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DOUBT BIPOLAR FUZZY SUBALGEBRAS AND IDEALS IN BCK/BCI-ALGEBRAS

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Abstract. The notions of doubt bipolar fuzzy subalgebras and (closed) doubt bipolar fuzzy ideals are introduced, and related properties are investigated. Characterizations of a doubt bipolar fuzzy subalgebra and a doubt bipolar fuzzy ideal are given, and relations between a doubt bipolar fuzzy subalgebra and a doubt bipolar fuzzy ideal are discussed. The concepts of homomorphic preimages and doubt images of doubt bipolar fuzzy ideals in BCK/BCI- algebras are investigated. Conditions for a doubt bipolar fuzzy ideal to be a closed doubt bipolar fuzzy ideal are provided.

1. Introduction

The study of BCK/BCI-algebras was introduced by Imai and Iséki [7, 6] in 1966 as a generalization of the concept of set-theoretic difference and propositional calculus. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. BCK/BCI-algebras are two important classes of logical algebras proposed by Iséki [8], Iséki et al. [9], Meng [22] and Meng et al. [23]. Since then, a great deal of literature has been produced on the theory of BCK/BCI-algebras.

The concept of fuzzy sets and various operations on it were first introduced by Zadeh [28]. This provided a useful mathematical tool for describing the systems that are too complex or ill defined. Since then, fuzzy sets have been applied to many branches of mathematics. The study of fuzzy sets and their applications to mathematical contexts have reached to what is now commonly called fuzzy mathematics. In 1994, Zhang [32] first initiated the concept of bipolar fuzzy sets as a generalization of traditional fuzzy sets [28]. Also, Lee [19, 20] proposed an extension of fuzzy sets, namely bipolar fuzzy sets. Fuzzy sets give a degree of membership of an element in a given set, bipolar fuzzy sets gives both a positive membership degree and a negative membership degree. The positive membership degree belongs to the interval [0, 1] and the negative membership degree belongs to the interval [-1, 0]. In the case of bipolar fuzzy sets, the membership degrees range is increased from the interval [0, 1] to the interval [-1, 1]. As the basis for the study of bipolar fuzzy set theory many operations and relations over bipolar fuzzy sets were introduced

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[29, 34, 33]. Recently, the theory of bipolar fuzzy sets becomes a vigorous area of research in different domains such as group theory, semigroup theory, ring theory, semiring theory, graph theory, engineering, physics, statics, medical science, social science, artificial intelligent, computer networks, expert systems, decision making and so on.

On the other hand, the concept of fuzzy sets was applied to the theory of groupoids and groups by Rosenfeld [25], where he introduced the fuzzy subgroup of a group. Later, Biswas [3] introduced the notion of anti fuzzy subgroups of groups. Since then, the literature of various algebraic structures has been fuzzified. Xi [27] applied the concept of fuzzy set to BCK-algebras and investigated related properties. Further, fuzzy aspects of several notions in BCK/BCI-algebras were studied by Jun (together with Kim, Meng, Song and Xin) [10, 12, 13, 15, 24]. However, Huang [4] fuzzified BCI-algebras in little different ways. Consequently, Jun [11] established the definition of a doubt fuzzy subalgebra and a doubt fuzzy ideal in BCK/BCI-algebras to avoid the confusion created in [4] Huang's definition of fuzzy BCI-algebras and gave some results about it. After that, Zhan and Tan [31, 30] introduced the notion of doubt fuzzy p-ideals in BCI-algebra and the notion of doubt fuzzy H-ideals in BCK-algebras.

Recently, based on the results of bipolar fuzzy sets, more and more researchers have devoted themselves to studying some bipolar fuzzy algebraic structures. Lee [18] applied the bipolar fuzzy set theory to BCK/BCI-algebras. He introduced the notions of bipolar fuzzy subalgebras and bipolar fuzzy ideals in BCK/BCI-algebras, and then the notion of bipolar fuzzy set theory was applied to BCK/BCI-algebras, BCH-algebras, KU-algebras, K-algebras, Lie algebras and hyper BCK-algebras etc (see [1, 2, 14, 16, 17, 21, 26]). To the best of our knowledge no works are available on doubt bipolar fuzzy subalgebras and ideals in BCK/BCI-algebras. For this reason we are motivated to develop these theories for BCK/BCI-algebras.

In this paper, we discuss a bipolar fuzzy set with an application to BCK/BCI-algebras. We introduce the notions of doubt bipolar fuzzy subalgebras and (closed) doubt bipolar fuzzy ideals in BCK/BCI-algebras, and investigate related properties. Then we present some characterizations of a doubt bipolar fuzzy subalgebra and a doubt bipolar fuzzy ideal by means of doubt positive t-level cut set and doubt negative s-level cut set. Moreover, relations between a doubt bipolar fuzzy subalgebra and a doubt bipolar fuzzy ideal are discussed. We investigate the homomorphic preimages and doubt images of doubt bipolar fuzzy ideals in BCK/BCI-algebras under some conditions. Also, we provide conditions for a doubt bipolar fuzzy ideal to be a closed doubt bipolar fuzzy ideal.

2. Preliminaries

In this section, we include some basic definitions and preliminary facts about BCK/BCI-algebras which are essential for our results. Throughout this paper, X always denotes a BCK/BCI-algebra without any specifications. We give here only those concepts of BCK/BCI-algebras which are important for our treatment, and for details about the theory of these algebras we may refer to [5, 23, 7, 6, 8, 9].

By a BCI-algebra [6] we mean an algebra $(X; *, 0)$ of type (2, 0) satisfying the following axioms for all $x, y, z \in X$:

(I) $((x * y) * (x * z)) * (z * y) = 0$,

(II) $(x * (x * y)) * y = 0$,

(III) $x * x = 0$,
(IV) $x * y = 0$ and $y * x = 0$ imply $x = y$.

If a BCI-algebra X satisfies $0 * x = 0$, then X is called a BCK-algebra. A partial ordering $\leq$ on a BCK/BCI-algebra X can be defined by $x \leq y$ if and only if $x * y = 0$. Any BCK/BCI-algebra X satisfies the following axioms for all $x, y, z \in X$:

(I1) $x * 0 = x$,
(I2) $(x * y) * z = (x * z) * y$,
(I3) $x * y \leq x$,
(I4) $(x * y) * z \leq (x * z) * (y * z)$,
(I5) $x \leq y \Rightarrow x * z \leq y * z,\ z * y \leq z * x$.

A non-empty subset I of a BCK/BCI-algebra X is called a subalgebra of X if $x * y \in I$ for any $x, y \in I$.

A non-empty subset S of a BCK/BCI-algebra X is called an ideal of X if

(1) $0 \in S$
(2) $x * y \in S$ and $y \in S$, then $x \in S$ for all $x, y \in X$.

Definition 2.1. [28] *A fuzzy set in a BCK/BCI-algebra X is a function $\mu : X \to [0, 1]$.*

Definition 2.2. [27] *A fuzzy set μ in a BCK/BCI-algebra X is called a fuzzy subalgebra of X if $\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in X$.*

Definition 2.3. [27] *A fuzzy set μ in a BCK/BCI-algebra X is called a fuzzy ideal of X if $\mu(0) \geq \mu(x)$ and $\mu(x) \geq \min\{\mu(x * y), \mu(y)\}$ for all $x, y \in X$.*

In order to resolve the contradiction popped up in Huang's [4] definition of a fuzzy BCI-algebra, Jun [11] introduced the definition of a doubt fuzzy subalgebra and a doubt fuzzy ideal in BCK/BCI-algebras, which are as follows:

Definition 2.4. [11] *A fuzzy set $A = \{(x, \mu_A(x)) \mid x \in X\}$ in X is called a doubt fuzzy subalgebra of X if $\mu_A(x * y) \leq \max\{\mu_A(x), \mu_A(y)\}$ for all $x, y \in X$.*

Definition 2.5. [11] *A fuzzy set $A = \{(x, \mu_A(x)) \mid x \in X\}$ in X is called a doubt fuzzy ideal of X if $\mu_A(0) \leq \mu_A(x)$ and $\mu_A(x) \leq \max\{\mu_A(x * y), \mu_A(y)\}$ for all $x, y \in X$.*

The proposed work is done on bipolar fuzzy sets. The formal definition of a bipolar fuzzy set is given below:

Definition 2.6. [19, 33] *Let X be a non-empty set. A bipolar fuzzy set A in X is an object having the form*

$$A = \{(x, \mu_A^P(x), \mu_A^N(x)) \mid x \in X\}$$

where $\mu_A^P : X \longrightarrow [0, 1]$ and $\mu_A^N : X \longrightarrow [-1, 0]$ are mappings.

We use the positive membership degree $\mu_A^P(x)$ to denote the satisfaction degree of an element x to the property corresponding to a bipolar fuzzy set A, and the negative membership degree $\mu_A^N(x)$ to denote the satisfaction degree of an element x to some implicit counter-property corresponding to a bipolar fuzzy set A. If $\mu_A^P(x) \neq 0$ and $\mu_A^N(x) = 0$, it is the situation that x is regarded as having only positive satisfaction for A. If $\mu_A^P(x) = 0$ and $\mu_A^N(x) \neq 0$, it is the situation that x does not satisfy the property of A but somewhat satisfies the counter property of

A. It is possible for an element x to be such that $\mu_A^P(x) \neq 0$ and $\mu_A^N(x) \neq 0$ when the membership function of the property overlaps that of its counter property over some portion of X.

For the sake of simplicity, we shall use the symbol $A = (\mu_A^P, \mu_A^N)$ for the bipolar fuzzy set $A = \{(x, \mu_A^P(x), \mu_A^N(x)) | x \in X\}$.

Definition 2.7. [18] *A bipolar fuzzy set $A = (\mu_A^P, \mu_A^N)$ in X is called a bipolar fuzzy subalgebra of X if it satisfies the following conditions for all $x, y \in X$:*

(1) $\mu_A^P(x * y) \geq \min\{\mu_A^P(x), \mu_A^P(y)\}$,
(2) $\mu_A^N(x * y) \leq \max\{\mu_A^N(x), \mu_A^N(y)\}$.

Definition 2.8. [18] *A bipolar fuzzy set $A = (\mu_A^P, \mu_A^N)$ in X is called a bipolar fuzzy ideal of X if it satisfies the following conditions for all $x, y \in X$:*

(1) $\mu_A^P(0) \geq \mu_A^P(x)$ *and* $\mu_A^N(0) \leq \mu_A^N(x)$,
(2) $\mu_A^P(x) \geq \min\{\mu_A^P(x * y), \mu_A^P(y)\}$,
(3) $\mu_A^N(x) \leq \max\{\mu_A^N(x, y), \mu_A^N(y)\}$.

3. Doubt bipolar fuzzy subalgebras

In this section, we introduce doubt bipolar fuzzy subalgebras in BCK/BCI-algebras and investigate some of their properties.

Definition 3.1. *Let $A = (\mu_A^P, \mu_A^N)$ be a bipolar fuzzy subset of X, then A is called a doubt bipolar fuzzy subalgebra of X if it satisfies the following conditions for all $x, y \in X$:*

(1) $\mu_A^P(x * y) \leq \max\{\mu_A^P(x), \mu_A^P(y)\}$,
(2) $\mu_A^N(x * y) \geq \min\{\mu_A^N(x), \mu_A^N(y)\}$.

Example 3.2. *Consider a $BCK-$algebra $X = \{0, a, b, c\}$ with the following Cayley table:*

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	a	0	b
c	c	c	c	0

Let $A = (\mu_A^P, \mu_A^N)$ be a bipolar fuzzy set in X defined by

$$\mu_A^P(x) = \begin{cases} 0.5, & \text{if } x = 0,\ a,\ c \\ 0.6, & \text{if } x = b, \end{cases}$$

and

$$\mu_A^N(x) = \begin{cases} -0.2, & \text{if } x = 0 \\ -0.3, & \text{if } x = a,\ b \\ -0.8, & \text{if } x = c. \end{cases}$$

By routine calculation, we know that $A = (\mu_A^P, \mu_A^N)$ is a doubt bipolar fuzzy subalgebra of X.

For a bipolar fuzzy set $A = (\mu_A^P, \mu_A^N)$ and $(t, s) \in [1, 0] \times [-1, 0]$, we defined

$$A_t^P = \{x \in X : \mu_A^P(x) \leq t\},$$

and

$$A_s^N = \{x \in X : \mu_A^N(x) \geq s\}$$

which are called the doubt positive t-level cut set of $A = (\mu_A^P, \mu_A^N)$ and the doubt negative s-level cut set of $A = (\mu_A^P, \mu_A^N)$, respectively. The set

$$A_{(t,s)} = \{x \in X \mid \mu_A^P(x) \leq t, \mu_A^N(x) \geq s\}$$

is called the doubt (t, s)-level cut set of $A = (\mu_A^P, \mu_A^N)$. Note that

$$A_{(t,s)} = A_t^P \cap A_s^N.$$

For every $\gamma \in [0, 1]$, the set $A_\gamma^P \cap A_{-\gamma}^N$ is called the doubt γ-level cut set of $A = (\mu_A^P, \mu_A^N)$.

Theorem 3.3. *Let $A = (\mu_A^P, \mu_A^N)$ be a bipolar fuzzy set over X and let $(t, s) \in [0, 1] \times [-1, 0]$. If $A = (\mu_A^P, \mu_A^N)$ is a doubt bipolar fuzzy subalgebra of X, then the nonempty doubt (t, s)-level cut set of A is a subalgebra of X.*

Proof. Let $(t, s) \in [0, 1] \times [-1, 0]$ and let $A_{(t,s)} \neq \emptyset$. If $x, y \in A_{(t,s)}$, then $\mu_A^P(x) \leq t$, $\mu_A^N(x) \geq s$, $\mu_A^P(y) \leq t$ and $\mu_A^N(y) \geq s$. It follows from Definition 3.1 that

$$\mu_A^P(x * y) \leq \max\{\mu_A^P(x), \mu_A^P(y)\} \leq t,$$

and

$$\mu_A^N(x * y) \geq \min\{\mu_A^N(x), \mu_A^N(y)\} \geq s.$$

Hence, $x * y \in A_{(t,s)}$, therefore $A_{(t,s)}$ is a subalgebra of X. □

Theorem 3.4. *Let $A = (\mu_A^P, \mu_A^N)$ be a bipolar fuzzy set over X and assume that $\emptyset \neq A_t^P$ and $\emptyset \neq A_s^N$ are subalgebras of X for all $(t, s) \in [0, 1] \times [-1, 0]$. Then $A = (\mu_A^P, \mu_A^N)$ is a doubt bipolar fuzzy subalgebra of X.*

Proof. Assume that $\emptyset \neq A_t^P$ and $\emptyset \neq A_s^N$ are subalgebras of X for all $(t, s) \in [0, 1] \times [-1, 0]$. If there exists $x', y', a', b' \in X$ such that

$$\mu_A^P(x' * y') > \max\{\mu_A^P(x'), \mu_A^P(y')\}$$

and

$$\mu_A^N(a' * b') < \min\{\mu_A^N(a'), \mu_A^N(b')\},$$

then by taking

$$\begin{aligned} t_o &= \frac{1}{2}[\mu_A^P(x' * y') + \max\{\mu_A^P(x'), \mu_A^P(y')\}], \\ s_o &= \frac{1}{2}[\mu_A^N(a' * b') + \min\{\mu_A^N(a'), \mu_A^N(b')\}], \end{aligned}$$

we have

$$\begin{aligned} \mu_A^P(x' * y') &> t_o > \max\{\mu_A^P(x'), \mu_A^P(y')\}, \\ \mu_A^N(a' * b') &< s_o < \min\{\mu_A^N(a'), \mu_A^N(b')\}. \end{aligned}$$

Hence, $x' * y' \notin A_{t_o}^P, x' \in A_{t_o}^P, y' \in A_{t_o}^P, a' * b' \notin A_{s_o}^N, a' \in A_{s_o}^N$ and $b' \in A_{s_o}^N$. This is a contradiction, Therefore $A = (\mu_A^P, \mu_A^N)$ is a doubt bipolar fuzzy subalgebra of X. □

Proposition 3.5. *If $A = (\mu_A^P, \mu_A^N)$ is a doubt bipolar fuzzy subalgebra of X, then $\mu_A^P(0) \leqslant \mu_A^P(x)$ and $\mu_A^N(0) \geqslant \mu_A^N(x)$ for all $x \in X$.*

Proof. For any $x \in X$, we have $\mu_A^P(0) = \mu_A^P(x * x) \leq \max\{\mu_A^P(x), \mu_A^P(x)\} = \mu_A^P(x)$ and $\mu_A^N(0) = \mu_A^N(x * x) \geq \min\{\mu_A^N(x), \mu_A^N(x)\} = \mu_A^N(x)$. This completes the proof. □

Proposition 3.6. *If every doubt bipolar fuzzy subalgebra $A = (\mu_A^P, \mu_A^N)$ of X satisfies*

$$\mu_A^P(x * y) \leq \mu_A^P((y)$$

and

$$\mu_A^N(x * y) \geq \mu_A^N((y)$$

for all $x, y \in X$, then $A = (\mu_A^P, \mu_A^N)$ is constant.

Proof. Note that in a BCK/BCI-algebra $X, x*0 = x$ for all $x \in X$, since $\mu_A^P(x*y) \leq \mu_A^P((y)$ and $\mu_A^N(x * y) \geq \mu_A^N((y)$, we have

$$\mu_A^P(x) = \mu_A^P(x * 0) \leq \mu_A^P(0),$$

and

$$\mu_A^N(x) = \mu_A^N(x * 0) \geq \mu_A^N(0).$$

It follows from Proposition 3.5 that $\mu_A^P(x) = \mu_A^P(0)$ and $\mu_A^N(x) = \mu_A^N(0)$ for all $x, y \in X$. Therefore, $A = (\mu_A^P, \mu_B^N)$ is constant. □

For elements x and y of a BCK/BCI-algebra X, let us write $x * y^n$ for $(...((x * y) * y) * ...) * y$ and $x^n * y$ for $x * (... * ((x * (x * y))...)$ where y and x occur n times respectively.

Proposition 3.7. *Let $A = (\mu_A^P, \mu_A^N)$ be a doubt bipolar fuzzy subalgebra of X and let $n \in \mathbb{N}$. Then for any $x \in X$, we have*

(1) *$\mu_A^P(x^n * x) \leq \mu_A^P(x)$ and $\mu_A^N(x^n * x) \geq \mu_A^N(x)$, if n is odd.*
(2) *$\mu_A^P(x^n * x) = \mu_A^P(x)$ and $\mu_A^N(x^n * x) = \mu_A^N(x)$, if n is even.*

Proof. (1) If n is odd, then $n = 2k - 1$ for some positive integer k. Let $x \in X$, then $\mu_A^P(x * x) = \mu_A^P(0) \leq \mu_A^P(x)$ and $\mu_A^N(x * x) = \mu_A^N(0) \geq \mu_A^N(x)$. Now assume that $\mu_A^P(x^{2k-1} * x) \leq \mu_A^P(x)$ and $\mu_A^N(x^{2k-1} * x) \geq \mu_A^N(x)$ for some positive integer k. Then,

$$\begin{aligned}
\mu_A^P(x^{2(k+1)-1} * x) &= \mu_A^P(x^{2k+1} * x) \\
&= \mu_A^P(x^{2k-1} * (x * (x * x))) \\
&= \mu_A^P(x^{2k-1} * (x * 0)) \\
&= \mu_A^P(x^{2k-1} * x) \\
&\leq \mu_A^P(x)
\end{aligned}$$

and

$$\begin{aligned}
\mu_A^N(x^{2(k+1)-1} * x) &= \mu_A^N(x^{2k+1} * x) \\
&= \mu_A^N(x^{2k-1} * (x * (x * x))) \\
&= \mu_A^N(x^{2k-1} * (x * 0)) \\
&= \mu_A^N(x^{2k-1} * x) \\
&\geq \mu_A^N(x).
\end{aligned}$$

This proves (1). Similarly, we can prove (2). □

4. Doubt bipolar fuzzy ideals

In this section, we introduce the notions of doubt bipolar fuzzy ideals and closed doubt bipolar fuzzy ideals in BCK/BCI-algebras. Several fundamental properties and theorems related to these concepts are also studied and investigated.

Definition 4.1. *A bipolar fuzzy set $A = (\mu_A^P, \mu_A^N)$ in X is called a doubt bipolar fuzzy ideal if it satisfies the following conditions for all $x, y \in X$:*

(1) $\mu_A^P(0) \leq \mu_A^P(x)$ *and* $\mu_A^N(0) \geq \mu_A^N(x)$,
(2) $\mu_A^P(x) \leq \max\{\mu_A^P(x * y), \mu_A^P(y)\}$,
(3) $\mu_A^N(x) \geq \min\{\mu_A^N(x * y), \mu_A^N(y)\}$.

Example 4.2. *Consider a $BCK-$algebra $X = \{0, a, b, c\}$ which is given in Example 3.2. Define a bipolar fuzzy set $A = (\mu_A^P, \mu_A^N)$ in X as follows:*

$$\mu_A^P(x) = \begin{cases} 0, & \text{if } x = 0 \\ 0.5, & \text{if } x = a,\ b \\ 1, & \text{if } x = c, \end{cases}$$

and

$$\mu_A^N(x) = \begin{cases} -0.2, & \text{if } \ x = 0 \\ -0.3, & \text{if } \ x = a,\ b \\ -0.8, & \text{if } \ x = c. \end{cases}$$

By routine calculation, we know that $A = (\mu_A^P, \mu_A^N)$ is a doubt bipolar fuzzy ideal of X.

Proposition 4.3. *Let $A = (\mu_A^P, \mu_A^N)$ be a doubt bipolar fuzzy ideal of X. If $\leq$ is a partial ordering on X, then $\mu_A^P(x) \leq \mu_A^P(y)$ and $\mu_A^N(x) \geq \mu_A^N(y)$ for all $x, y \in X$ such that $x \leq y$.*

Proof. Let $A = (\mu_A^P, \mu_A^N)$ be a doubt bipolar fuzzy ideal of X. It is known that $\leq$ is a partial ordering on X defined by $x \leq y$ if and only if $x * y = 0$ for all $x, y \in X$. Then

$$\begin{aligned} \mu_A^P(x) &\leq \max\{\mu_A^P(x * y), \mu_A^N(y)\} \\ &= \max\{\mu_A^P(0), \mu_A^P(y)\} \\ &= \mu_A^P(y) \end{aligned}$$

and

$$\begin{aligned} \mu_A^N(x) &\geq \min\{\mu_A^N(x * y), \mu_A^N(y)\} \\ &= \min\{\mu_A^N(0), \mu_A^N(y)\} \\ &= \mu_A^N(y). \end{aligned}$$

This completes the proof. □

Proposition 4.4. *Let $A = (\mu_A^P, \mu_A^N)$ be a doubt bipolar fuzzy ideal of X. Then*

$$\mu_A^P(x * y) \leq \mu_A^P((x * y) * y) \Leftrightarrow \mu_A^P((x * z) * (y * z)) \leq \mu_A^P((x * y) * z)$$

and

$$\mu_A^N(x * y) \geq \mu_A^N((x * y) * y) \Leftrightarrow \mu_A^N((x * z) * (y * z)) \geq \mu_A^N((x * y) * z)$$

for all $x, y, z \in X$.

Proof. Note that

$$\begin{aligned}((x*(y*z))*z)*z &= ((x*z)*(y*z))*z \\ &\leq (x*y)*z\end{aligned}$$

for all $x, y, z \in X$. Assume that $\mu_A^P(x*y) \leq \mu_A^P((x*y)*y)$ and $\mu_A^N(x*y) \geq \mu_A^N((x*y)*y)$ for all $x, y, z \in X$. It follows from (I2)and Proposition 4.3 that

$$\begin{aligned}\mu_A^P((x*z)*(y*z)) &= \mu_A^P((x*(y*z))*z) \\ &\leq \mu_A^P(((x*(y*z))*z)*z) \\ &\leq \mu_A^P((x*y)*z)\end{aligned}$$

and

$$\begin{aligned}\mu_A^N((x*z)*(y*z)) &= \mu_A^N((x*(y*z))*z) \\ &\geq \mu_A^N(((x*(y*z))*z)*z) \\ &\geq \mu_A^N((x*y)*z),\end{aligned}$$

for all $x, y, z \in X$.

Conversely, suppose that

$$\mu_A^P((x*z)*(y*z)) \leq \mu_A^P((x*y)*z) \tag{4.1}$$

and

$$\mu_A^N((x*z)*(y*z)) \geq \mu_A^N((x*y)*z) \tag{4.2}$$

for all $x, y, z \in X$. If we substitute z for y in Equations (4.1) and (4.2). Then

$$\begin{aligned}\mu_A^P(x*z) &= \mu_A^P((x*z)*0) \\ &= \mu_A^P((x*z)*(z*z)) \\ &\leq \mu_A^P((x*z)*z)\end{aligned}$$

and

$$\begin{aligned}\mu_A^N(x*z) &= \mu_A^N((x*z)*0) \\ &= \mu_A^N((x*z)*(z*z)) \\ &\geq \mu_A^N((x*z)*z),\end{aligned}$$

for all $x, z \in X$ by using (III) and (I1.) □

Proposition 4.5. *Let $A = (\mu_A^P, \mu_A^N)$ be a doubt bipolar fuzzy ideal of X. Then*

$$\mu_A^P(x*y) \leq \max\{\mu_A^P(x*z), \mu_A^P(z*y)\}$$

and

$$\mu_A^N(x*y) \geq \min\{\mu_A^N(x*z), \mu_A^N(z*y)\}$$

for all $x, y, z \in X$.

Proof. Note that $((x*y)*(x*z)) \leq (z*y)$ for all $x, y, x \in X$. It follows from Proposition 4.3, that

$$\mu_A^P((x*y)*(x*z)) \leq \mu_A^P(z*y)$$

and

$$\mu_A^N((x*y)*(x*z)) \geq \mu_A^N(z*y).$$

Now, by Definition 4.1, we have

$$\begin{aligned}\mu_A^P(x*y) &\leq \max\{\mu_A^P((x*y)*(x*z)), \mu_A^P(x*z)\}\\ &\leq \max\{\mu_A^P(x*z), \mu_A^P(z*y)\}\end{aligned}$$

and

$$\begin{aligned}\mu_A^N(x*y) &\geq \min\{\mu_A^N((x*y)*(x*z)), \mu_A^N(x*z)\}\\ &\geq \min\{\mu_A^N(x*z), \mu_A^N(z*y)\}\end{aligned}$$

for all $x, y, z \in X$. This completes the proof. □

Proposition 4.6. *Let $A = (\mu_A^P, \mu_A^N)$ be a doubt bipolar fuzzy ideal of X. Then*

$$\mu_A^P(x*(x*y)) \leq \mu_A^P(y)$$

and

$$\mu_A^N(x*(x*y)) \geq \mu_A^N(y)$$

for all $x, y \in X$.

Proof. Let $A = (\mu_A^P, \mu_A^N)$ be a doubt bipolar fuzzy ideal of X. Then for all $x, y \in X$, we have

$$\begin{aligned}\mu_A^P(x*(x*y)) &\leq \max\{\mu_A^P((x*(x*y))*y), \mu_A^P(y)\}\\ &= \max\{\mu_A^P((x*y)*(x*y)), \mu_A^P(y)\}\\ &= \max\{\mu_A^P(0), \mu_A^P(y)\}\\ &= \mu_A^P(y)\end{aligned}$$

and

$$\begin{aligned}\mu_A^N(x*(x*y)) &\geq \min\{\mu_A^N((x*(x*y))*y), \mu_A^N(y)\}\\ &= \min\{\mu_A^N((x*y)*(x*y)), \mu_A^N(y)\}\\ &= \min\{\mu_A^N(0), \mu_A^N(y)\}\\ &= \mu_A^N(y).\end{aligned}$$

This completes the proof. □

Theorem 4.7. *Let $A = (\mu_A^P, \mu_A^N)$ be a bipolar fuzzy set over X and let $(t, s) \in [0, 1] \times [-1, 0]$. If $A = (\mu_A^P, \mu_A^N)$ is a doubt bipolar fuzzy ideal of X, then the nonempty doubt (t, s)-level cut set of A is an ideal of X.*

Proof. Assume that $A_{(t,s)} \neq \emptyset$ for $(t, s) \in [0, 1] \times [-1, 0]$. Clearly, $0 \in A_{(t,s)}$. Let $x * y \in A_{(t,s)}$ and $y \in A_{(t,s)}$. Then $\mu_A^P(x*y) \leq t$, $\mu_A^N(x*y) \geq s$, $\mu_A^P(y) \leq t$ and $\mu_A^N(y) \geq s$. It follows from Definition 4.1 that

$$\mu_A^P(x) \leq \max\{\mu_A^P(x*y), \mu_A^P(y)\} \leq t$$

and

$$\mu_A^N(x) \geq \min\{\mu_A^N(x*y), \mu_A^N(y)\} \geq s,$$

so, $x \in A_{(t,s)}$. Therefore $A_{(t,s)}$ is an ideal of X. □

Theorem 4.8. *Let $A = (\mu_A^P, \mu_A^N)$ be a bipolar fuzzy set over X and assume that $\emptyset \neq A_t^P$ and $\emptyset \neq A_s^N$ are ideals of X for all $(t, s) \in [0, 1] \times [-1, 0]$. Then $A = (\mu_A^P, \mu_A^N)$ is a doubt bipolar fuzzy ideal of X.*

Proof. Assume that $\emptyset \neq A_t^P$ and $\emptyset \neq A_s^N$ are ideals of X for all $(t,s) \in [0,1]\times[-1,0]$. For any $x \in X$, let $\mu_A^P(x) = t$ and $\mu_A^N(x) = s$. Then $x \in A_t^P \cap A_s^N$, and so A_t^P and A_s^N are nonempty. Since A_t^P and A_s^N are ideals of X, so $0 \in A_t^P \cap A_s^N$. Hence, $\mu_A^P(0) \leq t = \mu_A^P(x)$ and $\mu_A^N(0) \geq s = \mu_A^N(x)$ for all $x \in X$. If there exists $x', y', a', b' \in X$ such that

$$\mu_A^P(x') > \max\{\mu_A^P(x' * y'), \mu_A^P(y')\}$$

and

$$\mu_A^N(a') < \min\{\mu_A^N(a' * b'), \mu_A^N(b')\},$$

then by taking

$$\begin{aligned} t_1 &= \frac{1}{2}[\mu_A^P(x') + \max\{\mu_A^P(x' * y'), \mu_A^P(y')\}], \\ s_1 &= \frac{1}{2}[\mu_A^N(a') + \min\{\mu_A^N(a' * b'), \mu_A^N(b')\}], \end{aligned}$$

we have

$$\begin{aligned} \mu_A^P(x') &> t_1 > \max\{\mu_A^P(x' * y'), \mu_A^P(y')\}, \\ \mu_A^N(a') &< s_1 < \min\{\mu_A^N(a' * b'), \mu_A^N(b')\}. \end{aligned}$$

Hence, $x' \notin A_{t_1}^P, x' * y' \in A_{t_1}^P, y' \in A_{t_1}^P, a' \notin A_{s_1}^N, a' * b' \in A_{s_1}^N$ and $b' \in A_{s_1}^N$. This is a contradiction, and so $\mu_A^P(x) \leq \max\{\mu_A^P(x * y), \mu_A^P(y)\}$ and $\mu_A^N(x) \geq \min\{\mu_A^N(x * y), \mu_A^N(y)\}$ for all $x, y \in X$. Therefore $A = (\mu_A^P, \mu_A^N)$ is a doubt bipolar fuzzy ideal of X. □

Proposition 4.9. *Let $A = (\mu_A^P, \mu_A^N)$ be a doubt bipolar fuzzy ideal of X. If the inequality $x * y \leq z$ holds in X, then $\mu_A^P(x) \leq \max\{\mu_A^P(y), \mu_A^P(z)\}$ and $\mu_A^N(x) \geq \min\{\mu_A^N(y), \mu_A^N(z)\}$ for all $x, y, z \in X$.*

Proof. Let $A = (\mu_A^P, \mu_A^N)$ be a doubt bipolar fuzzy ideal of X and let $x, y, z \in X$ be such that $x * y \leq z$. Then $(x * y) * z = 0$, and so

$$\begin{aligned} \mu_A^P(x) &\leq \max\{\mu_A^P(x * y), \mu_A^P(y)\} \\ &\leq \max\{\max\{\mu_A^P((x * y) * z), \mu_A^P(z)\}, \mu_A^P(y)\} \\ &= \max\{\max\{\mu_A^P(0), \mu_A^P(z)\}, \mu_A^P(y)\} \\ &= \max\{\mu_A^P(y), \mu_A^P(z)\} \end{aligned}$$

and

$$\begin{aligned} \mu_A^N(x) &\geq \min\{\mu_A^N(x * y), \mu_A^N(y)\} \\ &\geq \min\{\min\{\mu_A^N((x * y) * z), \mu_A^N(z)\}, \mu_A^N(y)\} \\ &= \min\{\min\{\mu_A^N(0), \mu_A^N(z)\}, \mu_A^N(y)\} \\ &= \min\{\mu_A^N(y), \mu_A^N(z)\}. \end{aligned}$$

This completes the proof. □

Theorem 4.10. *In a BCK-algebra X, every doubt bipolar fuzzy ideal of X is a doubt bipolar fuzzy subalgebra of X.*

Proof. Let $A = (\mu_A^P, \mu_A^N)$ be a doubt bipolar fuzzy ideal of a BCK- algebra X. For any $x, y \in X$, we have

$$\begin{aligned}
\mu_A^P(x * y) &\leq \max\{\mu_A^P((x * y) * x), \mu_A^P(x)\} \\
&= \max\{\mu_A^P((x * x) * y), \mu_A^P(x)\} \\
&= \max\{\mu_A^P(0 * y), \mu_A^P(x)\} \\
&= \max\{\mu_A^P(0), \mu_A^P(x)\} \\
&\leq \max\{\mu_A^P(x), \mu_A^P(y)\}
\end{aligned}$$

and

$$\begin{aligned}
\mu_A^N(x * y) &\geq \min\{\mu_A^N((x * y) * x), \mu_A^N(x)\} \\
&= \min\{\mu_A^N((x * x) * y), \mu_A^N(x)\} \\
&= \min\{\mu_A^N(0 * y), \mu_A^N(x)\} \\
&= \min\{\mu_A^N(0), \mu_A^N(x)\} \\
&\geq \min\{\mu_A^N(x), \mu_A^N(y)\}
\end{aligned}$$

Hence, $A = (\mu_A^P, \mu_A^N)$ is a doubt bipolar fuzzy subalgebra of a BCK-algebra X. □

Example 4.11. *In Example 4.2, $A = (\mu_A^P, \mu_A^N)$ is a doubt bipolar fuzzy ideal of X, so that $A = (\mu_A^P, \mu_A^N)$ is a doubt bipolar fuzzy subalgebra of X.*

The converse of Theorem 4.10 is not true in general. For example, the doubt bipolar fuzzy subalgebra $A = (\mu_A^P, \mu_A^N)$ in Example 3.2 is not a doubt bipolar fuzzy ideal of X, since $\mu_A^P(b) = 0.6, \mu_A^P(b) = 0.6 \nleq 0.5 = \max\{\mu_A^P(b * a), \mu_A^P(a)\}$.

We give a condition for a doubt bipolar fuzzy subalgebra to be a doubt bipolar fuzzy ideal in a BCK-algebra.

Theorem 4.12. *Let $A = (\mu_A^P, \mu_A^N)$ be a doubt bipolar fuzzy subalgebra of X. If the inequality $x * y \leq z$ holds in X, then $A = (\mu_A^P, \mu_A^N)$ is a doubt bipolar fuzzy ideal of X.*

Proof. Let $A = (\mu_A^P, \mu_A^N)$ be a doubt bipolar fuzzy subalgebra of X. Then from Proposition 3.5, $\mu_A^P(0) \leq \mu_A^P(x)$ and $\mu_A^N(0) \geq \mu_A^N(x)$, for all $x \in X$. As $x * y \leq z$ holds in X, then from Proposition 4.9, we get $\mu_A^P(x) \leq \max\{\mu_A^P(y), \mu_A^P(z)\}$ and $\mu_A^N(x) \geq \min\{\mu_A^N(y), \mu_A^N(z)\}$ for all $x, y, z \in X$.

Since $x * (x * y) \leq y$ for all $x, y \in X$, then $\mu_A^P(x) \leq \max\{\mu_A^P(x * y), \mu_A^P(y)\}$ and $\mu_A^N(x) \geq \min\{\mu_A^N(x * y), \mu_A^N(y)\}$. Hence, $A = (\mu_A^P, \mu_A^N)$ is a doubt bipolar fuzzy ideal of X. □

The following example shows that Theorem 4.10 is not valid in a BCI-algebra X, that is, if X is a BCI-algebra, then there is a doubt bipolar fuzzy ideal that is not a doubt bipolar fuzzy subalgebra.

Example 4.13. *Consider a BCI-algebra $X = Y \times \mathbb{Z}$, where $(Y, *, 0)$ is a BCI-algebra and $(\mathbb{Z}, -, 0)$ is the adjoint BCI-algebra of the additive group $(\mathbb{Z}, +, 0)$ of integers (see[5]). Let $A = (\mu_A^P, \mu_A^N)$ is a doubt bipolar fuzzy set of X given by*

$$\mu_A^P(x) = \begin{cases} 0, & \text{if } x \in Y \times (\mathbb{N} \cup \{0\}) \\ 0.5, & \text{if } x \notin Y \times (\mathbb{N} \cup \{0\}), \end{cases}$$

and

$$\mu_A^N(x) = \begin{cases} 0, & \text{if } x \in Y \times (\mathbb{N} \cup \{0\}) \\ -0.6, & \text{if } x \notin Y \times (\mathbb{N} \cup \{0\}), \end{cases}$$

for all $x \in X$, where $\mathbb{N}$ the set of all natural numbers. By routine calculation, we know that $A = (\mu_A^P, \mu_A^N)$ is a doubt bipolar fuzzy ideal of X. If we take $x = (0,0)$ and $y = (0,1)$ in $x \in Y \times (\mathbb{N} \cup \{0\})$, then $x * y = (0,0) * (0,1) = (0,-1) \notin Y \times (\mathbb{N} \cup \{0\})$. Hence,

$$\mu_A^P(x * y) = 0.5 \nleqslant 0 = \max\{\mu_A^P(x), \mu_A^P(y)\}$$

and

$$\mu_A^N(x * y) = -0.6 \ngeq 0 = \min\{\mu_A^N(x), \mu_A^N(y)\}.$$

Therefore, $A = (\mu_A^P, \mu_A^N)$ is not a doubt bipolar fuzzy subalgebra of X.

Definition 4.14. *Let $(X, *, 0)$ and $(X', *', 0')$ be two BCK/BCI-algebras, a homomorphism is a map $f : X \to X'$ satisfying $f(x*y) = f(x)*' f(y)$ for all $x, y \in X$.*

Definition 4.15. *Let $f : X \to X'$ be a homomorphism of BCK/BCI-algebras and let $A = (\mu_A^P, \mu_A^N)$ be a bipolar fuzzy set in X', then the bipolar fuzzy set $A_f = (\mu_{A_f}^P, \mu_{A_f}^N)$ in X define by $\mu_{A_f}^P = \mu_A^P of$ and $\mu_{A_f}^N = \mu_A^N of$ (i.e., $\mu_{A_f}^P(x) = \mu_A^P(f(x))$ and $\mu_{A_f}^N(x) = \mu_A^N(f(x))$ for all $x \in X$) is called the preimage of A under f.*

Theorem 4.16. *An onto homomorphic preimage of a doubt bipolar fuzzy ideal is a doubt bipolar fuzzy ideal.*

Proof. Let $f : X \to X'$ be an onto homomorphism of BCK/BCI-algebras, $A = (\mu_A^P, \mu_A^N)$ be a doubt bipolar fuzzy ideal in X', and $A_f = (\mu_{A_f}^P, \mu_{A_f}^N)$ be preimage of A under f. For any $x' \in X'$ there exist $x \in X$ such that $f(x) = x'$. We have $\mu_{A_f}^P(0) = \mu_A^P(f(0)) = \mu_A^P(0') \leq \mu_A^P(x') = \mu_A^P(f(x)) = \mu_{A_f}^P(x)$, and $\mu_{A_f}^N(0) = \mu_A^N(f(0)) = \mu_A^N(0') \geq \mu_A^N(x') = \mu_A^N(f(x)) = \mu_{A_f}^N(x)$. Let $x \in X$ and $y' \in X'$, then there exist $y \in X$ such that $f(y) = y'$. We have

$$\begin{aligned}
\mu_{A_f}^P(x) &= \mu_A^P(f(x)) \\
&\leq \max\{\mu_A^P(f(x) * y'), \mu_A^P(y')\} \\
&= \max\{\mu_A^P(f(x) * f(y)), \mu_A^P(f(y))\} \\
&= \max\{\mu_A^P(f(x * y)), \mu_A^P(f(y))\} \\
&= \max\{\mu_{A_f}^P(x * y), \mu_{A_f}^P(y)\}
\end{aligned}$$

and

$$\begin{aligned}
\mu_{A_f}^N(x) &= \mu_A^N(f(x)) \\
&\geq \min\{\mu_A^N(f(x) * y'), \mu_A^N(y')\} \\
&= \min\{\mu_A^N(f(x) * f(y)), \mu_A^N(f(y))\} \\
&= \min\{\mu_A^N(f(x * y)), \mu_A^N(f(y))\} \\
&= \min\{\mu_{A_f}^N(x * y), \mu_{A_f}^N(y)\}.
\end{aligned}$$

Hence, $A_f = (\mu_{A_f}^P, \mu_{A_f}^N)$ is a doubt bipolar fuzzy ideal of X. □

Definition 4.17. *Let $f : X \to X'$ be a homomorphism of BCK/BCI-algebras. If $A = (\mu_A^P, \mu_A^N)$ is a bipolar fuzzy set of a BCK/BCI-algebra X, then the doubt image of A under f, denoted by $f(A)$, is a bipolar fuzzy set of X' defined by*

$$f(A) = (f_{\inf}(\mu_A^P), f_{\sup}(\mu_A^N),$$

where

$$\begin{aligned} f_{\inf}(\mu_A^P)(y) &= \inf_{x\in f^{-1}(y)}(\mu_A^P(x)) \text{ and} \\ f_{\sup}(\mu_A^N)(y) &= \sup_{x\in f^{-1}(y)}(\mu_A^N(x)), \text{ for each } y\in X'. \end{aligned}$$

Definition 4.18. *Let $A=(\mu_A^P,\mu_A^N)$ be a doubt bipolar fuzzy BCK/BCI-algebras X, then we say that μ_A^P has inf property if for any subset S of X there exist $s_1\in S$ such that $\mu_A^P(s_1)=\inf_{r\in S}\mu_A^P(r)$ and we say that μ_A^N has sup property if for any subset T of X there exist $t_1\in T$ such that $\mu_A^N(t_1)=\sup_{k\in T}\mu_A^N(k)$.*

For the homomorphic doubt image of a doubt bipolar fuzzy ideal of BCK/BCI-algebras, we have the following theorem.

Theorem 4.19. *An onto homomorphic doubt image of a doubt bipolar fuzzy ideal with μ_A^P has* inf *property and μ_A^N has* sup *property is a doubt bipolar fuzzy ideal.*

Proof. Let $f:X\to X'$ be an onto homomorphism of BCK/BCI-algebras and let $A=(\mu_A^P,\mu_A^N)$ be a doubt bipolar fuzzy ideal of X with μ_A^P has inf property and μ_A^N has sup property. Since $A=(\mu_A^P,\mu_A^N)$ is a doubt bipolar fuzzy ideal of X, then we have $\mu_A^P(0)\le\mu_A^P(x)$ and $\mu_A^N(0)\ge\mu_A^N(x)$ for all $x\in X$. Note that $0\in f^{-1}(0')$ where 0 and $0'$ are the zeros of X and X', respectively. Thus, $f_{\inf}(\mu_A^P)(0')=\inf_{t\in f^{-1}(0')}(\mu_A^P(t))=\mu_A^P(0)\le\mu_A^P(x)$ and $f_{\sup}(\mu_A^N)(0')=\sup_{t\in f^{-1}(0')}(\mu_A^N(t))=\mu_A^N(0)\ge\mu_A^N(x)$, for all $x\in X$. Also, since $A=(\mu_A^P,\mu_A^N)$ has inf and sup properties. This implies that $f_{\inf}(\mu_A^P)(0')\le\inf_{t\in f^{-1}(x')}(\mu_A^P(t))=f_{\inf}(\mu_A^P)(x')$ and $f_{\sup}(\mu_A^N)(0')\ge\sup_{t\in f^{-1}(x')}(\mu_A^N(t))=f_{\sup}(\mu_A^N)(x')$, for any $x'\in X'$. For any $x',y'\in X'$, let $x_o\in f^{-1}(x'),y_o\in f^{-1}(y')$ be such that

$$\begin{aligned} \mu_A^P(x_o) &= \inf_{t\in f^{-1}(x')}(\mu_A^P(t)), \\ \mu_A^N(x_o) &= \sup_{t\in f^{-1}(x')}(\mu_A^N(t)), \\ \mu_A^P(y_o) &= \inf_{t\in f^{-1}(y')}(\mu_A^P(t)), \\ \mu_A^N(y_o) &= \sup_{t\in f^{-1}(y')}(\mu_A^N(t)), \end{aligned}$$

$$\begin{aligned} \mu_A^P(x_o*y_o) &= f_{\inf}(\mu_A^P)(f(x_o*y_o)) \\ &= f_{\inf}(\mu_A^P)(x'*y') \\ &= \inf_{(x_o*y_o)\in f^{-1}(x'*y')}\mu_A^P(x_o*y_o) \\ &= \inf_{t\in f^{-1}(x'*y')}\mu_A^P(t), \end{aligned}$$

and

$$\begin{aligned}\mu_A^N(x_o * y_o) &= f_{\inf}(\mu_A^N)(f(x_o * y_o)) \\ &= f_{\sup}(\mu_A^N)(x' * y') \\ &= \sup_{(x_o * y_o) \in f^{-1}(x' * y')} \mu_A^N(x_o * y_o) \\ &= \sup_{t \in f^{-1}(x' * y')} \mu_A^N(t).\end{aligned}$$

Then,

$$\begin{aligned} f_{\inf} \quad (\mu_A^P) \quad (x') &= \inf_{t \in f^{-1}(x')} \mu_A^P(t) = \mu_A^P(x_o) \\ &\leq \max\{\mu_A^P(x_o * y_o), \mu_A^P(y_o\} \\ &= \max\{\inf_{t \in f^{-1}(x' * y')} \mu_A^P(t), \inf_{t \in f^{-1}(y')} \mu_A^P(t)\} \\ &= \max\{f_{\inf}(\mu_A^P)(x' * y'), f_{\inf}(\mu_A^P)(y')\}\end{aligned}$$

and

$$\begin{aligned} f_{\sup} \quad (\mu_A^N) \quad (x') &= \sup_{t \in f^{-1}(x')} \mu_A^N(t) = \mu_A^N(x_o) \\ &\geq \min\{\mu_A^N(x_o * y_o), \mu_A^N(y_o\} \\ &= \min\{\sup_{t \in f^{-1}(x' * y')} \mu_A^N(t), \sup_{t \in f^{-1}(y')} \mu_A^N(t)\} \\ &= \min\{f_{\sup}(\mu_A^N)(x' * y'), f_{\sup}(\mu_A^N)(y')\}.\end{aligned}$$

Hence, the doubt image $f(A)$ of A under$f$ is a doubt bipolar fuzzy ideal of X'. □

Proposition 4.20. *Let $A = (\mu_A^P, \mu_A^N)$ be a doubt bipolar fuzzy ideal of X. Then the sets*

$$J = \{x \in X : \mu_A^P(x) = \mu_A^P(0)\}$$

and

$$K = \{x \in X : \mu_A^N(x) = \mu_N^P(0)\}$$

are ideals of X.

Proof. Obviously, $0 \in J$ and $0 \in K$. Hence, $J \neq \emptyset$ and $K \neq \emptyset$. Now, let $x, y \in J$ such that $x * y, y \in J$. Then $\mu_A^P(x * y) = \mu_A^P(0) = \mu_A^P(y)$. Now, $\mu_A^P(x) \leq \max\{\mu_A^P(x * y), \mu_A^P(y)\} = \mu_A^P(0)$. since $A = (\mu_A^P, \mu_A^N)$ be a doubt bipolar fuzzy ideal of $X, \mu_A^P(0) \leq \mu_A^P(x)$. Therefore, $\mu_A^P(0) = \mu_A^P(x)$. It follows that $x \in J$, for all $x, y \in X$. Therefore, J is an ideal of X. Similarly, we can prove that K is an ideal of X. □

For any elements $\omega_t, \omega_s \in X$, we consider the sets:

$$A_{\omega_t}^P = \{x \in X \mid \mu_A^P(x) \leq \mu_A^P(\omega_t)\}$$

and

$$A_{\omega_s}^N = \{x \in X \mid \mu_A^N(x) \geq \mu_A^N(\omega_s)\}.$$

Clearly, $\omega_t \in A_{\omega_t}^P$ and $\omega_s \in A_{\omega_s}^N$. So that $\omega_t \in A_{\omega_t}^P$ and $\omega_s \in A_{\omega_s}^N$ are nonempty sets of X.

Theorem 4.21. *Let ω_t and ω_s be any two elements of X. If $A = (\mu_A^P, \mu_A^N)$ is a doubt bipolar fuzzy ideal of X, then $A_{\omega_t}^P$ and $A_{\omega_s}^N$ are ideals of X.*

Proof. Clearly, $0 \in A^P_{\omega_t}$ and $0 \in A^N_{\omega_s}$. Let $x, y \in X$ be such that $x * y \in A^P_{\omega_t} \cap A^N_{\omega_s}$ and $y \in A^P_{\omega_t} \cap A^N_{\omega_s}$. Then $\mu^P_A(x*y) \leq \mu^P_A(\omega_t), \mu^P_A(y) \leq \mu^P_A(\omega_t), \mu^N_A(x*y) \geq \mu^N_A(\omega_s)$ and $\mu^N_A(y) \geq \mu^N_A(\omega_s)$. It follows that from Definition 4.1, that

$$\mu^P_A(x) \leq \max\{\mu^P_A(x*y), \mu^P_A(y)\} \leq \mu^P_A(\omega_t)$$

and

$$\mu^N_A(x) \geq \min\{\mu^N_A(x*y), \mu^N_A(y)\} \geq \mu^N_A(\omega_s).$$

Hence, $x \in A^P_{\omega_t} \cap A^N_{\omega_s}$, and therefore $A^P_{\omega_t}$ and $A^N_{\omega_s}$ are ideals of X. □

Theorem 4.22. *Let $\omega_t, \omega_s \in X$ and let $A = (\mu^P_A, \mu^N_A)$ be a bipolar fuzzy set over X. Then*

(1) *If $A^P_{\omega_t}$ and $A^N_{\omega_s}$ are ideals of X, then the following assertion is valid for all $x, y, z \in X$:*
   (A1) $\mu^P_A(x) \geq \max\{\mu^P_A(y*z), \mu^P_A(z)\} \Rightarrow \mu^P_A(x) \geq \mu^P_A(y)$,
   (A2) $\mu^N_A(x) \leq \min\{\mu^N_A(y*z), \mu^N_A(z)\} \Rightarrow \mu^N_A(x) \leq \mu^N_A(y)$.

(2) *If $A = (\mu^P_A, \mu^N_A)$ satisfies (A1), (A2) and*
   (A3) $\mu^P_A(0) \leq \mu^P_A(x)$ and $\mu^N_A(0) \geq \mu^N_A(x)$

for all $x \in X$. Then $A^P_{\omega_t}$ and $A^N_{\omega_s}$ are ideals for all $\omega_t \in Im(\mu^P_A)$ and $\omega_s \in Im(\mu^N_A)$.

Proof. (1) Assume that $A^P_{\omega_t}$ and $A^N_{\omega_s}$ are ideals of X for $\omega_t, \omega_s \in X$. Let $x, y, z \in X$ be such that $\mu^P_A(x) \geq \max\{\mu^P_A(y*z), \mu^P_A(z)\}$ and $\mu^N_A(x) \leq \min\{\mu^N_A(y*z), \mu^N_A(z)\}$. Then $y*z \in A^P_{\omega_t} \cap A^N_{\omega_s}$ and $z \in A^P_{\omega_t} \cap A^N_{\omega_s}$, where $\omega_t = \omega_s = x$. Since $A^P_{\omega_t}$ and $A^N_{\omega_s}$ are ideals of X, it follows that $y \in A^P_{\omega_t} \cap A^N_{\omega_s}$ for $\omega_t = \omega_s = x$. Hence, $\mu^P_A(y) \leq \mu^P_A(\omega_t) = \mu^P_A(x)$ and $\mu^N_A(y) \geq \mu^N_A(\omega_s) = \mu^N_A(x)$.

(2) Let $\omega_t \in Im(\mu^P_A)$ and $\omega_s \in Im(\mu^N_A)$ and suppose that $A = (\mu^P_A, \mu^N_A)$ satisfies (A1), (A2) and (A3). Clearly, $0 \in A^P_{\omega_t} \cap A^N_{\omega_s}$ by (A3). Let $x, y \in X$ be such that $x*y \in A^P_{\omega_t} \cap A^N_{\omega_s}$ and $y \in A^P_{\omega_t} \cap A^N_{\omega_s}$. Then $\mu^P_A(x*y) \leq \mu^P_A(\omega_t), \mu^P_A(y) \leq \mu^P_A(\omega_t), \mu^N_A(x*y) \geq \mu^N_A(\omega_s)$ and $\mu^N_A(y) \geq \mu^N_A(\omega_s)$, which implies that

$$\max\{\mu^P_A(x*y), \mu^P_A(y)\} \leq \mu^P_A(\omega_t)$$

and

$$\min\{\mu^N_A(x*y), \mu^N_A(y)\} \geq \mu^N_A(\omega_s).$$

It follows from (A1) and (A2) that $\mu^P_A(\omega_t) \geq \mu^P_A(x)$ and $\mu^N_A(\omega_s) \leq \mu^N_A(x)$. Thus, $x \in A^P_{\omega_t} \cap A^N_{\omega_s}$, and therefore $A^P_{\omega_t}$ and $A^N_{\omega_s}$ are ideals of X. □

In a BCI-algebra X, a doubt bipolar fuzzy ideal $A = (\mu^P_A, \mu^N_A)$ need not be a doubt bipolar fuzzy subalgebra of X, see Example 4.13. If the doubt bipolar fuzzy ideal $A = (\mu^P_A, \mu^N_A)$ is also a doubt bipolar fuzzy subalgebra of X, then it has better algebraic properties. So that, in the following definition we introduce the concept of closed doubt bipolar fuzzy ideals of X.

Definition 4.23. *Let X be a BCI-algebra. A doubt bipolar fuzzy ideal $A = (\mu^P_A, \mu^N_A)$ of X is said to be closed if it is also a doubt bipolar fuzzy subalgebra of X.*

Example 4.24. *Consider a BCI-algebra $X = \{0, a, b, c, d\}$ with the following Cayley table:*

$*$	0	a	b	c	d
0	0	0	b	c	d
a	a	0	b	c	d
b	b	b	0	d	c
c	c	c	d	0	b
d	d	d	c	b	0

Let $A = (\mu_A^P, \mu_A^N)$ be a bipolar fuzzy set in X defined by

$$\mu_A^P(x) = \begin{cases} 0.1, & \text{if} \quad x = 0 \\ 0.3, & \text{if} \quad x = a \\ 0.4, & \text{if} \quad x = b \\ 0.6, & \text{if} \quad x = c,\ d, \end{cases}$$

and

$$\mu_A^N(x) = \begin{cases} -0.3, & \text{if} \quad x = 0 \\ -0.4, & \text{if} \quad x = a \\ -0.8, & \text{if} \quad x = b,\ d \\ -0.6, & \text{if} \quad x = c. \end{cases}$$

By routine calculation, we know that $A = (\mu_A^P, \mu_A^N)$ is a closed doubt bipolar fuzzy ideal of X.

Theorem 4.25. *Let X be a BCI-algebra. For any $\alpha_1, \alpha_2 \in [0, 1]$ and $\beta_1, \beta_2 \in [-1, 0]$ with $\alpha_1 < \alpha_2$ and $\beta_1 > \beta_2$, let $A = (\mu_A^P, \mu_A^N)$ be a bipolar fuzzy set of X given as follows:*

$$\mu_A^P(x) = \begin{cases} \alpha_1, & \text{if} \quad x \in X_+ \\ \alpha_2, & \text{if} \quad x \notin X_+, \end{cases}$$

and

$$\mu_A^P(x) = \begin{cases} \beta_1, & \text{if} \quad x \in X_+ \\ \beta_2, & \text{if} \quad x \notin X_+, \end{cases}$$

where $X_+ = \{x \in X \mid 0 \leq x\}$. Then $A = (\mu_A^P, \mu_A^N)$ is a closed doubt bipolar fuzzy ideal of X.

Proof. Since $0 \in X_+$, we have $\mu_A^P(0) = \alpha_1 \leq \mu_A^P(x)$ and $\mu_A^N(0) = \beta_1 \geq \mu_A^N(x)$ for all $x \in X$. Let $x, y \in X$. If $x \in X_+$, then

$$\mu_A^P(x) = \alpha_1 \leq \max\{\mu_A^P(x * y), \mu_A^P(y)\}$$

and

$$\mu_A^N(x) = \beta_1 \geq \min\{\mu_A^N(x * y), \mu_A^N(y)\}.$$

Suppose that $x \notin X_+$. If $x * y \in X_+$, then $y \notin X_+$, and if $y \in X_+$, then $x * y \notin X_+$. In either case, we get

$$\mu_A^P(x) = \alpha_2 = \max\{\mu_A^P(x * y), \mu_A^P(y)\}$$

and

$$\mu_A^N(x) = \beta_2 = \min\{\mu_A^N(x * y), \mu_A^N(y)\}.$$

For any $x, y \in X$, if any one of x and y does not belong to X_+, then

$$\mu_A^P(x * y) \leq \alpha_2 = \max\{\mu_A^P(x), \mu_A^P(y)\}$$

and

$$\mu_A^N(x * y) \geq \beta_2 = \min\{\mu_A^N(x), \mu_A^N(y)\}.$$

If $x, y \in X_+$, then $x * y \in X_+$. Hence,

$$\mu_A^P(x * y) = \alpha_1 = \max\{\mu_A^P(x), \mu_A^P(y)\}$$

and

$$\mu_A^N(x * y) = \beta_1 = \min\{\mu_A^N(x), \mu_A^N(y)\}.$$

Therefore, $A = (\mu_A^P, \mu_A^N)$ is a closed doubt bipolar fuzzy ideal of X. □

Proposition 4.26. *Let X be a BCI-algebra. If $A = (\mu_A^P, \mu_A^N)$ is a closed doubt bipolar fuzzy ideal of X, then*

$$\mu_A^P(0 * x) \leq \mu_A^P(x) \text{ and } \mu_A^N(0 * x) \geq \mu_A^N(x) \tag{4.3}$$

for all $x \in X$.

Proof. For any $x \in X$, we have

$$\begin{aligned} \mu_A^P(0 * x) &\leq \max\{\mu_A^P(0), \mu_A^P(x)\} \\ &\leq \max\{\mu_A^P(x), \mu_A^P(x)\} \\ &= \mu_A^P(x) \end{aligned}$$

and

$$\begin{aligned} \mu_A^N(0 * x) &\geq \min\{\mu_A^N(0), \mu_A^N(x)\} \\ &\geq \min\{\mu_A^N(x), \mu_A^N(x)\} \\ &= \mu_A^N(x). \end{aligned}$$

This completes the proof. □

Now, we provide conditions for a doubt bipolar fuzzy ideal to be a doubt bipolar fuzzy subalgebra and hence a closed doubt bipolar fuzzy ideal of a BCI-algebra X.

Theorem 4.27. *Let X be a BCI-algebra. If $A = (\mu_A^P, \mu_A^N)$ is a doubt bipolar fuzzy ideal of X that satisfies the condition of Equation (4.3), then $A = (\mu_A^P, \mu_A^N)$ is a doubt bipolar fuzzy subalgebra and hence is a closed doubt bipolar fuzzy ideal of X.*

Proof. Note that $(x * y) * x \leq 0 * y$ for all $x, y \in X$. Using Proposition 4.9 and Equation (4.3), we have

$$\begin{aligned} \mu_A^P(x * y) &\leq \max\{\mu_A^P(x), \mu_A^P(0 * y)\} \\ &\leq \max\{\mu_A^P(x), \mu_A^P(y)\} \end{aligned}$$

and

$$\begin{aligned} \mu_A^N(x * y) &\geq \min\{\mu_A^N(x), \mu_A^N(0 * y)\} \\ &\geq \min\{\mu_A^N(x), \mu_A^N(y)\}. \end{aligned}$$

Hence, $A = (\mu_A^P, \mu_A^N)$ is a doubt bipolar fuzzy subalgebra and therefore is a closed doubt bipolar fuzzy ideal of X. □

5. Conclusions

Doubt bipolar fuzzy subalgebras and (closed) doubt bipolar fuzzy ideals with special properties play an important role in investigating the structure of an algebraic system. In this work, we discussed a bipolar fuzzy set with an application to BCK/BCI-algebras. We introduced the notions of doubt bipolar fuzzy subalgebras and (closed) doubt bipolar fuzzy ideals in BCK/BCI-algebras, and investigated related properties. We considered some characterizations of a doubt bipolar fuzzy subalgebra and a doubt bipolar fuzzy ideal in BCK/BCI-algebras by means of doubt positive t-level cut set and doubt negative s-level cut set. Relations between a doubt bipolar fuzzy subalgebra and a doubt bipolar fuzzy ideal were provided. Also, homomorphic preimages and doubt images of doubt bipolar fuzzy ideals in BCK/BCI-algebras were discussed. Finally, we provided conditions for a doubt bipolar fuzzy ideal to be a closed doubt bipolar fuzzy ideal. We believe that our results presented in this paper will give a foundation for further study the algebraic structure of BCK/BCI-algebras.

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