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C-REGULAR TOPOLOGICAL SPACES

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Abstract. A topological space X is called *C-regular* if there exists a one-to-one function f from X onto a regular space Y such that the restriction $f_{|K} : K \longrightarrow f(K)$ is a homeomorphism for each compact subspace $K \subseteq X$. We discuss this property and illustrate the relationships between C-regularity and some other properties such as submetrizability, local compactness, L-regularity, C-normality, epinormality, epiregularity, σ-compactness and semiregularity.

1. Introduction

We define a new topological property called *C-regular.* Unlike C-normality[2], we prove that C-regularity is a topological property which is multiplicative, additive, and hereditary. We show that C-regularity and C-normality are independent. Also we investigate the function witnesses the C-regularity when it is continuous and when it is not. We introduce the notion of L-regularity. Throughout this paper, we denote by $\mathbb{N}$ the set of all positive integers, and an order pair by $\langle x, y\rangle$. We denoted the first infinite ordinal by ω_0 and the first uncountable ordinal by ω_1. A T_3 space is a T_1 regular space, a Tychonoff ($T_{3\frac{1}{2}}$) space is a T_1 completely regular space, and a T_4 space is a T_1 normal space. For a subset B of a space X, int$B$ denotes the interior of B and $\overline{B}$ denote the closure of B. A space X is locally compact if for each $y \in X$ and each open neighborhood U of y there exists an open neighborhood V of y such that $y \in V \subseteq \overline{V} \subseteq U$ and $\overline{V}$ is compact, we do not assume the space to be T_2 in the definitions of local compactness and compactness.

2. C-Regularity

Definition 2.1. A topological space X is called *C-regular* if there exists a bijective function f from X into a regular space Y such that the restriction $f_{|K} : K \longrightarrow f(K)$ is a homeomorphism for each compact subspace $K \subseteq X$.

By the definition, it is clear that a compact C-regular space must be regular see Theorem 2.12 below. Obviously, any regular space is C-regular, hence so is any

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Tychonoff space, just by taking $Y = X$ and f to be the identity function, but the converse is not true in general. For example, the Half-Disc space [13] is C-regular which is not regular. It is C-regular because it is submetrizable see Theorem 2.9. C-regularity does not imply regularity even with first countability. For example, Smirnov's deleted sequence topology [13] is first countable and C-regular being submetrizable but not regular.

If X is C-regular and $f : X \longrightarrow Y$ is a witness of the C-regularity of X, then f may not be continuous. Here is an example.

Example 2.2. Consider $\mathbb{R}$ with the countable complement topology $\mathcal{C}C$ [13]. Since the only compact subspace are the finite subspaces and $(\mathbb{R}, \mathcal{C}C)$ is T_1, then the compact subspace are discrete. If we let D be the discrete topology on $\mathbb{R}$, then clearly the identity function from $(\mathbb{R}, \mathcal{C}C)$ to $(\mathbb{R}, D)$ is a witness of the C-regularity of $(\mathbb{R}, \mathcal{C}C)$ which is not continuous. ■

Recall that a space X is *Fréchet* if for any subset B of X and any $x \in \overline{B}$ there exist a sequence $(b_n)_{n\in\mathbb{N}}$ of points of B such that $b_n \longrightarrow x$, see [7].

Theorem 2.3. If X is C-regular and Fréchet, then any function witnessing its C-regularity is continuous.

Proof. Let X be a Fréchet C-regular space and $f : X \longrightarrow Y$ be a witness of the C-regularity of X. Take $B \subseteq X$ and pick $y \in f(\overline{B})$. There is a unique $x \in X$ such that $f(x) = y$, thus $x \in \overline{B}$. Since X is Fréchet, then there exists a sequence $(b_n) \subseteq B$ such that $b_n \longrightarrow x$. As the subspace $K = \{x\} \cup \{b_n : n \in \mathbb{N}\}$ of X is compact the induced map $f_{|_K} : K \longrightarrow f(K)$ is a homeomorphism. Let $W \subseteq Y$ be any open neighborhood of y. Then $W \cap f(K)$ is an open neighborhood of y in the subspace $f(K)$. Since $f_{|_K}$ is a homeomorphism, then $f^{-1}(W \cap f(K)) = f^{-1}(W) \cap K$ is an open neighborhood of x in K, then there exists $m \in \mathbb{N}$ such that $b_n \in f^{-1}(W \cap f(K))$ $\forall n \geqslant m$, hence $f(b_n) \in (W \cap f(K))$ $\quad \forall n \geqslant m$, then $W \cap f(B) \neq \emptyset$. Hence $y \in \overline{f(B)}$ and $f(\overline{B}) \subseteq \overline{f(B)}$. Thus f is continuous. □

Since any first countable space is Fréchet [7], we conclude the following corollary:

Corollary 2.4. If X is C-regular first countable and $f : X \longrightarrow Y$ witnessing the C-regularity of X, then f is continuous.

Recall that a space X is a *k-space* if X is T_2 and it is a quotient image of a locally compact space [7]. By the theorem: " *a function f from a k-space X into a space Y is continuous if and only if $f_{|_Z} : Z \longrightarrow Y$ is continuous for each compact subspace Z of X*", [7, 3.3.21]. We conclude the following:

Corollary 2.5. If X is a C-regular k-space and $f : X \longrightarrow Y$ witnessing the C-regularity of X, then f is continuous.

Recall that a topological space X is called *C-normal* if there exists a one-to-one function f from X onto a normal space Y such that the restriction $f_{|_K} : K \longrightarrow f(K)$ is a homeomorphism for each compact subspace $K \subseteq X$[2].

Theorem 2.6. Every C-regular Fréchet Lindelöf space is C-normal.

Proof. Let X be any C-regular Fréchet Lindelöf space. Pick a regular space Y and a bijective function $f : X \longrightarrow Y$ such that the restriction $f_{|_K} : K \longrightarrow f(K)$ is a homeomorphism for each compact subspace $K \subseteq X$. By Theorem 2.3, f is continuous. Since the continuous image of a Lindelöf space is Lindelöf [7, 3.8.7], we conclude that Y is Lindelöf, hence normal as any regular Lindelöf space is normal [7, 3.8.2]. Therefore, X is C-normal. $\square$

C-normality and C-regularity are independent from each other. Here is an example of a C-normal which is not C-regular.

Example 2.7. Consider $\mathbb{R}$ with its *left ray topology* $\mathcal{L}$ [13]. So, $\mathcal{L} = \{\emptyset, \mathbb{R}\} \cup \{(-\infty, x) : x \in \mathbb{R}\}$. Since any two non-empty closed sets must intersect, then $(\mathbb{R}, \mathcal{L})$ is normal, hence C-normal [2]. Now, suppose that $(\mathbb{R}, \mathcal{L})$ is C-regular. Pick a regular space Y and a bijective function $f : \mathbb{R} \longrightarrow Y$ such that the restriction $f_{|_K} : K \longrightarrow f(K)$ is a homeomorphism for each compact subspace $K \subseteq \mathbb{R}$. It is well-known that a subspace K of $(\mathbb{R}, \mathcal{L})$ is compact if and only if K has a maximal element. Thus $(-\infty, 3]$ is compact, hence $f_{|_{(-\infty,3]}} : (-\infty, 3] \longrightarrow f((-\infty, 3]) \subset Y$ is a homeomorphism. i.e. $(-\infty, 3]$ as a subspace of $(\mathbb{R}, \mathcal{L})$ is regular which is a contradiction as $[2, 3]$ is closed in $(-\infty, 3]$ and $0 \notin [2, 3]$ and any non-empty open sets in $(-\infty, 3]$ must intersect. Therefore, $(\mathbb{R}, \mathcal{L})$ cannot be C-regular. $\blacksquare$

Here is an example of a C-regular space which is not C-normal.

Example 2.8. Consider the infinite Tychonoff product space $G = D^{\omega_1} = \prod_{\alpha \in \omega_1} D$, where $D = \{0, 1\}$ considered with the discrete topology. Let H be the subspace of G consisting of all points of G with at most countably many non-zero coordinates. Put $M = G \times H$. Raushan Buzyakova proved that M cannot be mapped onto a normal space Z by a bijective continuous function [6]. Using Buzyakova's result and the fact that M is a k-space, we conclude that M is a Tychonoff space which is not C-normal [11]. Since M is Tychonoff, then it is C-regular. $\blacksquare$

Recall that a topological space (X, τ) is called *submetrizable* if there exists a metric d on X such that the topology τ_d on X generated by d is coarser than τ, i.e., $\tau_d \subseteq \tau$, see [9].

Theorem 2.9. Every submetrizable space is C-regular.

Proof. Let τ' be a metrizable topology on X such that $\tau' \subseteq \tau$. Then (X, τ') is regular and the identity function $id_X : (X, \tau) \longrightarrow (X, \tau')$ is a bijective and continuous. If K is any compact subspace of (X, τ), then $id_X(K)$ is Hausdorff being a subspace of the metrizable space (X, τ'), and the restriction of the identity function on K onto $id_X(K)$ is a homeomorphism by [7, 3.1.13]. $\square$

The converse of Theorem 2.9 is not true in general. For example, the Tychonoff Plank $((\omega_1+1)\times(\omega_0+1))\setminus\{\langle\omega_1,\omega_0\rangle\}$ is C-regular being Hausdorff locally compact, but it is not submetrizable, because if it was, then $(\omega_1+1)\times\{0\}\subseteq((\omega_1+1)\times(\omega_0+1))\setminus\{\langle\omega_1,\omega_0\rangle\}$ is submetrizable, because submetrizablity is hereditary, but $((\omega_1+1)\times\{0\}\cong\omega_1+1$ and ω_1+1 is not submetrizable.

Since any Hausdorff locally compact space is Tychonoff. hence regular, then we have the following theorem.

Theorem 2.10. Every Hausdorff locally compact space is C-regular.

The converse of Theorem 2.10 is not true in general as the Dieudonné Plank [13] is Tychonoff, hence C-regular but not locally compact. Hausdorffness is essential in Theorem 2.10. Here is an example of a locally compact space which is neither C-regular nor Hausdorff.

Example 2.11. The particular point topology $\tau_{\sqrt{2}}$ on $\mathbb{R}$, see [13], where the particular point is $\sqrt{2}\in\mathbb{R}$, is not C-regular. It is well-known that $(\mathbb{R},\tau_{\sqrt{2}})$ is neither T_1 nor regular space. If $B\subseteq\mathbb{R}$, then $\{\{x,\sqrt{2}\}:x\in B\}$ is an open cover for B, thus a subset B of $\mathbb{R}$ is compact if and only if it is finite. To show that $(\mathbb{R},\tau_{\sqrt{2}})$ is not C-regular, suppose that $(\mathbb{R},\tau_{\sqrt{2}})$ is C-regular. Let Y be a regular space and $f:\mathbb{R}\longrightarrow Y$ be a bijective function such that the restriction $f_{|_K}:K\longrightarrow f(K)$ is a homeomorphism for each compact subspace K of $(\mathbb{R},\tau_{\sqrt{2}})$. For the space Y, we have only two cases:

Case 1: If Y is T_1-space. Take $K=\{x,\sqrt{2}\}$, such that $x\neq\sqrt{2}$, hence K is a compact subspace of $(\mathbb{R},\tau_{\sqrt{2}})$. By assumption $f_{|_K}:K\longrightarrow f(K)=\{f(x),f(\sqrt{2})\}$ is a homeomorphism. Because $f(K)$ is a finite subspace of Y and Y is T_1, then $f(K)$ is discrete subspace of Y. Therefore, we obtain that $f_{|_K}$ is not continuous and this a contradiction as $f_{|_K}$ is a homeomorphism.

Case2: If Y is not T_1-space. We will claim that the topology on Y is coarser than the particular point topology with $f(\sqrt{2})$ as its particular point. To prove this claim, we suppose not, then there exists a non-empty open set $U\subset Y$ such that $f(\sqrt{2})\notin U$. Take $y\in U$ and let $x\in\mathbb{R}$ be the unique real number such that $f(x)=y$. Consider $\{x,\sqrt{2}\}$, note that $x\neq\sqrt{2}$ since $f(x)=y\in U$, f is one-to-one $f(\sqrt{2})\notin U$. Let $f_{|_{\{x,\sqrt{2}\}}}:\{x,\sqrt{2}\}\longrightarrow\{y,f(\sqrt{2})\}$. Now, $\{y\}$ is open in the subspace $\{y,f(\sqrt{2})\}$ of Y because $\{y\}=U\cap\{y,f(\sqrt{2})\}$, but $f^{-1}(\{y\})=\{x\}$ and $\{x\}$ is not open in the subspace $\{x,\sqrt{2}\}$ of $(\mathbb{R},\tau_{\sqrt{2}})$, which means $f_{|_{\{x,\sqrt{2}\}}}$ is not continuous, this is a contradiction, and the above claim is proved. But any particular point space consisting of more than one point cannot be regular, so we get a contradiction as Y is assumed to be regular. Thus $(\mathbb{R},\tau_{\sqrt{2}})$ is not C-regular. ■

Proposition 2.12. If X is a compact non-regular space, then X cannot be C-regular.

We conclude that from the above theorem, $\mathbb{R}$ with the finite complement topology is not C-regular.

Proposition 2.13. If X is a T_1-space such that the only compact subspace are the finite subspace, then X is C-regular.

Proposition 2.14. C-regularity is a topological property.

Theorem 2.15. C-regularity is an additive property.

Proof. Let X_s be a C-regular space for each $s \in S$. We prove that their sum $\oplus_{s\in S}X_s$ is C-regular. For each $s \in S$, pick a regular space Y_s and a bijective function $f_s : X_s \longrightarrow Y_s$ such that $f_{s_{|K_s}} : K_s \longrightarrow f_s(K_s)$ is a homeomorphism for each compact subspace K_s of X_s. Because Y_s is regular for each $s \in S$, then the sum $\oplus_{s\in S}Y_s$ is regular, [7, 2.2.7]. Consider the function sum [7, 2.2.E] $f = \oplus_{s\in S}f_s : \oplus_{s\in S}X_s \longrightarrow \oplus_{s\in S}Y_s$ defined by $f(x) = f_s(x)$ if $x \in X_s, s \in S$. A subspace $K \subseteq \oplus_{\alpha\in\Lambda}X_\alpha$ is compact if and only if the set $S_0 = \{s \in S : K \cap X_s \neq \emptyset\}$ is finite and $K \cap X_s$ is compact in X_s for each $s \in S_0$. If $K \subseteq \oplus_{s\in S}X_s$ is compact. then $(\oplus_{s\in S}f_s)_{|K}$ is a homeomorphism since $f_{s_{|K\cap X_s}}$ is a homeomorphism for each $s \in S_0$. □

Theorem 2.16. C-regularity is a multiplicative property.

Proof. Let X_s be a C-regular space for each $s \in S$. Pick a regular space Y_s and a bijective function $f_s : X_s \longrightarrow Y_s$ such that $f_{s_{|K_s}} : K_s \longrightarrow f_s(K_s)$ is a homeomorphism for each compact subspace K_s of X_s. Since Y_s is regular for each $s \in S$, then the Cartesian product $\prod_{s\in S} Y_s$ is regular [7, 2.3.11]. Define $f : \prod_{s\in S} X_s \longrightarrow \prod_{s\in S} Y_s$ by $f((x_s) : s \in S) = (f_s(x_s) : s \in S)$ for each $s \in S$, then f is bijective. Let $K \subseteq \prod_{s\in S} X_s$ be any compact subspace and let p_s be the usual projection, then $p_s(K) \subseteq X_s$ is compact. Now, $K \subseteq \prod_{s\in S} p_s(K) = K^\star$ is compact, by the Tychonoff theorem. Hence $f_{|K^\star} = \prod_{s\in S} f_s\,_{|p_s(K)}$ is a homeomorphism. Thus $f_{|K}$ is a homeomorphism, because the restriction of a homeomorphism is a homeomorphism. □

Theorem 2.17. C-regularity is a hereditary property.

Proof. Let A be any subspace of C-regular space X, then there exists a regular space Y and $f : X \longrightarrow Y$ be a witness of the C-regularity of X. Let $B = f(A) \subseteq Y$. Then B is regular, being a subspace of a regular space. Now we have $f_{|A} : A \longrightarrow B$ is a bijective function. Since any compact subspace of A is compact in X and $f_{|A}{}_{|K} = f_{|K}$, we conclude that A is C-regular. □

Frome Theorem 2.16 and Theorem 2.17, we conclude the following corollary.

Corollary 2.18. $\prod_{s\in S} X_s$ is C-regular if and only if X_s is C-regular $\forall s \in S$.

3. C-Regularity and Other Properties

We introduce another new topological property called *L-regular* .

Definition 3.1. A topological space X is called *L-regular* if there exists a one-to-one function f from X onto a regular space Y such that the restriction $f_{|_L} : L \longrightarrow f(L)$ is a homeomorphism for each Lindelöf subspace $L \subseteq X$.

By the definition it is clear that a Lindelöf L-regular space must be regular. Since any compact space is Lindelöf, then any L-regular space is C-regular. The converse is not true in general. Obviously, no Lindelöf non-regular space is L-regular. So, no countable complement topology on uncountable set X is L-regular, but it is C-regular, see Example 2.2.

Proposition 3.2. L-regularity is a topological property.

Proposition 3.3. L-regularity is an additive property.

Proposition 3.4. L-regularity is a multiplicative property.

Proposition 3.5. L-regularity is a hereditary property.

Proposition 3.6. $\prod_{s\in S} X_s$ is L-regular if and only if X_s is L-regular $\forall s \in S$.

A function $f : X \longrightarrow Y$ witnessing the L-regularity of X need not be continuous. But it will be if X is of countable tightness. Recall that a space X is of *countable tightness* if for each subset B of X and each $x \in \overline{B}$, there exists a countable subset B_0 of B such that $x \in \overline{B_0}$ [7].

Theorem 3.7. If X is L-regular and of countable tightness and $f : X \longrightarrow Y$ is a witness of the L-regularity of X, then f is continuous.

Proof. Let A be any non-empty subset of X. Let $y \in f(\overline{A})$ be arbitrary. Let $x \in X$ be the unique element such that$f(x) = y$. Then $x \in \overline{A}$. Pick a countable subset $A_0 \subseteq A$ such that $x \in \overline{A_0}$. Let $B = \{x\} \cup A_0$; then B is a Lindelöf subspace of X and hence $f_{|_B} : B \longrightarrow f(B)$ is a homeomorphism. Now, let $V \subseteq Y$ be any open neighborhood of y; then $V \cap f(B)$ is open in the subspace $f(B)$ containing y. Thus $f^{-1}(V)\cap B$ is open in the subspace B containing x. Thus $(f^{-1}(V)\cap B)\cap A_0 \neq \emptyset$. So $(f^{-1}(V)\cap B)\cap A \neq \emptyset$. Hence $\emptyset \neq f((f^{-1}(V)\cap B)\cap A) \subseteq f(f^{-1}(V)\cap A) = V \cap f(A)$. Thus $y \in \overline{f(A)}$. Therefore, f is continuous. $\square$

Recall that if $(x_n)_{n\in\mathbb{N}}$ is a sequence in a topological space X, then the *convergency set of* (x_n) is defined by $C(x_n) = \{x \in X : x_n \longrightarrow x\}$ and a topological space X is *sequential* if for any $A \subseteq X$ we have that A is closed if and only if $C(x_n) \subseteq A$ for any sequence $(x_n) \subseteq A$, see [7]. We have the following implications, see [7, 1.6.14, 1.7.13].

First countability $\Rightarrow$ Fréchet $\Rightarrow$ Sequential $\Rightarrow$ Countable tightness.

Corollary 3.8. If X is L-regular and first countable (Fréchet, Sequential) and $f : X \longrightarrow Y$ is a witness of the L-regularity of X, then f is continuous.

Theorem 3.9. If X is C-regular space such that each Lindelöf subspace is contained in a compact subspace, then X is L-regular.

Proof. Assume that X is any C-regular space where if L is any Lindelöf subspace of X, then there exists a compact subspace K where $L \subseteq K$. Let Y be a regular space and $f : X \longrightarrow Y$ be a witness of the C-regularity of X. Now, let L be any Lindelöf subspace of X. Pick a compact subspace K of X where $L \subseteq K$, then $f_{|K} : K \longrightarrow f(K)$ is a homeomorphism, thus $f_{|_L} : L \longrightarrow f(L)$ is a homeomorphism as $(f_{|K})_{|_L} = f_{|_L}$. $\square$

Now, we study some relationships between C-regularity and some other properties. Recall that a topological space (X, τ) is called *epinormal* if there is a coarser topology τ' on X such that (X, τ') is T_4 [3]. A topological space (X, τ) is called *epiregular* if there is a coarser topology τ' on X such that (X, τ') is T_3 [4]. By a similar proof as that of Theorem 2.9 above, we can prove the following corollaries:

Corollary 3.10. Any epinormal space is C-regular.

Corollary 3.11. Any epiregular space is C-regular.

Any indiscrete space which has more than one element is an example of C-regular space which is neither epiregular nor epinormal.

Let X be any Hausdorff non-k-space. Let $kX = X$. Define a topology on kX as follows: a subset of kX is open if and only if its intersection with any compact subspace C of the space X is open in C. kX with this topology is Hausdorff and k-space such that X and kX have the same compact subspace and the same topology on these subspace [5], we conclude the following:

Theorem 3.12. If X is Hausdorff but not k-space, then X is C-normal if and only if kX is C-normal.

Corollary 3.13. If X is Hausdorff but not k-space, then X is C-regular if and only if kX is C-regular.

C-normality and σ-compactness are independent from each other. For example, uncountable discrete space is C-normal being T_4, but not σ-compact. The modified Fort space is σ-compact but not C-normal because it is a compact non-normal space. Also C-regularity and σ-compactness are independent. For example the rational sequence space [13] is C-regular being Tychonoff, but not σ-compact. $\mathbb{R}$ with the finite complement topology is not C-regular, but it is σ-compact being compact. By using Theorem 2.6, we have the following corollary.

Corollary 3.14. Any C-regular Fréchet σ-compact space is C-normal.

C-regularity and first countability do not imply Hausdorffness, for example the Odd-Even space [13]. C-regularity does not imply Semiregularity, for example the double pointed reals space, see [13], is C-regular being regular, but not Semiregular, we still do not know if Semiregularity implies C-regularity or not.

Let X be any topological space. Let $X' = X \times \{a\}$. Note that $X \cap X' = \emptyset$. Let $A(X) = X \cup X'$. For simplicity, for an element $x \in X$, we will denote the element $\langle x, a \rangle$ in X' by x' and for a subset $E \subseteq X$ let $E' = \{x' : x \in E\} = E \times \{a\} \subseteq X'$. For each $x' \in X'$, let $\mathcal{B}(x') = \{\{x'\}\}$. For each $x \in X$, let $\mathcal{B}(x) = \{U \cup (U' \setminus \{x'\}) : U$ is open in X with $x \in U\}$. Let $\mathcal{T}$ denote the unique topology on $A(X)$ which has $\{\mathcal{B}(x) : x \in X\} \cup \{\mathcal{B}(x') : x' \in X'\}$ as its neighborhood system. $A(X)$ with this topology is called the *Alexandroff Duplicate of X.*

Theorem 3.15. If X is C-regular, then its Alexandroff Duplicate $A(X)$ is also C-regular.

Proof. Let X be any C-regular space. Pick a regular space Y and $f : X \longrightarrow Y$ be a witness of the C-regularity of X. Consider the Alexandroff Duplicate spaces $A(X)$ and $A(Y)$ of X and Y respectively. Since Y is regular, then $A(Y)$ is regular. Now, define $g : A(X) \longrightarrow A(Y)$ by $g(a) = f(a)$ if $a \in X$, and if $a \in X'$, let b be the unique element in X, where $b' = a$, hence define $g(a) = (f(b))'$. Thus g is a bijective function. A subspace $K \subseteq A(X)$ is compact if and only if $K \cap X$ is compact in X, and for each open set U in X with $K \cap X \subseteq U$ we have that $(K \cap X') \setminus U'$ is finite. Let $K \subseteq A(X)$ be any compact subspace. To prove that $g_{|_K} : K \longrightarrow g(K)$ is a homeomorphism, let $a \in K$ be arbitrary. If $a \in K \cap X'$, pick $b \in X$ be the unique element such that $b' = a$. For the smallest basic open neighborhood $\{(f(b))'\}$ of the point $g(a)$, then we have that $\{a\}$ is open in K and $g(\{a\}) \subseteq \{(f(b))'\}$. If $a \in K \cap X$, then let W be any open set in Y such that $g(a) = f(a) \in W$. Now, consider $H = (W \cup (W' \setminus \{f(a)'\})) \cap g(K)$ which is a basic open neighborhood of $f(a)$ in $g(K)$. Because $f_{|_{K \cap X}} : K \cap X \longrightarrow f(K \cap X)$ is a homeomorphism, then there exists an open set U in X with $a \in U$ and $f_{|_{K \cap X}}(U \cap K) \subseteq W$. Consider $(U \cup (U' \setminus \{a'\})) \cap K = G$ is open in K such that $a \in G$ and $g_{|_K}(G) \subseteq H$. Hence, $g_{|_K}$ is continuous. Now, we prove that $g_{|_K}$ is open. Let $V \cup (V' \setminus \{v'\})$ such that $v \in V$ is open in X, be any basic open set in $A(X)$, hence $(V \cap K) \cup ((V' \cap K) \setminus \{v'\})$ is a basic open set in K. Because $X \cap K$ is compact in X, then $g_{|_K}(V \cap (X \cap K)) = f_{|_{X \cap K}}(V \cap (X \cap K))$ is open in $Y \cap f(K \cap X)$ as $f_{|_{X \cap K}}$ is a homeomorphism. Therefore, $V \cap K$ is open in $Y \cap f(X \cap K)$. And, $g((V' \cap K) \setminus \{v'\})$ is open in $Y' \cap g(K)$ being a set of isolated points. hence $g_{|_K}$ is an open function. Thus $g_{|_K}$ is a homeomorphism. □

A similar proof as in [12], we get the following theorem.

Theorem 3.16. If X is L-regular, then its Alexandroff Duplicate $A(X)$ is also L-regular.

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