
&\preceq^X k\left(1-(1-k^2)\alpha\beta\gamma\right)||x_n - p||_E.
\end{aligned}$$

It follows that

$$||x_n - p||_E \preceq^X k^n\theta^n||x_0 - p||_E,\ \forall n \in \mathbb{N},$$

where $\theta = 1-(1-k^2)\alpha\beta\gamma \in [0,1)$. Since $(E, \succeq^E)$ is Archimedean, the sequence $a_n = k^n\theta^n||x_0 - p||_E$ is order-descending to 0, therefore x_n is order-convergent to p.

On the other hand, if $\{\bar{x}_n\}$ denotes the Picard iterative process, the corresponding inequality stands

$$||\bar{x}_n - p||_E \preceq^X k^n||x_0 - p||_E,\ \forall n \in \mathbb{N}.$$

Denote $b_n = k^n||x_0 - p||_E$. According to definition 5.1, $\{a_n\}$ is faster descending to 0 than $\{b_n\}$, therefore the Thakur iterative process $\{x_n\}$ is faster order-convergent than the Picard iterative process $\{\bar{x}_n\}$. □

6. Conclusion

The main contribution of this work is a new approach on normed vector spaces. By taking ordered vector space valued mappings, satisfying similar properties as a standard norm, it results an *order-norm* and properly defined related concepts like order-metric space, order-convergence, order-Cauchy sequence, order-completeness, order-Banach space. Several fixed point theorems for mappings satisfying Hardy-Rogers-type inequalities on order-metric spaces are proved. Two major results are particularly significant. One of them defines a class of order-Banach spaces, starting from Dedekind σ-complete Riesz spaces and leads to consistent examples. The other important outcome proves that Thakur iterative processes order-converge faster than Picard iterations, for Hardy-Rogers contractive mappings.

In fact, this paper provides only a selection of possible fixed point or convergence results related to the newly defined order-Banach spaces. Once the order-metric framework properly settled, one may consider other types of contractive conditions (for example order-nonexpansive mapping, order-Zamfirescu mapping and so on) or any other iterative process (Mann, Ishikawa, Agarwal and other) so that ultimately to extend various results from standard Banach spaces.

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