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A NEW THREE-STEP ITERATION METHOD FOR α-NONEXPANSIVE MAPPINGS AND VARIATIONAL INEQUALITIES

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Abstract. In this paper, we find a common element for the set of fixed points of an α-nonexpansive mapping and the set of solutions for variational inequalities by some three-steps iteration scheme. Moreover, the strong convergence to common element of two sets under some constraints are established.

1. Introduction

Let K be a nonempty closed and convex set in a real Hilbert space, whose inner product and norm are denoted by $\langle .,. \rangle$ and $\|.\|$, respectively. Let $T : K \to K$ be a nonlinear operator and S be a nonexpansive operator. Let P_K be the projection from H onto the convex set K. A mapping $T : K \to H$ is called monotone, if $\langle Tu - Tv, u - v \rangle \geq 0$.
T is an α-inverse strongly monotone, if there exists a positive real number α such that

$$\langle Tu - Tv, u - v \rangle \geq \alpha \|Tu - Tv\|^2,$$

for all $u, v \in K$. A self mapping S on K is called nonexpansive if

$$\|Su - Sv\| \leqslant \|u - v\| \qquad \text{for all} \quad u, v \in K.$$

We take into account the following problem:

$$\textit{Find } u \in K \ \textit{ such that } \ \langle Tu, v - u \rangle \geq 0, \qquad \forall v \in K, \tag{1.1}$$

which is introduced and studied by Stampacchia [9] in 1964. Iterative methods has been extensively studied by many authors (see [1, 2, 6] and the references therein). The following definitions and results from [7] are needed to go on:

Lemma 1.1. *For each $z \in H$, there exists $u \in K$ such that*

$$\langle u - z, v - u \rangle \geq 0,$$

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for all $v \in K$, if and only if, $u = P_K[z]$ in which P_K is the projection from H onto the closed convex set K.

Using Lemma 1.1, one can show that the variational inequality (1.1) is equivalent to a fixed-point problem.

Lemma 1.2. *The function $u \in K$ is a solution of the variational inequality* (1.1)*, if and only if, $u \in K$ satisfies the relation*

$$u = P_k[u - \rho Tu],$$

where $\rho > 0$ is a constant.

It is clear from Lemma 1.2 that the variational inequalities and the fixed point problems are equivalent. This alternative equivalent formulation plays a significant role in the studies of variational inequalities and related optimization problems.
Let S be a nonexpansive mapping. We denote the set of all fixed points of S by $F(S)$ and the set of all solutions of the variational inequalities (1.1) by $VI(K,T)$.

We can characterize the problem. If $x^* \in F(S) \cap VI(K,T)$, then by Lemma 1.2, we have

$$x^* = Sx^* = P_K[x^* - \rho Tx^*] = SP_K[x^* - \rho Tx^*],$$

where $\rho > 0$ is a constant.
This fixed point formulation is used to suggest the following iterative methods for finding a common element for two different sets of fixed points of nonexpansive mappings and solutions of the variational inequalities.

Algorithm 1.3. *For given $x_0 \in K$, consider the sequence x_n by the following iterative scheme:*

$$z_n = (1 - c_n)x_n + c_n SP_K[x_n - \rho Tx_n], \tag{1.2}$$

$$y_n = (1 - b_n)x_n + b_n SP_K[z_n - \rho Tz_n], \tag{1.3}$$

$$x_{n+1} = (1 - a_n)x_n + a_n SP_K[y_n - \rho Ty_n], \tag{1.4}$$

where $a_n, b_n, c_n \in [0,1]$, for all $n \geq 0$ and S is a nonexpansive operator. Algorithm 1.3 is a three-steps predictor-corrector method. For $S = I$ (the identity operator) Algorithm 1.3 has been investigated by Noor [4].

Note that, for $c_n \equiv 0$, Algorithm 1.3 reduces as follows:

Algorithm 1.4. *For an arbitrarily chosen initial point $x_0 \in K$, consider the sequence x_n by the following iterative scheme:*

$$y_n = (1 - b_n)x_n + b_n SP_K[x_n - \rho Tx_n],$$

$$x_{n+1} = (1 - a_n)x_n + a_n SP_K[y_n - \rho Ty_n],$$

where $a_n, b_n \in [0,1]$, for all $n \geq 0$ and S is the nonexpansive operator. Algorithm 1.4 is called two-steps iterative method, which has been considered and studied by Huang and Noor [4]. *For $b_n \equiv 1$, $a_n \equiv 1$, Algorithm 1.4 reduces as follows:*

Algorithm 1.5. *For an arbitrarily chosen initial point $x_0 \in K$, consider the sequence x_n by the following iterative scheme:*

$$y_n = SP_K[x_n - \rho Tx_n],$$

$$x_{n+1} = SP_K[y_n - \rho Ty_n].$$

Remark. *It is worth mentioning that our Algorithm 1.5 is a two-steps method, which may be regarded as a predictor-corrector method. Moreover, Algorithm 1.4 covers the case in Algorithm 1.5 and Algorithm 1.5 can be written as*

$$x_{n+1} = P_K[P_K[x_n - \rho T x_n] - \rho T P_K[x_n - \rho T x_n]],$$

which is called extragradient Algorithm. We refer to [4]*, for more details about the convergence of Algorithm 1.5.*

For $b_n \equiv 0, c_n \equiv 0$, Algorithm 1.3 is simplified in the following iterative method:

Algorithm 1.6. *For given $x_0 \in K$, consider the sequence x_n by the following iterative scheme:*

$$x_{n+1} = (1 - a_n)x_n + a_n S P_K[x_n - \rho T x_n], \tag{1.5}$$

which is known as one-step iteration method. We refer to Huang and Noor [4] *for the convergence of Algorithm 1.6.*

In particular, three-step method is quite general and includes several new and recent known algorithms for solving variational inequalities and nonexpansive mappings.

Theorem 1.7. [8, Theorem 3.1] *Let K be a closed convex subset of a real Hilbert space H. Let $T : K \to H$ be an α-inverse-monotone and let S be a nonexpansive mapping from K into itself such that $F(S) \cap VI(K,T) \neq \emptyset$. Let $u \in C$ be fixed and $x_0 \in C$ be given and let the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ are generated by the following:*

$$\begin{cases} y_n = \sigma_n x_n + (1 - \sigma_n) S P_K[x_n - \rho_n T x_n] \\ z_n = \mu_n x_n + (1 - \mu_n) S P_K[y_n - \rho_n T y_n] \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n S P_K[z_n - \rho_n T z_n], n \geqslant 0, \end{cases} \tag{1.6}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\mu_n\}$ and $\{\sigma_n\}$ are five sequences in $[0,1]$ and $\{\rho_n\}$ is a sequence in $[0, 2\alpha]$. As long as, $\rho_n \in [a,b]$, for some a, b in which $0 < a < b < 2\alpha$ and

(i) $\alpha_n + \beta_n + \gamma_n = 1$,
(ii) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$
(iii) $0 < \liminf_{n\to\infty} \beta_n \leqslant \limsup_{n\to\infty} \beta_n < 1$ *and* $lim_{n\to\infty}\mu_n = 1$
(iv) $lim_{n\to\infty}(\rho_{n+1} - \rho_n) = 0$,

then $\{x_n\}$ is defined by (1.6) *which converges strongly to $x^* \in P_{F(S)\cap VI(K,T)}u$.*

2. Preliminaries

Gobel and Pineda [3], studied α-nonexpansive mapping, which is a generalization of the nonexpansive one. Let $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$, where $\alpha_i \geq 0, i = 1, 2, \cdots, n$ and $\sum_{i=1}^{n} \alpha_i = 1$ and let K be a nonempty closed and convex subset of a Banach space X. A mapping $T : K \to K$ is said to be α-nonexpansive if

$$\sum_{i=1}^{n} \alpha_i \|T^i x - T^i y\| \leq \|x - y\|, \tag{2.1}$$

for all $x, y \in K$.

For some technical reasons we always assume that $\alpha_1 > 0$. In this case the mapping T satisfies the Lipschitz condition

$$\|Tx - Ty\| \leqslant \frac{1}{\alpha_1}\|x - y\|,$$

for all $x, y \in K$.

Taking into account (2.1), one can investigate that considering $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ conducive to nonexpansivity of

$$T_\alpha x = \sum_{i=1}^{n} \alpha_i T^i x,$$

for all $x \in K$. However, nonexpansivity of T_α is much weaker than (2.1), it does not yield the continuity of T.

Lemma 2.1. [10] *Suppose that $\{\delta_k\}_{k=0}^{k=\infty}$ is a nonnegative sequence satisfies in the following inequality*

$$\delta_{k+1} \leq (1 - \lambda_k)\delta_k + \sigma_k, \qquad k \geq 0,$$

where $\lambda_k \in [0, 1]$, $\sum_{k=0}^{\infty} \lambda_k = \infty$ and $\sigma_k = o(\lambda_k)$. Then $\lim_{k\to\infty} \delta_k = 0$.

Theorem 2.2. [5] *Let K be a closed convex subset of Banach space X and let $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$, for all $n \in \mathbb{N}$, be such that $\alpha_i \geq 0, i = 1, 2, \cdots, n, \alpha_1 > 0$ and $\sum_{i=1}^{n} \alpha_i = 1$. Let T be an α-nonexpansive mapping from K into itself. If $\alpha_1 > \frac{1}{\sqrt[n-1]{2}}$, then $F(T) = F(T_\alpha)$.*

3. Main results

In this section, we introduced a new definition which is generalization of the α-inverse-monotone mapping.

Definition 3.1. *We say that a mapping $T : K \to H$ satisfies in θ-property, if there exist $\rho > 0$ and a function $0 < \theta(\rho) < \infty$ such that*

$$\|(x - y) - \rho(Tx - Ty)\| \leq \theta(\rho)\|x - y\|,$$

for all $x, y \in K$.

Example 3.2. *Let $H = K := l^2$ and $Tx = \frac{x}{\rho+1}$, for some $\rho > 0$. Then we put $\theta(\rho) = \frac{1}{\rho+1} < 1$.*

Example 3.3. *Let $H = K := l^2$, $Tx := x$ and $0 < \rho < 1$. Then $\theta(\rho) = 1 - \rho < 1$.*

Example 3.4. *Let $H = [0, \infty)$ and $K = [0, 1]$. Define $T : K \to H$ as follows:*

$$T(x) = \frac{x^2}{1 + x} \quad \forall x \in [0, 1].$$

Put $\theta(\rho) = 1 + 3\rho$. Clearly,

$$\begin{aligned}
\|(x - y) - \rho(Tx - Ty)\| &= \|(x - y) - \rho(\frac{x^2}{1 + x} - \frac{y^2}{1 + y})\| \\
&\leq \|x - y\| + \rho\|\frac{x^2}{1 + x} - \frac{y^2}{1 + y}\| \\
&\leq \|x - y\| + \rho\|x^2(1 + y) - y^2(1 + x)\| \\
&\leq (1 + 3\rho)\|x - y\|
\end{aligned}$$

Therefore, T satisfies in θ-property.

Example 3.5. *Let $H = [0, \infty)$ and $K = [0, 1]$. The mappings $T(x) = x$, $T(x) = 1-x$ and $T(x) = x^2$ satisfy in θ-property by $\theta(\rho) = 1-\rho$, for $0 < \rho < 1$, $\theta(\rho) = 1+\rho$ and $\theta(\rho) = 1$, respectively.*

Theorem 3.6. *Let K be a closed convex subset of a real Hilbert space H. Let $T : K \to H$ satisfies in θ-property and S be an α-nonexpansive mapping from K into itself such that $\alpha_1 > \frac{1}{\sqrt[n-1]{2}}$ and $F(S) \cap VI(K, T) \neq \emptyset$. Let $\{x_n\}$ be a sequence defined by Algorithm 1.3, for any initial point $x_0 \in K$ where $0 < \theta(\rho) < 1$, $a_n, b_n, c_n \in [0, 1]$ and $\sum_{n=0}^{\infty} a_n = \infty$. Then the sequence x_n, which is derived from Algorithm 1.3, converges strongly to $x^* \in F(S) \cap VI(K, T)$.*

Proof. Pick $x^* \in F(S) \cap VI(K, T)$. Since $F(S) = F(S_\alpha)$, by Theorem 2.2, we have $x^* \in K \cap F(S_\alpha) \cap VI(K, T)$. Therefore,

$$x^* = (1 - c_n)x^* + c_n S_\alpha P_K[x^* - \rho T x^*] \tag{3.1}$$
$$= (1 - b_n)x^* + b_n S_\alpha P_K[x^* - \rho T x^*] \tag{3.2}$$
$$= (1 - a_n)x^* + a_n S_\alpha P_K[x^* - \rho T x^*], \tag{3.3}$$

where $a_n, b_n, c_n \in [0, 1]$. We shall show that $x_{n+1} \to x^*$ as $n \to \infty$. From (1.4), (3.3) and the nonexpansive property of the projection P_K and the nonexpansive mapping S_α, we have

$$\begin{aligned}\|x_{n+1} - x^*\| &= \|(1 - a_n)x_n + a_n S_\alpha P_K[y_n - \rho T y_n] - (1 - a_n)x^* - a_n S_\alpha P_K[x^* - \rho T x^*]\| \\ &\leq (1 - a_n)\|x_n - x^*\| + a_n\|S_\alpha P_K[y_n - \rho T y_n] - S_\alpha P_K[x^* - \rho T x^*]\| \\ &\leq (1 - a_n)\|x_n - x^*\| + a_n\|y_n - x^* - \rho(T y_n - T x^*)\|. \end{aligned} \tag{3.4}$$

From θ-property on T,

$$\|y_n - x^* - \rho(T y_n - T x^*)\| \leq \theta(\rho)\|y_n - x^*\|, \tag{3.5}$$

where $\theta(\rho) < 1$.

Combining (3.4) and (3.5), we have

$$\|x_{n+1} - x^*\| \leq (1 - a_n)\|x_n - x^*\| + a_n\theta(\rho)\|y_n - x^*\| \tag{3.6}$$

From (1.3), (3.2) and α-nonexpansivity of the operators S_α and P_K, we have

$$\begin{aligned}\|y_n - x^*\| &= \|(1 - b_n)x_n + b_n S_\alpha P_K[z_n - \rho T z_n] - (1 - b_n)x^* + b_n S_\alpha P_K[x^* - \rho T x^*]\| \\ &\leq (1 - b_n)\|x_n - x^*\| + b_n\|S_\alpha P_K[z_n - \rho T z_n] - S_\alpha P_K[x^* - \rho T x^*]\| \\ &\leq (1 - b_n)\|x_n - x^*\| + b_n\|[z_n - \rho T z_n] - [x^* - \rho T x^*]\|. \end{aligned} \tag{3.7}$$

Now from θ-property on T, it yields that

$$\|z_n - x^* - \rho[T z_n - T x^*]\| \leq \theta(\rho)\|z_n - x^*\|. \tag{3.8}$$

Analogously, (1.2) and (3.1) conclude that

$$\|z_n - x^*\| \leq (1-c_n)\|x_n - x^*\| + c_n\theta(\rho)\|x_n - x^*\| = (1 - c_n(1-\theta(\rho)))\|x_n - x^*\| \leq \|x_n - x^*\|. \tag{3.9}$$

Then from (3.7) and (3.8), we have

$$\begin{aligned}\|y_n - x^*\| &\leq (1 - b_n)\|x_n - x^*\| + b_n\theta(\rho)\|z_n - x^*\| \\ &\leq (1 - b_n)\|x_n - x^*\| + b_n\theta(\rho)\|x_n - x^*\| \\ &\leq \|x_n - x^*\|. \end{aligned} \tag{3.10}$$

From (3.6), (3.9) and (3.10), one can obtain

$$\begin{aligned}\|x_{n+1}-x^*\| &\leq (1-a_n)\|x_n-x^*\|+a_n\theta(\rho)\|y_n-x^*\| \\ &\leq (1-a_n)\|x_n-x^*\|+a_n\theta(\rho)\|x_n-x^*\| = [1-a_n(1-\theta(\rho))]\|x_n-x^*\|,\end{aligned}$$

and hence by Lemma 2.1, $lim_{n\to\infty}\|x_n-x^*\|=0$. □

Example 3.7. *Let $H := l^2$, $K := B_{l^2}$, where B_{l^2} denote the closed unit ball of l^2. Define the mapping $T : H \to K$ by:*

$$T(x) = (x-\frac{1}{2})(\frac{1}{1+\rho}),$$

for all $x \in B_{l^2}$, $S : K \to K$ by $S(x) = 1-x$ for all $x \in K$ and the projection operator $P_K : H \to K$ define as follows:

$$P_K(x) = \frac{x}{\max\{1, \|x\|_{l^2}\}},$$

for all $x \in K$. Clearly S is an $(\frac{1}{\sqrt{2}}, 1-\frac{1}{\sqrt{2}})$-nonexpansive mapping and $\frac{1}{2} \in F(S)\cap VI(K,T)$. If $a_n = \frac{1}{n+1}$, $b_n = \frac{1}{2^n}$, $c_n = \frac{1}{3^n}$ and $x_0 = 0$, then the sequence x_n, which is derived from Algorithm 1.3, converges strongly to $\frac{1}{2} \in F(S)\cap VI(K,T)$.

Solving the above example in Mathematica software as follows:

```
Clear[" `*"]
  ρ = 1; X[0] = 0; n = 0;
  a[n_] = 1/(n + 1);
  b[n_] = 1/2^n;
  c[n_] = 1/3^n;
S[x_] = 1/Sqrt[2] (1 - x) + (1 - 1/Sqrt[2]) x
        (1 - x)/Sqrt[2] + (1 - 1/Sqrt[2]) x
T[x_] =1/(ρ + 1)(x - 1/2);

Subscript[P, k][x_] = x;

Z[n_] :=(1 - c[n])X[n] + c[n]S[Subscript[P, k][X[n] - ρT[X[n]]]]// N

Y[n_] :=(1 - b[n])X[n] + b[n]S[Subscript[P, k][Z[n] - ρT[Z[n]]]]// N

X[(n + 1)_] :=(1 - a[n])X[n] + a[n]S[Subscript[P, k][Y[n] - ρT[Y[n]]]]//Simplify // N

For[i = 0, i <= 8, i + 1 + +, X[i + 1] = (1 - a[i])X[i] + a[i]S[Subscript[P, k][Y[i] - ρT[Y[i]]]]

        Table[X[i+n], {i, 0, 10}]

{0, 0.504442, 0.499961, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5}
```

If $c_n \equiv 0$ then the following result is a special case of Theorem 3.6.

Corollary 3.8. *Let K be a closed convex subset of a real Hilbert space H. Let $T : K \to H$ satisfies in θ-property and S be an α-nonexpansive mapping of K into K such that $\alpha_1 > \frac{1}{\sqrt[n-1]{2}}$ and $F(S)\cap VI(K,T) \neq \emptyset$. Let $\{x_n\}$ be a sequence*

defined by Algorithm 1.4, for any initial point $x_0 \in K$, with condition $0 < \theta(\rho) < 1$, $a_n, b_n \in [0,1]$ and $\sum_{n=0}^{\infty} a_n = \infty$, then $\{x_n\}$ obtained from Algorithm 1.4 converges strongly to $x^* \in F(S) \cap VI(K,T)$.

Now, we prove the strong convergence theorem of Algorithem 1.6.

Theorem 3.9. *Let K be a closed convex subset of a real Hilbert space H. Let $T : K \to H$ satisfies in θ-property and S be an α-nonexpansive mapping of K into K such that $\alpha_1 > \frac{1}{\sqrt[n-1]{2}}$ and $F(S) \cap VI(K,T) \neq \emptyset$. Let $\{x_n\}$ be the approximate solution obtained from Algorithm 1.6, for any initial point $x_0 \in K$, with condition $0 < \theta(\rho) < 1$. Then the sequence $\{x_n\}$ converges strongly to $x^* \in F(S) \cap VI(K,T)$.*

Proof. Consider

$$\|x_n - x^* - \rho[Tx_n - Tx^*]\| \leq \theta(\rho)\|x_n - x^*\|. \tag{3.11}$$

From (1.5), (3.3),(3.11) and the α-nonexpansiv property of the oprators S_α and P_K, we have

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \|(1-a_n)x_n + a_n S_\alpha P_K[x_n - \rho T x_n] - (1-a_n)x^* - a_n S_\alpha P_K[x^* - \rho T x^*]\| \\
&\leq (1-a_n)\|x_n - x^*\| + a_n\|S_\alpha P_K[x_n - \rho T x_n] - S_\alpha P_K[x^* - \rho T x^*]\| \\
&\leq (1-a_n)\|x_n - x^*\| + a_n\|x_n - x^* - \rho(Tx_n - Tx^*)\| \\
&\leq (1-a_n)\|x_n - x^*\| + a_n\theta(\rho)\|x_n - x^*\| \\
&= [1 - a_n(1-\theta(\rho))]\|x_n - x^*\|.
\end{aligned}$$

Therefore by Lemma 2.1, $lim_{n\to\infty}\|x_n - x^*\| = 0$. □

4. Another New Algorithm

The following equality is established in a Hilbert space

$$\|(x-y) - \rho(Tx - Ty)\|^2 = \|x-y\|^2 + \rho^2\|Tx - Ty\|^2 - 2\rho\langle x - y, Tx - Ty\rangle.$$

If the mapping $T : K \to H$ be an α-inverse strongly monotone, then

$$\|x-y\|^2 + \rho^2\|Tx - Ty\|^2 - \|(x-y) - \rho(Tx - Ty)\|^2 \geq 2\rho\alpha\|Tx - Ty\|^2.$$

Hence

$$(\rho^2 - 2\rho\alpha)\|Tx - Ty\|^2 + \|x-y\|^2 \geq \|(x-y) - \rho(Tx - Ty)\|^2.$$

That is, if $\alpha < \frac{\rho}{2}$, then T is not satiesfies in θ-property. The following theorem shows that, if T satiesfies in θ-property, then T is an α-inverse strongly monotone mapping. Therefore it is a concolusion of Theorem 1.7.

Theorem 4.1. *Let K be a closed convex subset of a real Hilbert space H. Let $T : K \to H$ satisfies in θ-property with condition $0 < \theta(\rho) < 1$ and S be an α-nonexpansive mapping of K into K such that $\alpha_1 > \frac{1}{\sqrt[n-1]{2}}$ and $F(S) \cap VI(K,T) \neq \emptyset$. For fixed $u \in K$ and given $x_0 \in K$ arbitrary, let the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ are generated by*

$$\begin{cases} y_n = \sigma_n x_n + (1-\sigma_n)S_\alpha P_K[x_n - \rho_n T x_n] \\ z_n = \mu_n x_n + (1-\mu_n)S_\alpha P_K[y_n - \rho_n T y_n] \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n S_\alpha P_K[z_n - \rho_n T z_n], n \geqslant 0, \end{cases} \tag{4.1}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\mu_n\}$ and $\{\sigma_n\}$ are five sequences in $[0,1]$ and $\theta(\rho_n)$ is a sequence in $(0,1)$. If for these sequences $\rho_n \in (0,2)$ and

(i) $\alpha_n + \beta_n + \gamma_n = 1$,
(ii) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$
(iii) $0 < \liminf_{n\to\infty} \beta_n \leqslant \limsup_{n\to\infty} \beta_n < 1$ *and* $\lim_{n\to\infty} \mu_n = 1$
(iv) $lim_{n\to\infty}(\rho_{n+1} - \rho_n) = 0$.

Then $\{x_n\}$ *defined by* (4.1) *converges strongly to* $x^* \in P_{F(S)\cap VI(K,T)}u$.

Proof. For all $x, y \in K$ and $\rho_n \in (0, 2)$, we have

$$\begin{aligned}\|(I - \rho_n T)x - (I - \rho_n T)y\| &= \|(x - y) - \rho_n(Tx - Ty)\| \\ &\leq \theta(\rho_n)\|x - y\| \leqslant \|x - y\|, \end{aligned} \tag{4.2}$$

that is $(I - \rho_n T)$ is nonexpansive. On the other hand

$$\|(I - \rho_n T)x - (I - \rho_n T)y\|^2 = \|x - y\|^2 + \rho_n^2\|Tx - Ty\|^2 - 2\rho_n\langle x - y, Tx - Ty\rangle,$$

therefore

$$\langle x - y, Tx - Ty\rangle \geqslant \frac{\rho_n}{2}\|(Tx - Ty)\|^2.$$

That is, T is a $\frac{\rho_n}{2}$-inverse strongly momotone mapping. Hence by Theorem 1.7 we can obtain the desired result. □

If $S = I$ then the following result is a special case of Theorem 4.1.

Corollary 4.2. *Let K be a closed convex subset of a real Hilbert space H. Let $T : K \to H$ satisfies in θ-property with condition $0 < \theta(\rho) < 1$ and $VI(K, T) \neq \emptyset$. For fixed $u \in K$ and given $x_0 \in K$ arbitrary, let the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ are generated by*

$$\begin{cases} y_n = \sigma_n x_n + (1 - \sigma_n)P_K[x_n - \rho_n T x_n] \\ z_n = \mu_n x_n + (1 - \mu_n)P_K[y_n - \rho_n T y_n] \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n P_K[z_n - \rho_n T z_n], n \geqslant 0, \end{cases} \tag{4.3}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\mu_n\}$ and $\{\sigma_n\}$ are five sequences in $[0, 1]$ and $\theta(\rho_n)$ is a sequence in $(0, 1)$. If for these sequences $\rho_n \in (0, 2)$ and

(i) $\alpha_n + \beta_n + \gamma_n = 1$,
(ii) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$
(iii) $0 < \liminf_{n\to\infty} \beta_n \leqslant \limsup_{n\to\infty} \beta_n < 1$ *and* $lim_{n\to\infty}\mu_n = 1$
(iv) $lim_{n\to\infty}(\rho_{n+1} - \rho_n) = 0$.

Then $\{x_n\}$ *defined by* (4.3) *converges strongly to* $x^* \in P_{VI(K,T)}u$.

Theorem 4.3. *Let K be a closed convex subset of a real Hilbert space H. Let $T : K \to H$ satisfies in θ-property with condition $0 < \theta(\rho) < 1$ and S be an α-nonexpansive mapping of K into K such that $\alpha_1 > \frac{1}{\sqrt[n-1]{2}}$ and $F(S) \cap T^{-1}0 \neq \emptyset$. For fixed $u \in K$ and given $x_0 \in K$ arbitrary, let the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ are generated by*

$$\begin{cases} y_n = \sigma_n x_n + (1 - \sigma_n)S_\alpha P_K[x_n - \rho_n T x_n] \\ z_n = \mu_n x_n + (1 - \mu_n)S_\alpha P_K[y_n - \rho_n T y_n] \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n S_\alpha P_K[z_n - \rho_n T z_n], n \geqslant 0, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\mu_n\}$ and $\{\sigma_n\}$ are five sequences in $[0, 1]$ and $\theta(\rho_n)$ is a sequence in $(0, 1)$. If for these sequences $\rho_n \in (0, 2)$ and

(i) $\alpha_n + \beta_n + \gamma_n = 1$,
(ii) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$
(iii) $0 < \liminf_{n\to\infty} \beta_n \leqslant \limsup_{n\to\infty} \beta_n < 1$ *and* $lim_{n\to\infty}\mu_n = 1$

(iv) $lim_{n\to\infty}(\rho_{n+1} - \rho_n) = 0$.

Then $\{x_n\}$ *defined by* (4.1) *converges strongly to* $x^* \in P_{F(S)\cap T^{-1}0}u$.

Proof. Since $T^{-1}0 = VI(K,T)$, putting $P_K = I$, by Theorem 4.1, we can obtain the conclusion. This completes the proof. □

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