C-REGULAR TOPOLOGICAL SPACES

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Abstract. A topological space $X$ is called $C$-regular if there exists a one-to-one function $f$ from $X$ onto a regular space $Y$ such that the restriction $f|_K : K \to f(K)$ is a homeomorphism for each compact subspace $K \subseteq X$. We discuss this property and illustrate the relationships between $C$-regularity and some other properties such as submetrizability, local compactness, $L$-regularity, $C$-normality, epinormality, epiregularity, $\sigma$-compactness and semiregularity.

1. Introduction

We define a new topological property called $C$-regular. Unlike $C$-normality\cite{1}, we prove that $C$-regularity is a topological property which is multiplicative, additive, and hereditary. We show that $C$-regularity and $C$-normality are independent. Also we investigate the function witnesses the $C$-regularity when it is continuous and when it is not. We introduce the notion of $L$-regularity. Throughout this paper, we denote by $\mathbb{N}$ the set of all positive integers, and an order pair by $(x, y)$. We denoted the first infinite ordinal by $\omega_0$ and the first uncountable ordinal by $\omega_1$. A $T_3$ space is a $T_1$ regular space, a Tychonoff ($T_{3\frac{1}{2}}$) space is a $T_1$ completely regular space, and a $T_4$ space is a $T_1$ normal space. For a subset $B$ of a space $X$, $\text{int}B$ denotes the interior of $B$ and $\overline{B}$ denote the closure of $B$. A space $X$ is locally compact if for each $y \in X$ and each open neighborhood $U$ of $y$ there exists an open neighborhood $V$ of $y$ such that $y \in V \subseteq \overline{V} \subseteq U$ and $\overline{V}$ is compact, we do not assume the space to be $T_2$ in the definitions of local compactness and compactness.

2. C-Regularity

Definition 2.1. A topological space $X$ is called $C$-regular if there exists a bijective function $f$ from $X$ into a regular space $Y$ such that the restriction $f|_K : K \to f(K)$ is a homeomorphism for each compact subspace $K \subseteq X$.

By the definition, it is clear that a compact $C$-regular space must be regular see Theorem 2.12 below. Obviously, any regular space is $C$-regular, hence so is any...
Corollary 2.5. If subspace $Z \subseteq X$ is a locally compact space then by the theorem: "a function $f$ from a k-space $X$ into a space $Y$ is continuous if and only if $f|_Z : Z \to Y$ is continuous for each compact subspace $Z$ of $X"$, \cite{13} 3.3.21. We conclude the following:

Corollary 2.4. If $X$ is a C-regular k-space and $f : X \to Y$ witnessing the C-regularity of $X$, then $f$ is continuous.

Recall that a space $X$ is a k-space if $X$ is $T_2$ and it is a quotient image of a locally compact space \cite{7}. By the theorem: "a function $f$ from a k-space $X$ into a space $Y$ is continuous if and only if $f|_Z : Z \to Y$ is continuous for each compact subspace $Z$ of $X"$, \cite{7} 3.3.21. We conclude the following:

Example 2.2. Consider $\mathbb{R}$ with the countable complement topology $CC$ \cite{13}. Since the only compact subspace are the finite subspaces and $(\mathbb{R}, CC)$ is $T_1$, then the compact subspace are discrete. If we let $D$ be the discrete topology on $\mathbb{R}$, then clearly the identity function from $(\mathbb{R}, CC)$ to $(\mathbb{R}, D)$ is a witness of the C-regularity of $(\mathbb{R}, CC)$ which is not continuous.

Recall that a space $X$ is Fréchet if for any subset $B$ of $X$ and any $x \in \overline{B}$ there exist a sequence $(b_n)_{n \in \mathbb{N}}$ of points of $B$ such that $b_n \to x$, see \cite{7}.

Theorem 2.3. If $X$ is C-regular and Fréchet, then any function witnessing its C-regularity is continuous.

Proof. Let $X$ be a Fréchet C-regular space and $f : X \to Y$ be a witness of the C-regularity of $X$. Take $B \subseteq X$ and pick $y \in f(\overline{B})$. There is a unique $x \in X$ such that $f(x) = y$, thus $x \in \overline{B}$. Since $X$ is Fréchet, then there exists a sequence $(b_n) \subseteq B$ such that $b_n \to x$. As the subspace $K = \{x\} \cup \{b_n : n \in \mathbb{N}\}$ of $X$ is compact the induced map $f|_K : K \to f(K)$ is a homeomorphism. Let $W \subseteq Y$ be any open neighborhood of $y$. Then $W \cap f(K)$ is an open neighborhood of $y$ in the subspace $f(K)$. Since $f|_K$ is a homeomorphism, then $f^{-1}(W \cap f(K)) = f^{-1}(W) \cap K$ is an open neighborhood of $x$ in $K$, then there exists $m \in \mathbb{N}$ such that $b_n \in f^{-1}(W) \cap K$ \forall $n \geq m$, hence $f(b_n) \in (W \cap f(K)) \forall n \geq m$, then $W \cap f(B) \neq \emptyset$. Hence $y \in \overline{f(B)}$ and $f(\overline{B}) \subseteq \overline{f(B)}$. Thus $f$ is continuous. 

Since any first countable space is Fréchet \cite{7}, we conclude the following corollary:
Recall that a topological space \( X \) is called \( C\)-normal if there exists a one-to-one function \( f \) from \( X \) onto a normal space \( Y \) such that the restriction \( f|_K : K \rightarrow f(K) \) is a homeomorphism for each compact subspace \( K \subseteq X \) [2].

**Theorem 2.6.** Every \( C\)-regular Fréchet Lindelöf space is \( C\)-normal.

**Proof.** Let \( X \) be any \( C\)-regular Fréchet Lindelöf space. Pick a regular space \( Y \) and a bijective function \( f : X \rightarrow Y \) such that the restriction \( f|_K : K \rightarrow f(K) \) is a homeomorphism for each compact subspace \( K \subseteq X \). By Theorem 2.3, \( f \) is continuous. Since the continuous image of a Lindelöf space is Lindelöf [7 3.8.7], we conclude that \( Y \) is Lindelöf, hence normal as any regular Lindelöf space is normal [7 3.8.2]. Therefore, \( X \) is \( C\)-normal. \( \square \)

\( C\)-normality and \( C\)-regularity are independent from each other. Here is an example of a \( C\)-normal which is not \( C\)-regular.

**Example 2.7.** Consider \( \mathbb{R} \) with its left ray topology \( \mathcal{L} \) [13]. So, \( \mathcal{L} = \{ \emptyset, \mathbb{R} \} \cup \{ (-\infty, x) : x \in \mathbb{R} \} \). Since any two non-empty closed sets must intersect, then \( (\mathbb{R}, \mathcal{L}) \) is normal, hence \( C\)-normal [2]. Now, suppose that \( (\mathbb{R}, \mathcal{L}) \) is \( C\)-regular. Pick a regular space \( Y \) and a bijective function \( f : \mathbb{R} \rightarrow Y \) such that the restriction \( f|_K : K \rightarrow f(K) \) is a homeomorphism for each compact subspace \( K \subseteq \mathbb{R} \). It is well-known that a subspace \( K \) of \( (\mathbb{R}, \mathcal{L}) \) is compact if and only if \( K \) has a maximal element. Thus \( (-\infty, 3] \) is compact, hence \( f|_{(-\infty, 3]} : (-\infty, 3] \rightarrow f((-\infty, 3]) \subseteq Y \) is a homeomorphism. i.e. \( (-\infty, 3] \) as a subspace of \( (\mathbb{R}, \mathcal{L}) \) is regular which is a contradiction as [2,3] is closed in \( (-\infty, 3] \) and \( 0 \notin [2,3] \) and any non-empty open sets in \( (-\infty, 3] \) must intersect. Therefore, \( (\mathbb{R}, \mathcal{L}) \) cannot be \( C\)-regular. \( \blacksquare \)

Here is an example of a \( C\)-regular space which is not \( C\)-regular.

**Example 2.8.** Consider the infinite Tychonoff product space \( G = D^{\omega_1} = \prod_{\alpha \in \omega_1} D \), where \( D = \{ 0, 1 \} \) considered with the discrete topology. Let \( H \) be the subspace of \( G \) consisting of all points of \( G \) with at most countably many non-zero coordinates. Put \( M = G \times H \). Raushan Buzyakova proved that \( M \) cannot be mapped onto a normal space \( Z \) by a bijective continuous function [4]. Using Buzyakova’s result and the fact that \( M \) is a \( k\)-space, we conclude that \( M \) is a Tychonoff space which is not \( C\)-normal [11]. Since \( M \) is Tychonoff, then it is \( C\)-regular. \( \blacksquare \)

Recall that a topological space \((X, \tau)\) is called \textit{submetrizable} if there exists a metric \( d \) on \( X \) such that the topology \( \tau_d \) on \( X \) generated by \( d \) is coarser than \( \tau \), i.e., \( \tau_d \subseteq \tau \), see [9].

**Theorem 2.9.** Every submetrizable space is \( C\)-regular.

**Proof.** Let \( \tau' \) be a metrizable topology on \( X \) such that \( \tau' \subseteq \tau \). Then \((X, \tau')\) is regular and the identity function \( id_X : (X, \tau) \rightarrow (X, \tau') \) is a bijective and continuous. If \( K \) is any compact subspace of \((X, \tau)\), then \( id_X(K) \) is Hausdorff being a subspace of the metrizable space \((X, \tau')\), and the restriction of the identity function on \( K \) onto \( id_X(K) \) is a homeomorphism by [7 3.1.13]. \( \square \)
The converse of Theorem 2.9 is not true in general. For example, the Tychonoff Plank \(((\omega_1 + 1) \times (\omega_0 + 1)) \setminus \{(\omega_1, \omega_0)\}\) is C-regular being Hausdorff locally compact, but it is not submetrizable, because if it was, then \((\omega_1 + 1) \times \{0\} \subseteq ((\omega_1 + 1) \times (\omega_0 + 1)) \setminus \{(\omega_1, \omega_0)\}\) is submetrizable, because submetrizability is hereditary, but \(((\omega_1 + 1) \times \{0\} \cong \omega_1 + 1\) and \(\omega_1 + 1\) is not submetrizable.

Since any Hausdorff locally compact space is Tychonoff, hence regular, then we have the following theorem.

**Theorem 2.10.** Every Hausdorff locally compact space is C-regular.

The converse of Theorem 2.10 is not true in general as the Dieudonné Plank is Tychonoff, hence C-regular but not locally compact. Hausdorffness is essential in Theorem 2.10. Here is an example of a locally compact space which is neither C-regular nor Hausdorff.

**Example 2.11.** The particular point topology \(\tau_{\sqrt{2}}\) on \(\mathbb{R}\), see [13], where the particular point is \(\sqrt{2} \in \mathbb{R}\), is not C-regular. It is well-known that \((\mathbb{R},\tau_{\sqrt{2}})\) is neither \(T_1\) nor regular space. If \(B \subseteq \mathbb{R}\), then \(\{\{x, \sqrt{2}\} : x \in B\}\) is an open cover for \(B\), thus a subset \(B\) of \(\mathbb{R}\) is compact if and only if it is finite. To show that \((\mathbb{R},\tau_{\sqrt{2}})\) is not C-regular, suppose that \((\mathbb{R},\tau_{\sqrt{2}})\) is C-regular. Let \(Y\) be a regular space and \(f : \mathbb{R} \rightarrow Y\) be a bijective function such that the restriction \(f|_K : K \rightarrow f(K)\) is a homeomorphism for each compact subspace \(K\) of \((\mathbb{R},\tau_{\sqrt{2}})\). For the space \(Y\), we have only two cases:

Case 1: If \(Y\) is \(T_1\)-space. Take \(K = \{x, \sqrt{2}\}\), such that \(x \neq \sqrt{2}\), hence \(K\) is a compact subspace of \((\mathbb{R},\tau_{\sqrt{2}})\). By assumption \(f|_K : K \rightarrow f(K) = \{f(x), f(\sqrt{2})\}\) is a homeomorphism. Because \(f(K)\) is a finite subspace of \(Y\) and \(Y\) is \(T_1\), then \(f(K)\) is discrete subspace of \(Y\). Therefore, we obtain that \(f|_K\) is not continuous and this a contradiction as \(f|_K\) is a homeomorphism.

Case 2: If \(Y\) is not \(T_1\)-space. We will claim that the topology on \(Y\) is coarser than the particular point topology with \(f(\sqrt{2})\) as its particular point. To prove this claim, we suppose not, then there exists a non-empty open set \(U \subseteq Y\) such that \(f(\sqrt{2}) \notin U\). Take \(y \in U\) and let \(x \in \mathbb{R}\) be the unique real number such that \(f(x) = y\). Consider \(\{x, \sqrt{2}\}\), note that \(x \neq \sqrt{2}\) since \(f(x) = y \in U\), \(f\) is one-to-one \(f(\sqrt{2}) \notin U\). Let \(f|_{(x, \sqrt{2})} : \{x, \sqrt{2}\} \rightarrow \{y, f(\sqrt{2})\}\). Now, \(\{y\}\) is open in the subspace \(y, f(\sqrt{2})\) of \(Y\) because \(\{y\} = U \cap \{y, f(\sqrt{2})\}\), but \(f^{-1}(\{y\}) = \{x\}\) and \(\{x\}\) is not open in the subspace \(x, \sqrt{2}\) of \((\mathbb{R},\tau_{\sqrt{2}})\), which means \(f|_{(x, \sqrt{2})}\) is not continuous, this is a contradiction, and the above claim is proved. But any particular point space consisting of more than one point cannot be regular, so we get a contradiction as \(Y\) is assumed to be regular. Thus \((\mathbb{R},\tau_{\sqrt{2}})\) is not C-regular.

**Proposition 2.12.** If \(X\) is a compact non-regular space, then \(X\) cannot be C-regular.

We conclude that from the above theorem, \(\mathbb{R}\) with the finite complement topology is not C-regular.
Proposition 2.13. If $X$ is a $T_1$-space such that the only compact subspace are the finite subspace, then $X$ is $C$-regular.

Proposition 2.14. $C$-regularity is a topological property.

Theorem 2.15. $C$-regularity is an additive property.

Proof. Let $X_s$ be a $C$-regular space for each $s \in S$. We prove that their sum $\oplus_{s \in S} X_s$ is $C$-regular. For each $s \in S$, pick a regular space $Y_s$ and a bijective function $f_s : X_s \to Y_s$ such that $f_s|_{K_s} : K_s \to f_s(K_s)$ is a homeomorphism for each compact subspace $K_s$ of $X_s$. Because $Y_s$ is regular for each $s \in S$, then the sum $\oplus_{s \in S} Y_s$ is regular, [7, 2.2.7]. Consider the function sum $\oplus_{s \in S} f_s : \oplus_{s \in S} X_s \to \oplus_{s \in S} Y_s$ defined by $f(x) = f_s(x)$ if $x \in X_s, s \in S$. A subspace $K \subseteq \oplus_{s \in A} X_s$ is compact if and only if the set $S_0 = \{s \in S : K \cap X_s \neq \emptyset\}$ is finite and $K \cap X_s$ is compact in $X_s$ for each $s \in S_0$. If $K \subseteq \oplus_{s \in S} X_s$ is compact, then $(\oplus_{s \in S} f_s)|_K$ is a homeomorphism since $f_s|_{K \cap X_s}$ is a homeomorphism for each $s \in S_0$.

Theorem 2.16. $C$-regularity is a multiplicative property.

Proof. Let $X_s$ be a $C$-regular space for each $s \in S$. Pick a regular space $Y_s$ and a bijective function $f_s : X_s \to Y_s$ such that $f_s|_{K_s} : K_s \to f_s(K_s)$ is a homeomorphism for each compact subspace $K_s$ of $X_s$. Since $Y_s$ is regular for each $s \in S$, then the Cartesian product $\prod_{s \in S} Y_s$ is regular [4, 2.3.11]. Define $f : \prod_{s \in S} X_s \to \prod_{s \in S} Y_s$ by $f(x_s) : s \in S) = (f_s(x_s) : s \in S)$ for each $s \in S$, then $f$ is bijective. Let $K \subseteq \prod_{s \in S} X_s$ be any compact subspace and let $p_s$ be the usual projection, then $p_s(K) \subseteq X_s$ is compact. Now, $K \subseteq \prod_{s \in S} p_s(K) = K^*$ is compact, by the Tychonoff theorem. Hence $f|_{K^*} = \prod_{s \in S} f_s|_{p_s(K)}$ is a homeomorphism. Thus $f|_{K^*}$ is a homeomorphism, because the restriction of a homeomorphism is a homeomorphism.

Theorem 2.17. $C$-regularity is a hereditary property.

Proof. Let $A$ be any subspace of $C$-regular space $X$, then there exists a regular space $Y$ and $f : X \to Y$ be a witness of the $C$-regularity of $X$. Let $B = f(A) \subseteq Y$. Then $B$ is regular, being a subspace of a regular space. Now we have $f|_A : A \to B$ is a bijective function. Since any compact subspace of $A$ is compact in $X$ and $f|_{A|_K} = f|_K$, we conclude that $A$ is $C$-regular.

From Theorem 2.16 and Theorem 2.17 we conclude the following corollary.

Corollary 2.18. $\prod_{s \in S} X_s$ is $C$-regular if and only if $X_s$ is $C$-regular $\forall s \in S$. 
3. C-Regularity and Other Properties

We introduce another new topological property called \( L \)-regular.

**Definition 3.1.** A topological space \( X \) is called \( L \)-regular if there exists a one-to-one function \( f \) from \( X \) onto a regular space \( Y \) such that the restriction \( f|_L : L \to f(L) \) is a homeomorphism for each Lindelöf subspace \( L \subseteq X \).

By the definition it is clear that a Lindelöf \( L \)-regular space must be regular. Since any compact space is Lindelöf, then any \( L \)-regular space is \( C \)-regular. The converse is not true in general. Obviously, no Lindelöf non-regular space is \( L \)-regular. So, no countable complement topology on uncountable set \( X \) is \( L \)-regular, but it is \( C \)-regular, see Example 2.2.

**Proposition 3.2.** \( L \)-regularity is a topological property.

**Proposition 3.3.** \( L \)-regularity is an additive property.

**Proposition 3.4.** \( L \)-regularity is a multiplicative property.

**Proposition 3.5.** \( L \)-regularity is a hereditary property.

**Proposition 3.6.** \( \prod_{s \in S} X_s \) is \( L \)-regular if and only if \( X_s \) is \( L \)-regular \( \forall s \in S \).

A function \( f : X \to Y \) witnessing the \( L \)-regularity of \( X \) need not be continuous. But it will be if \( X \) is of countable tightness. Recall that a space \( X \) is of countable tightness if for each subset \( B \) of \( X \) and each \( x \in B \), there exists a countable subset \( B_0 \) of \( B \) such that \( x \in B_0 \) [1].

**Theorem 3.7.** If \( X \) is \( L \)-regular and of countable tightness and \( f : X \to Y \) is a witness of the \( L \)-regularity of \( X \), then \( f \) is continuous.

**Proof.** Let \( A \) be any non-empty subset of \( X \). Let \( y \in f(A) \) be arbitrary. Let \( x \in X \) be the unique element such that \( f(x) = y \). Then \( x \in f(A) \). Pick a countable subset \( A_0 \subseteq A \) such that \( x \in f(A_0) \). Let \( B = \{x\} \cup A_0 \); then \( B \) is a Lindelöf subspace of \( X \) and hence \( f|_B : B \to f(B) \) is a homeomorphism. Now, let \( V \subseteq Y \) be any open neighborhood of \( y \); then \( V \cap f(B) \) is open in the subspace \( f(B) \) containing \( y \). Thus \( f^{-1}(V) \cap B \) is open in the subspace \( f(B) \) containing \( y \). Thus \( f^{-1}(V) \cap A_0 \) is open in the subspace \( f(B) \) containing \( y \). Thus \( f^{-1}(V) \cap A_0 \neq \emptyset \). Hence \( \emptyset \neq f(f^{-1}(V) \cap A) \subseteq f(f^{-1}(V) \cap A) = V \cap f(A) \). Hence \( y \in f(A) \). Therefore, \( f \) is continuous. \( \square \)
Recall that if \((x_n)_{n \in \mathbb{N}}\) is a sequence in a topological space \(X\), then the convergence set of \((x_n)\) is defined by \(C(x_n) = \{x \in X : x_n \to x\}\) and a topological space \(X\) is sequential if for any \(A \subseteq X\) we have that \(A\) is closed if and only if \(C(x_n) \subseteq A\) for any sequence \((x_n) \subseteq A\), see [7]. We have the following implications, see [7, 1.6.14, 1.7.13].

First countability \(\Rightarrow\) Fréchet \(\Rightarrow\) Sequential \(\Rightarrow\) Countable tightness.

**Corollary 3.8.** If \(X\) is \(L\)-regular and first countable (Fréchet, Sequential) and \(f : X \to Y\) is a witness of the \(L\)-regularity of \(X\), then \(f\) is continuous.

**Theorem 3.9.** If \(X\) is \(C\)-regular space such that each Lindelöf subspace is contained in a compact subspace, then \(X\) is \(L\)-regular.

*Proof.* Assume that \(X\) is any \(C\)-regular space where if \(L\) is any Lindelöf subspace of \(X\), then there exists a compact subspace \(K\) where \(L \subseteq K\). Let \(Y\) be a regular space and \(f : X \to Y\) be a witness of the \(C\)-regularity of \(X\). Now, let \(L\) be any Lindelöf subspace of \(X\). Pick a compact subspace \(K\) of \(X\) where \(L \subseteq K\), then \(f|_K : K \to f(K)\) is a homeomorphism, thus \(f|_L : L \to f(L)\) is a homeomorphism as \((f|_K)|_L = f|_L\). \(\square\)

Now, we study some relationships between \(C\)-regularity and some other properties. Recall that a topological space \((X, \tau)\) is called epinormal if there is a coarser topology \(\tau'\) on \(X\) such that \((X, \tau')\) is \(T_4\) [3]. A topological space \((X, \tau)\) is called epiregular if there is a coarser topology \(\tau'\) on \(X\) such that \((X, \tau')\) is \(T_3\) [4]. By a similar proof as that of Theorem 2.9 above, we can prove the following corollaries:

**Corollary 3.10.** Any epinormal space is \(C\)-regular.

**Corollary 3.11.** Any epiregular space is \(C\)-regular.

Any indiscrete space which has more than one element is an example of \(C\)-regular space which is neither epiregular nor epinormal.

Let \(X\) be any Hausdorff non-\(k\)-space. Let \(kX = X\). Define a topology on \(kX\) as follows: a subset of \(kX\) is open if and only if its intersection with any compact subspace \(C\) of the space \(X\) is open in \(C\). \(kX\) with this topology is Hausdorff and \(k\)-space such that \(X\) and \(kX\) have the same compact subspace and the same topology on these subspace [5], we conclude the following:

**Theorem 3.12.** If \(X\) is Hausdorff but not \(k\)-space, then \(X\) is \(C\)-normal if and only if \(kX\) is \(C\)-normal.

**Corollary 3.13.** If \(X\) is Hausdorff but not \(k\)-space, then \(X\) is \(C\)-regular if and only if \(kX\) is \(C\)-regular.
C-normality and σ-compactness are independent from each other. For example, uncountable discrete space is C-normal being $T_4$, but not σ-compact. The modified Fort space is σ-compact but not C-normal because it is a compact non-normal space. Also C-regularity and σ-compactness are independent. For example the rational sequence space \[13\] is C-regular being Tychonoff, but not σ-compact. \(\mathbb{R}\) with the finite complement topology is not C-regular, but it is σ-compact being compact. By using Theorem 2.6 we have the following corollary.

**Corollary 3.14.** Any C-regular Fréchet σ-compact space is C-normal.

Let $X$ be any topological space. Let $X' = X \times \{a\}$. Note that $X \cap X' = \emptyset$. Let $A(X) = X \cup X'$. For simplicity, for an element $x \in X$, we will denote the element $\langle x, a \rangle$ in $X'$ by $x'$ and for a subset $E \subseteq X$ let $E' = \{x' : x \in E\} = E \times \{a\} \subseteq X'$. For each $x' \in X'$, let $\mathcal{B}(x') = \{\{x'\}\}$. For each $x \in X$, let $\mathcal{B}(x) = \{U \cup (U' \setminus \{x'\}) : U \text{ is open in } X \text{ with } x \in U\}$. Let $\mathcal{T}$ denote the unique topology on $A(X)$ which has $\{\mathcal{B}(x) : x \in X\} \cup \{\mathcal{B}(x') : x' \in X'\}$ as its neighborhood system. $A(X)$ with this topology is called the *Alexandroff Duplicate of $X$*.

**Theorem 3.15.** If $X$ is C-regular, then its Alexandroff Duplicate $A(X)$ is also C-regular.

**Proof.** Let $X$ be any C-regular space. Pick a regular space $Y$ and $f : X \rightarrow Y$ be a witness of the C-regularity of $X$. Consider the Alexandroff Duplicate spaces $A(X)$ and $A(Y)$ of $X$ and $Y$ respectively. Since $Y$ is regular, then $A(Y)$ is regular. Now, define $g : A(X) \rightarrow A(Y)$ by $g(a) = f(a)$ if $a \in X$, and if $a \in X'$, let $b$ be the unique element in $X$, where $b' = a$, hence define $g(a) = (f(b))'$. Thus $g$ is a bijective function. A subspace $K \subseteq A(X)$ is compact if and only if $K \cap X$ is compact in $X$, and for each open set $U$ in $X$ with $K \cap X \subseteq U$ we have that $(K \cap X') \setminus U'$ is finite. Let $K \subseteq A(X)$ be any compact subspace. To prove that $g|_{K} : K \rightarrow g(K)$ is a homeomorphism, let $a \in K$ be arbitrary. If $a \in K \cap X'$, pick $b \in X$ be the unique element such that $b' = a$. For the smallest basic open neighborhood $\{(f(b))'\}$ of the point $g(a)$, then we have that $\{a\}$ is open in $K$ and $g(\{a\}) \subseteq \{(f(b))'\}$. If $a \in K \cap X$, then let $W$ be any open set in $Y$ such that $g(a) = f(a) \in W$. Now, consider $H = (W \cup (W' \setminus \{f(a')\})) \cap g(K)$ which is a basic open neighborhood of $f(a)$ in $g(K)$.

Because $f|_{K \cap X} : K \cap X \rightarrow f(K \cap X)$ is a homeomorphism, then there exists an open set $U$ in $X$ with $a \in U$ and $f|_{K \cap X}(U \cap K) \subseteq W$. Consider $(U \cup (U' \setminus \{a'\})) \cap K = G$ is open in $K$ such that $a \in G$ and $g|_{K}(G) \subseteq H$. Hence, $g|_{K}$ is continuous. Now, we prove that $g|_{K}$ is open. Let $V \cup (V' \setminus \{v'\})$ such that $v \in V$ is open in $X$, be any basic open set in $A(X)$, hence $(V \cap K) \cup ((V' \cap K) \setminus \{v'\})$ is a basic open set in $K$. Because $X \cap K$ is compact in $X$, then $g|_{K}(V \cap (X \cap K)) = f|_{K \cap X}(V \cap (X \cap K))$ is open in $Y \cap f(K \cap X)$ as $f|_{K \cap X}$ is a homeomorphism. Therefore, $V \cap K$ is open in $Y \cap f(X \cap K)$. And, $g((V' \cap K) \setminus \{v'\})$ is open in $Y' \cap g(K)$ being a set of isolated points. hence $g|_{K}$ is an open function. Thus $g|_{K}$ is a homeomorphism. \(\square\)
A similar proof as in [12], we get the following theorem.

**Theorem 3.16.** If $X$ is $L$-regular, then its Alexandroff Duplicate $A(X)$ is also $L$-regular.

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