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ON THE SINGULARITIES OF GAUSSIAN RECTIFYING SURFACE AND SPECIAL CURVES

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Abstract. In this paper, we study the singularities of the rectifying surface. Also, we give the criteria for a general helix and a slant helix regarding to the image of the singular set of the rectifying surface. Furthermore, we discuss the elementary differential geometry of the rectifying surface.

1. Introduction

The theory of elementary differential geometry of curves and surfaces in $\mathbb{R}^3$ is a classical subject. Nevertheless it is still an enjoyable topic for the mathematicians and geometers. The theory of ruled surface is a nice topic in the field of differential geometry. The theory of surfaces in $\mathbb{R}^3$ has many applications in the reality. Perhaps the most active area of the theory of curves and surfaces in $\mathbb{R}^3$ is the singularity theory. The singularity of the ruled surfaces is an attractive aspect of research. There are several authors who have studied the ruled surfaces and their singularities [1, 2, 7, 8, 9]. In [6, 11, 12, 16], the authors gave beneficial criteria of singularities of wave fronts.
This paper consists of four sections. The first section gives the basic description about the topic of this paper. The second section gives some basic notations of the theory of curves and surfaces in $\mathbb{R}^3$. The third section examines the elementary differential geometry of developable ruled surfaces M_{BT} and M_{TB}. The fourth section studies the singularities of M_{BT} and M_{TB} and presents the geometric conditions for them to have certain singularities such as cuspidal edges, swallowtails and cuspidal butterflies singularities. In addition, in section four, the criteria of a helix and a slant helix are given in terms of the image of the singular set of M_{BT} and M_{TB}.

2. Preliminaries

In this section, we introduce some basic notations of the theory of curves and surfaces in $\mathbb{R}^3$. For more detailed properties, we refer the reader to [4, 13, 14, 17]. Firstly, we present some basic concepts of the theory of curves in $\mathbb{R}^3$. A curve in $\mathbb{R}^3$ is considered to be the image of a map from $I \subseteq \mathbb{R}$ into 3-dimensional space $\mathbb{R}^3$

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$\big(i.e., \gamma : I \to \mathbb{R}^3\big)$. If $\gamma'(t) \neq 0$ for all $t \in I$, then the space curve $\gamma : I \to \mathbb{R}^3$ is a regular space curve. A regular space curve γ has curvature ,κ_γ, given by

$$\kappa_\gamma = \frac{\|\gamma' \times \gamma''\|}{\|\gamma'\|^3},$$

where $\times$ is the vector cross product in $\mathbb{R}^3$. Moreover, γ has torsion ,τ_γ, given by

$$\tau_\gamma = \frac{det(\gamma', \gamma'', \gamma''')}{\|\gamma' \times \gamma''\|^2} = \frac{(\gamma' \times \gamma'') \cdot \gamma'''}{\|\gamma' \times \gamma''\|^2},$$

where $\cdot$ is the (canonical inner) product in $\mathbb{R}^3$. The unit tangent vector of a regular space curve γ is given by

$$T_\gamma = \frac{\gamma'}{\|\gamma'\|}.$$

Let $\kappa_\gamma \neq 0$. Then the unit principal normal vector N_γ and the unit binormal vector B_γ of a regular space curve γ are defined as

$$N_\gamma = \frac{T'_\gamma}{\|T'_\gamma\|} \text{ and } B_\gamma = T_\gamma \times N_\gamma.$$

The unit vectors T_γ, N_γ *and* B_γ together are called, $\{T_\gamma, N_\gamma, B_\gamma\}$, frame or the Serret-Frenet frame along γ. The relations between each generator of the abovementioned are given by $T_\gamma = N_\gamma \times B_\gamma$, $N_\gamma = B_\gamma \times T_\gamma$ *and* $B_\gamma = T_\gamma \times N_\gamma$. For any point p on a regular space curve γ with $\kappa_\gamma(p) \neq 0$, there are three planes spanned by T_γ, N_γ *and* B_γ. The plane spanned by B_γ *and* N_γ with normal vector T_γ is called the normal plane, the plane spanned by T_γ *and* B_γ with normal vector N_γ is called the rectifying plane and the plane spanned by N_γ *and* T_γ with normal vector B_γ is called the osculating plane. A regular space curve γ is said to be a unit speed curve if $\|\gamma'\| = 1$. If γ is a unit speed curve, then the Serret-Frenet equations are

$$\begin{aligned} T'_\gamma &= \kappa_\gamma N_\gamma \\ N'_\gamma &= -\kappa_\gamma T_\gamma + \tau_\gamma B_\gamma \\ B'_\gamma &= -\tau_\gamma N_\gamma, \end{aligned}$$

where κ_γ and τ_γ are the curvature and the torsion of γ, respectively. For a regular space curve γ, the spherical indicatrices are the curves T_γ, N_γ and B_γ, that lie on unit sphere. If κ_γ *and* τ_γ are non-vanishing, then the spherical indicatrices curves are regular and have curvatures and torsions as the following:

(1) $$\kappa_T = \frac{\sqrt{\kappa_\gamma^2 + \tau_\gamma^2}}{\kappa_\gamma} \textit{ and } \tau_T = \frac{\kappa_\gamma \left(\dfrac{\tau_\gamma}{\kappa_\gamma}\right)'}{\kappa_\gamma^2 + \tau_\gamma^2}.$$

(2) $$\kappa_N = \frac{\sqrt{\left(\kappa_\gamma^2 + \tau_\gamma^2\right)^3 + \left(\kappa_\gamma \tau'_\gamma - \kappa'_\gamma \tau_\gamma\right)^2}}{\left(\kappa_\gamma^2 + \tau_\gamma^2\right)^{3/2}} \textit{ and}$$

$$\tau_N = \frac{\left(\kappa_\gamma^2 + \tau_\gamma^2\right)\left(\kappa_\gamma \tau''_\gamma - \kappa''_\gamma \tau_\gamma\right) - 3\left(\kappa_\gamma \tau'_\gamma - \kappa'_\gamma \tau_\gamma\right)\left(\kappa_\gamma \kappa'_\gamma + \tau_\gamma \tau'_\gamma\right)}{\left(\kappa_\gamma^2 + \tau_\gamma^2\right)^3 + \left(\kappa_\gamma \tau'_\gamma - \kappa'_\gamma \tau_\gamma\right)^2}.$$

(3) $$\kappa_B = \frac{\sqrt{\kappa_\gamma^2 + \tau_\gamma^2}}{|\tau_\gamma|} \textit{ and } \tau_B = \frac{-\tau_\gamma \left(\dfrac{\kappa_\gamma}{\tau_\gamma}\right)'}{\kappa_\gamma^2 + \tau_\gamma^2}.$$

The arc-length function of a regular space curve γ based at $t_0 \in I$ is defined by $s = \int_{t_0}^{t} \|\gamma'(t)\| dt$. Let γ be a regular space curve. Then it is called a helix if its tangent vector makes a fixed angle, θ, with a fixed direction w $(i.e.,\ T_\gamma \cdot w = constant)$. If the curvature and the torsion of γ are non-vanishing, then the criteria of γ to be a helix is $\tau_\gamma/\kappa_\gamma = c$, where c is a constant. Let γ be a regular space curve. Then it is called a slant helix if its principal normal vector makes a fixed angle, θ, with a fixed direction w $(i.e.,\ N_\gamma \cdot w = constant)$. γ is called a Bertrand curve if there exist a regular space curve β such that $N_\beta = \pm N_\gamma$. Let γ be a regular space curve with $\kappa_\gamma \neq 0$. Then the criteria of γ to be a Bertrand curve is the condition $A\kappa_\gamma + B\tau_\gamma = 1$, where A *and* B are constants. A regular space curve is called a rectifying curve if its position vector always lies in the rectifying plane. B. Y. Chen gave the position vector of a rectifying curve by $\gamma(s) = \lambda T_\gamma + \mu B_\gamma$ such that $\lambda = s + c_1$ and $\mu = c_2$ where c_1, c_2 are non-zero constants and s is the arc-length of γ (see [5]). In addition, he showed that the criteria of a rectifying curve is given by $\tau_\gamma/\kappa_\gamma = as + b$, where a, b are non-zero constants.

Next, we introduce some basic concepts of the theory of surfaces in $\mathbb{R}^3$. A surface in $\mathbb{R}^3$ is considered to be the image of a map from $U \subseteq \mathbb{R}^2$ into 3-dimensional space $\mathbb{R}^3$ $\left(i.e.,\ M : U \subseteq \mathbb{R}^2 \to \mathbb{R}^3\right)$. A surface $M : U \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ is called a regular surface if M is smooth, injective and regular at every point. Let M is a regular surface defined by $M(s,t)$. Then the unit normal of M is given by

$$N_M = \frac{M_s \times M_t}{\|M_s \times M_t\|},$$

where M_s is the partial derivative of M with respect to s and M_t is the partial derivative of M with respect to t. A M is called a ruled surface if it is generated by straight lines and it is given by $M(t,u) = \gamma(t) + u\delta(t)$ where γ is called the base curve and δ is called the director curve. The first fundamental form of M is denoted by I_M and defined by

$$I_M = Eds^2 + 2Fdsdt + Gdt^2.$$

The symbols E, F *and* G are the coefficients of I_M, such that $E = M_s \cdot M_s$, $F = M_s \cdot M_t$ and $G = M_t \cdot M_t$. They could be written by a matrix as follows:

$$I_{coeff} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}.$$

The determinant of I_{coeff} is given by $\det(I_{coeff}) = EG - F^2$. The second fundamental form of M is denoted by II_M which is given by

$$II_M = lds^2 + 2mdsdt + ndt^2.$$

The symbols l, m *and* n are the coefficients of II_M, where $l = M_{ss} \cdot N_M$, $m = M_{st} \cdot N_M$ and $n = M_{tt} \cdot N_M$. They could be written by a matrix as follows:

$$II_{coeff} = \begin{pmatrix} l & m \\ m & n \end{pmatrix}.$$

The determinant of II_{coeff} is given by $\det(II_{coeff}) = ln - m^2$. The Gaussian curvature K_M of M and the Mean curvature H_M of M are given by

$$K_M = \frac{\det(II_{coeff})}{\det(I_{coeff})},$$
$$H_M = \frac{En - 2Fm + Gl}{2\det(I_{coeff})}.$$

If $K_M = 0$ at any point, then M is called a developable surface and if $H_M = 0$ at any point, then M is called a minimal surface. M is a piece of plane if both K_M and H_M are equal to zero at any point.

3. Elementary differential geometry of M_{BT} and M_{TB} surfaces

Let $\gamma : I \to \mathbb{R}^3$ be a regular space curve with $\kappa_\gamma \neq 0$ and let $\{T_\gamma,\ N_\gamma, B_\gamma\}$ be the Serret- Frenet frame along γ. In [2], the authors used these curves and presented two developable ruled surfaces $M_{TN} = T_\gamma + uN_\gamma$ *and* $M_{BN} = B_\gamma + uN_\gamma$. In the second surface if we put T_γ instead of N_γ we get $M_{BT} = B_\gamma + uT_\gamma$, which is called the Gaussian rectifying surface [5]. The authors in [15] studied the singularities of M_{BT}, we present them in Section 4. Now, if we swap B_γ and T_γ in the Gaussian rectifying surface then, we obtain $M_{TB} = T_\gamma + uB_\gamma$. Our target in this section is to examine the elementary geometric characteristics of M_{BT} and M_{TB} such as the first and the second fundamental forms, the Gaussian curvature and the mean curvature. By a direct calculation, we provide the following proposition.

Proposition 3.1. *If γ is a regular space curve parametrized by its arc-length s with $\kappa_\gamma \neq 0$, then the surfaces M_{BT} and M_{TB} have the following:*

(1) $I_{M_{BT}} = (u\kappa_\gamma - \tau_\gamma)^2 ds^2 + du^2$.

(2) $I_{M_{TB}} = (-u\tau_\gamma + \kappa_\gamma)^2 ds^2 + du^2$.

(3) $II_{M_{BT}} = \dfrac{-\tau_\gamma \left(u\kappa_\gamma - \tau_\gamma\right)^2}{\sqrt{u^2\kappa_\gamma^2 + \tau_\gamma^2}} ds^2$.

(4) $II_{M_{TB}} = \dfrac{-\kappa_\gamma(\kappa_\gamma - u\tau_\gamma)^2}{\sqrt{\kappa_\gamma^2 + u^2\tau_\gamma^2}} ds^2$.

(5) $K_{M_{BT}} = 0$ *and* $H_{M_{BT}} = \dfrac{-\tau_\gamma}{2\sqrt{\tau_\gamma^2 + u^2\kappa_\gamma^2}}$.

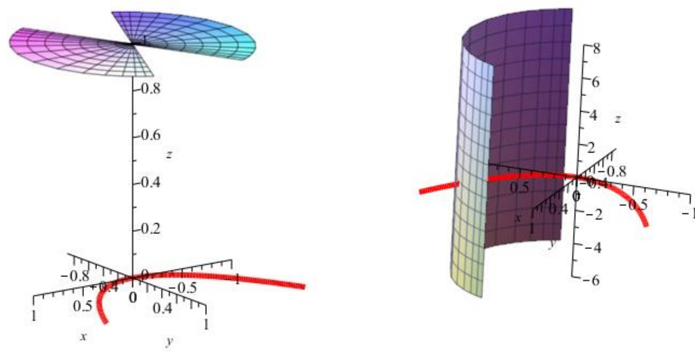
(6) $K_{M_{TB}} = 0$ *and* $H_{M_{TB}} = \dfrac{-\kappa_\gamma}{2\sqrt{\kappa_\gamma^2 + u^2\tau_\gamma^2}}$.

From Proposition 3.1, we can easily see that the surfaces M_{BT} and M_{TB} are developable surfaces because their Gaussian curvatures are equal to zero and each point is a parabolic point. Away from the singular points, if γ is a plane curve, then the surface M_{BT} is a piece of plane, $B_\gamma \equiv$constant, T_γ is a plane vector and the surface M_{TB} is a unit circular cylinder. In the following example, we clarify this issue.

Example 3.2. *Let γ be a regular space curve given by*

$$\gamma(t) = \left(t, t^2, 0\right).$$

By a direct calculation, we have that $\tau_\gamma = 0$. Therefore, M_{BT} is a piece of plane and M_{TB} is a cylinder (see Figure 1).



(A) M_{BT} surface and the red line is the curve γ.

(B) M_{TB} surface and the red line is the curve γ.

FIGURE 1. The surfaces M_{BT} *and* M_{TB} of Example 3.2.

For the surface M_{BT} if $u = 0$, then the curve B_γ is considered as the locus of constant mean curvature for M_{BT}, precisely $H_{M_{BT}} = -1/2$, and we illustrate that in the following example.

Example 3.3. *Let γ be a regular space curve given by*

$$\gamma(t) = \left(t, t^2, t^3\right).$$

At $u = 0$, the surface M_{BT} has constant mean curvature $H_{M_{BT}} = -1/2$ which implies that the curve B_γ is the locus of the constant mean curvature for M_{BT} (see Figure 2).

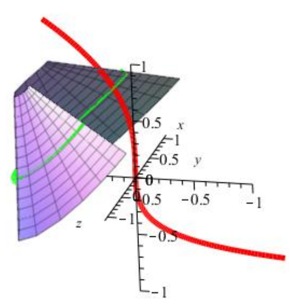
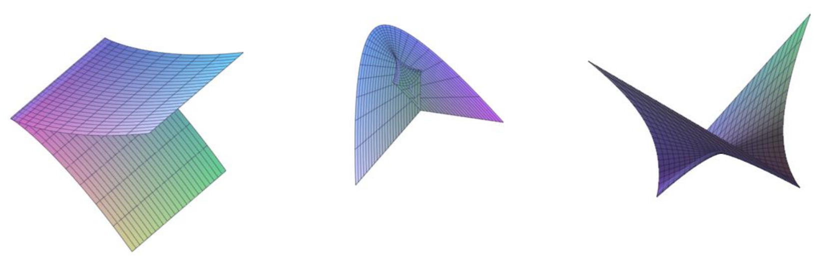


FIGURE 2. The surface M_{BT}, the green line is B_γ and the red line is the curve γ.

4. Singularities of M_{BT} and M_{TB} surfaces

In this section, we study the singularities of M_{TB} *and* M_{BT} surfaces by giving the geometric conditions for their singularities. First of all, we need to define some important terminologies of the singularity theory in $\mathbb{R}^3$. Let $f : \mathbb{N}^n \to \mathbb{M}^m$ be a smooth map, where $\mathbb{N}^n$ and $\mathbb{M}^m$ are two manifolds of dimensions n and m respectively. Then the rank of f is defined as the rank of df. At a point p, the map f is a singular if the $rank\,(f) < min\,(n, m)$. Let $f,\ g : (\mathbb{R}^p, 0) \to (\mathbb{R}^q, 0)$ be two map germs. We say that $f,\ g$ are $\mathcal{A}$-*equivalent* if there exist diffeomorphism germs $\psi : (\mathbb{R}^p, 0) \to (\mathbb{R}^p, 0)$ *and* $\phi : (\mathbb{R}^q, 0) \to (\mathbb{R}^q, 0)$ such that $\phi \circ f = g \circ \psi$. A smooth map $M : U \to \mathbb{R}^3$ is called a *wave front* if there is a unit vector $v \in \mathbb{R}^3$ along M such that $L_M = (M, v) : U \to T_1\mathbb{R}^3$ is a Legendrian immersion with respect to the canonical contact structure on $T_1\mathbb{R}^3$, where $T_1\mathbb{R}^3$ is the unit tangent bundle over $\mathbb{R}^3$. For a wave front M, a function $\lambda : U \to \mathbb{R}$ defined by $\lambda(x, y) = det(M_x, M_y, v)$ is called the *density function*, where (x, y) is a local coordinate system of U (see [12]). A singular point $p \in U$ of M is said to be a $non-degenerate$ if $d\lambda(p) \neq 0$. If p is a $non-degenerate$ singular point of a wave front M, then there is a direction $\eta(t) \in T_{\gamma(t)}U$ such that $dM(\eta(t)) = 0$. The direction $\eta(t)$ is called the *null vector field* (see [12]). The cuspidal edge (CE), the swallowtail (SW) and the cuspidal butterfly (CBF) singularities are map-germs $\mathcal{A}$-equivalent to $CE : (x, y) \longrightarrow (x^3, x^2, y),\ SW : (x, y) \longrightarrow (3x^4 + x^2y, 4x^3 + 2xy, y)$ *and* $CBF : (x, y) \longrightarrow (4x^5 + x^2y, 5x^4 + 2xy, y)$ at the origin, respectively (see Figure 3).



(A) Cuspidal edge (B) Swallowtail (C) Cuspidal butterfly

Figure 3

Proposition 4.1. *Let $\gamma : I \to \mathbb{R}^3$ be a unit speed curve with $\kappa_\gamma \neq 0$ and $\tau_\gamma \neq 0$. The map $M_{TB} : I \times \mathbb{R} \to \mathbb{R}^3$, given by $M_{TB} = T_\gamma + uB_\gamma$, is a wave front.*

Proof. The partial derivative of the surface M_{TB} can be calculated as follows

$$\frac{\partial M_{TB}(s,u)}{\partial s} = N_\gamma(\kappa_\gamma - u\tau_\gamma)\ and\ \frac{\partial M_{TB}(s,u)}{\partial u} = B_\gamma,$$

so that we have

$$\frac{\partial M_{TB}(s,u)}{\partial s} \times \frac{\partial M_{TB}(s,u)}{\partial u} = T_\gamma(\kappa_\gamma - u\tau_\gamma).$$

From the above formula, we observe that the unit vector T_γ is always normal to M_{TB}. The Legendrian mapping over M_{TB} is defined by $L_{M_{TB}} = (M_{TB}, T_\gamma)$:

$I \times \mathbb{R} \to \mathbb{R}^3 \times S^2$. The jacobian matrix of $L_{M_{TB}}$ is given by

$$JL_{M_{TB}} = \begin{pmatrix} (\kappa_\gamma - u\tau_\gamma) N_\gamma & \kappa_\gamma N_\gamma \\ B_\gamma & 0 \end{pmatrix}.$$

Since $\kappa_\gamma \neq 0$, the jacobian matrix of $L_{M_{TB}}$ has a maximal rank. Therefore, $L_{M_{TB}}$ is an immersion which implies that M_{TB} is a wave front. □

Proposition 4.2. *Let $\gamma : I \to \mathbb{R}^3$ be a unit speed curve with $\kappa_\gamma \neq 0$ and $\tau_\gamma \neq 0$. The map $M_{BT} : I \times \mathbb{R} \to \mathbb{R}^3$, given by $M_{BT} = B_\gamma + uT_\gamma$, is a wave front.*

Proof. By the similar steps to the proof of Proposition 4.1 we can easily show this proposition. □

Arnold [3] studied the wavefronts and showed that the generic singularities of wavefronts in $\mathbb{R}^3$ are cuspidal edges or swallowtails. The criteria for cuspidal edges and swallowtails are given in [11] as:

Theorem 4.3 ([11])**.** *Let $M : U \to \mathbb{R}^3$ be a wave front and $p \in U$ be a non-degenerate singular point of M. Then the following assertions hold:*

(1) M at p is $\mathcal{A}$-equivalent to a cuspidal edge if and only if $\eta\lambda(p) \neq 0$.
(2) M at p is $\mathcal{A}$-equivalent to a swallowtail if and only if $\eta\lambda(p) = 0$ and $\eta\eta\lambda(p) \neq 0$.

Moreover, the wavefronts have other singularities rather than cuspidal edge and swallowtail, for instance, cuspidal butterfly singularities. Izumiya and Saji [10] presented the criteria for cuspidal butterfly singularities as the following theorem.

Theorem 4.4 ([10])**.** *Let $M : U \to \mathbb{R}^3$ be a wave front and $p \in U$ be a non-degenerate singular point of M. Then the wave front M at p is $\mathcal{A}$-equivalent to a cuspidal butterfly if and only if $\eta\lambda(p) = 0$, $\eta\eta\lambda(p) = 0$ and $\eta\eta\eta\lambda(p) \neq 0$.*

Now, we give the geometric conditions for M_{TB} *and* M_{BT} surfaces to have cuspidal edges, swallowtails and cuspidal butterflies singularities. Katsumi, Izumiya and Yamasaki [15] studied the cuspidal edges and swallowtails for M_{BT} surface as in Theorem 4.9 in this paper.

Remark 4.5. *The singular set of M_{TB} is the set $\{(s, \kappa_\gamma(s)/\tau_\gamma(s)) \,|\, s \in I\}$ and the singular set of M_{BT} is the set $\{(s, \tau_\gamma(s)/\kappa_\gamma(s)) \,|\, s \in I\}$.*

Theorem 4.6. *Let $\gamma : I \to \mathbb{R}^3$ be a unit speed space curve with $\kappa_\gamma \neq 0$ and $\tau_\gamma \neq 0$. Then M_{TB} is diffeomorphic to a cuspidal edge at $p = (s_0, \kappa_\gamma(s_0)/\tau_\gamma(s_0))$ if and only if $(\kappa_\gamma/\tau_\gamma)' \neq 0$ at s_0.*

Proof. By a direct calculation, the density function λ of the wave front M_{TB} is given by $\lambda = \kappa_\gamma - u\tau_\gamma$ and the null vector field η is given by $\eta = \partial/\partial s$. Thus, by using Theorem 4.3, we show that M_{TB} is diffeomorphic to a cuspid edge if and only if $\eta\lambda \neq 0$ if and only if $\partial/\partial s(\kappa_\gamma - u\tau_\gamma) \neq 0$ that is $\kappa'_\gamma - u\tau'_\gamma \neq 0$, where $u = \kappa_\gamma/\tau_\gamma$. It is equivalent to $\dfrac{\tau_\gamma\kappa'_\gamma - \kappa_\gamma\tau'_\gamma}{\tau_\gamma} \neq 0$. This means that $(\kappa_\gamma/\tau_\gamma)' \neq 0$ at s_0. □

Theorem 4.7. *Let $\gamma : I \to \mathbb{R}^3$ be a unit speed space curve with $\kappa_\gamma \neq 0$ and $\tau_\gamma \neq 0$. If $(\kappa_\gamma/\tau_\gamma)' = 0$ at s_0, then M_{TB} is diffeomorphic to a swallowtail at $p = (s_0, \kappa_\gamma(s_0)/\tau_\gamma(s_0))$ if and only if $(\kappa_\gamma/\tau_\gamma)'' \neq 0$ at s_0.*

Proof. By the proof of Theorem 4.6, we have $\lambda = \kappa_\gamma - u\tau_\gamma$ *and* $\eta = \partial/\partial s$. Therefore, $\eta\lambda = \kappa'_\gamma - u\tau'_\gamma$ and $\eta\eta\lambda = \kappa''_\gamma - u\tau''_\gamma$. At a singular point $p = (s_0, \kappa_\gamma(s_0)/\tau_\gamma(s_0))$, $\eta\lambda = \tau_\gamma \left(\frac{\kappa_\gamma}{\tau_\gamma}\right)'$ and $\eta\eta\lambda = \kappa''_\gamma - \left(\dfrac{\kappa_\gamma}{\tau_\gamma}\right)\tau''_\gamma$. By the assumption $\left(\frac{\kappa_\gamma}{\tau_\gamma}\right)' = 0$ at s_0, it can be easily shown that $\eta\eta\lambda \neq 0$ at p if and only if $(\kappa_\gamma/\tau_\gamma)'' \neq 0$ at s_0. Therefore, M_{TB} is diffeomorphic to a swallowtail at p if and only if $(\kappa_\gamma/\tau_\gamma)'' \neq 0$ at s_0, which completes the proof. □

Theorem 4.8. *Let* $\gamma : I \to \mathbb{R}^3$ *be a unit speed space curve with* $\kappa_\gamma \neq 0$ *and* $\tau_\gamma \neq 0$. *If* $(\kappa_\gamma/\tau_\gamma)' = 0$ *and* $(\kappa_\gamma/\tau_\gamma)'' = 0$ *at* s_0, *then* M_{TB} *is diffeomorphic to a cuspidal butterfly at* $p = (s_0, \kappa_\gamma(s_0)/\tau_\gamma(s_0))$ *if and only if* $(\kappa_\gamma/\tau_\gamma)''' \neq 0$ *at* s_0.

Proof. By the proof of Theorem 4.6, we have $\lambda = \kappa_\gamma - u\tau_\gamma$ *and* $\eta = \partial/\partial s$. Therefore, $\eta\lambda = \kappa'_\gamma - u\tau'_\gamma$, $\eta\eta\lambda = \kappa''_\gamma - u\tau''_\gamma$ and $\eta\eta\eta\lambda = \kappa'''_\gamma - u\tau'''_\gamma$. At a singular point $p = (s_0, \kappa_\gamma(s_0)/\tau_\gamma(s_0))$, $\eta\lambda = \tau_\gamma\left(\frac{\kappa_\gamma}{\tau_\gamma}\right)'$, $\eta\eta\lambda = \kappa''_\gamma - \left(\dfrac{\kappa_\gamma}{\tau_\gamma}\right)\tau''_\gamma$ and $\eta\eta\eta\lambda = \kappa'''_\gamma - \left(\dfrac{\kappa_\gamma}{\tau_\gamma}\right)\tau'''_\gamma$. Therefore, by using Theorem 4.4, we show that M_{TB} is diffeomorphic to a cuspidal butterfly at p if and only if $\eta\lambda = 0$, $\eta\eta\lambda = 0$ *and* $\eta\eta\eta\lambda \neq 0$. So by the assumptions $(\kappa_\gamma/\tau_\gamma)' = 0$ *and* $(\kappa_\gamma/\tau_\gamma)'' = 0$ at s_0 which implies that $\eta\lambda = 0$ *and* $\eta\eta\lambda = 0$, it can be easily shown that $\eta\eta\eta\lambda \neq 0$ if and only if $(\kappa_\gamma/\tau_\gamma)''' \neq 0$ at s_0. Since $\eta\lambda = \eta\eta\lambda = 0$, M_{TB} is diffeomorphic to a cuspidal butterfly if and only if $\eta\eta\eta\lambda \neq 0$ which is equivalent to $(\kappa_\gamma/\tau_\gamma)''' \neq 0$ at s_0. This ends the proof. □

Theorem 4.9 ([15]). *Let* $\gamma : I \to \mathbb{R}^3$ *be a unit speed space curve with* $\kappa_\gamma \neq 0$ *and* $\tau_\gamma \neq 0$. *Then the following assertions hold:*

(1) M_{BT} *is diffeomorphic to a cuspidal edge at* $p = (s_0, \tau_\gamma(s_0)/\kappa_\gamma(s_0))$ *if and only if* $(\tau_\gamma/\kappa_\gamma)' \neq 0$ *at* s_0.
(2) M_{BT} *is diffeomorphic to a swallowtail at* $p = (s_0, \tau_\gamma(s_0)/\kappa_\gamma(s_0))$ *if and only if* $(\tau_\gamma/\kappa_\gamma)' = 0$ *and* $(\tau_\gamma/\kappa_\gamma)'' \neq 0$ *at* s_0.

Theorem 4.10. *Let* $\gamma : I \to \mathbb{R}^3$ *be a unit speed space curve with* $\kappa_\gamma \neq 0$ *and* $\tau_\gamma \neq 0$. *If* $(\tau_\gamma/\kappa_\gamma)' = 0$ *and* $(\tau_\gamma/\kappa_\gamma)'' = 0$ *at* s_0, *then* M_{BT} *is diffeomorphic to a cuspidal butterfly at* $p = (s_0, \tau_\gamma(s_0)/\kappa_\gamma(s_0))$ *if and only if* $(\tau_\gamma/\kappa_\gamma)''' \neq 0$ *at* s_0.

Proof. The proof is given by the method similar to the proof of Theorems 4.8. □

In the rest of this section, we present the criteria for a helix and a slant helix in terms of the image of the singular sets of M_{TB} *and* M_{BT}.

Theorem 4.11. *Let* $\gamma : I \to \mathbb{R}^3$ *be a unit speed space curve with* $\kappa_\gamma \neq 0$ *and* $\tau_\gamma \neq 0$. *Then* γ *is a helix if and only if the image of the singular set of* $M_{TB} = T_\gamma + uB_\gamma$ *is a single point (i.e.,* M_{TB} *is a cone).*

Proof. Since the singular set of M_{TB} is the set $\{(s, \kappa_\gamma(s)/\tau_\gamma(s)) \,|\, s \in I\}$, the image of the singular set of M_{TB} is given by $\Omega_{TB} = T_\gamma + (\kappa_\gamma/\tau_\gamma) B_\gamma$. By taking the derivative of Ω_{TB}, we find that $\Omega'_{TB} = (\kappa_\gamma/\tau_\gamma)' B_\gamma$. To show that Ω_{TB} is a single point, it is enough to show that $\Omega'_{TB} = 0$ for all $s \in I$. Now, $\Omega'_{TB} = 0$ for all $s \in I$ if and only if $(\kappa_\gamma/\tau_\gamma)' B_\gamma = 0$ for all $s \in I$, which is equivalent to $(\kappa_\gamma/\tau_\gamma)' = 0$ for all $s \in I$. This means that $(\kappa_\gamma/\tau_\gamma) = c$, where c is a constant, which is the condition that $\gamma(s)$ is a helix. □

Theorem 4.12. *Let $\gamma : I \to \mathbb{R}^3$ be a unit speed space curve with $\kappa_\gamma \neq 0$ and $\tau_\gamma \neq 0$. Then γ is a helix if and only if the image of the singular set of $M_{BT} = B_\gamma + uT_\gamma$ is a single point (i.e., M_{BT} is a cone).*

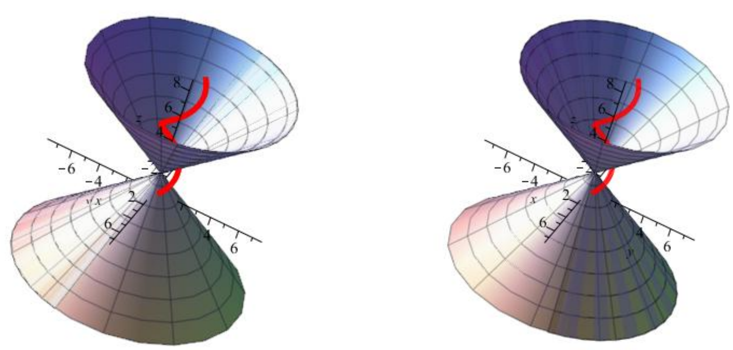
Proof. The proof is given by the method similar to the proof of Theorem 4.11. $\square$

Example 4.13. *Let γ be a regular space curve in $\mathbb{R}^3$ given by*

$$\gamma(t) = \left(cos(t/\sqrt{2}), sin(t/\sqrt{2}), t/\sqrt{2}\right).$$

By a direct calculation, we easily get that $\kappa_\gamma = 1/2$ and $\tau_\gamma = 1/2$. Since $\kappa_\gamma/\tau_\gamma = \tau_\gamma/\kappa_\gamma = 1$, γ is a helix. Hence, the surfaces M_{TB} and M_{BT} are cones (see Figure 4) given by

$$M_{TB} = \left(1/\sqrt{2}\sin\left(t/\sqrt{2}\right)(u-1), -1/\sqrt{2}\cos\left(t/\sqrt{2}\right)(u-1), 1/\sqrt{2}(u+1)\right),$$
$$M_{BT} = \left(-1/\sqrt{2}\sin\left(t/\sqrt{2}\right)(u-1), 1/\sqrt{2}\cos\left(t/\sqrt{2}\right)(u-1), 1/\sqrt{2}(u+1)\right).$$



(A) M_{TB} surface and the red line is the curve γ

(B) M_{BT} surface and the red line is the curve γ.

FIGURE 4. The surfaces M_{TB} *and* M_{BT} of Example 4.13.

Theorem 4.14. *Let $\gamma : I \to \mathbb{R}^3$ be a unit speed space curve with $\kappa_\gamma \neq 0$ and $\tau_\gamma \neq 0$. Then γ is a slant helix if and only if the image of the singular set of M_{TB}, given by $\Omega_{TB} = T_\gamma + (\kappa_\gamma/\tau_\gamma) B_\gamma$, is a slant helix.*

Proof. The unit tangent of Ω_{TB} is given by

$$T_{\Omega_{TB}} = \frac{\Omega'_{TB}}{\|\Omega'_{TB}\|} = B_\gamma$$

and the unit principal normal of Ω_{TB} is given by

$$N_{\Omega_{TB}} = \frac{T'_{\Omega_{TB}}}{\|T'_{\Omega_{TB}}\|} = -N_\gamma.$$

So by the definition of a slant helix, Ω_{TB} is a slant helix if and only if $N_{\Omega_{TB}} \cdot w = constant$, where w is a fixed direction, which is equivalent to $N_\gamma \cdot w = constant$, which is the condition that γ is a slant helix. $\square$

Theorem 4.15. *Let $\gamma : I \to \mathbb{R}^3$ be a unit speed space curve with $\kappa_\gamma \neq 0$ and $\tau_\gamma \neq 0$. Then γ is a slant helix if and only if the image of the singular set of M_{BT} is a slant helix.*

Proof. The proof is given by the method similar to the proof of Theorem 4.14. □

Theorem 4.16. *Let $\gamma : I \to \mathbb{R}^3$ be a unit speed space curve with $\kappa_\gamma \neq 0$ and $\tau_\gamma \neq 0$. If γ is a rectifying curve, then at any singular point the surface M_{BT} is diffeomorphic to a cuspidal edge.*

Proof. Let $\gamma : I \to \mathbb{R}^3$ be a unit speed space curve with $\kappa_\gamma \neq 0$ and $\tau_\gamma \neq 0$. If γ is a rectifying curve, then $(\tau_\gamma/\kappa_\gamma) = as + b$, where a and b are non-zero constants (see [5]). By taking the derivative, we get $(\tau_\gamma/\kappa_\gamma)' = a$, which implies that $(\tau_\gamma/\kappa_\gamma)' \neq 0$ for all $s \in I$. For any singular point $p = (s_0, \tau_\gamma(s_0)/\kappa_\gamma(s_0))$, we have $(\tau_\gamma/\kappa_\gamma)' = a \neq 0$ at s_0. Therefore, M_{BT} is diffeomorphic to a cuspidal edge at any singular point. □

Remark 4.17. *If M_{BT} has a cuspidal edge singularity at any singular point, it does not mean that γ is a rectifying curve.*

Proposition 4.18. *Let $\gamma : I \to \mathbb{R}^3$ be a unit speed space curve with $\kappa_\gamma \neq 0$ and $\tau_\gamma \neq 0$ and let γ be a Bertrand curve. Then M_{BT} is diffeomorphic to a cuspidal edge at $p = (s_0, \tau_\gamma(s_0)/\kappa_\gamma(s_0))$ if and only if $\tau'_\gamma \neq 0$ at s_0.*

Proof. Let $\gamma : I \to \mathbb{R}^3$ be a Bertrand curve with $\kappa_\gamma \neq 0$ and $\tau_\gamma \neq 0$. In [8], Izumiya and Takeuchi showed that there is a real number $A \neq 0$ such that $A\left(\tau'_\gamma \kappa_\gamma - \kappa'_\gamma \tau_\gamma\right) = \tau'_\gamma$. This is equivalent to

$$A\kappa_\gamma^2 \left(\tau_\gamma/\kappa_\gamma\right)' = \tau'_\gamma.$$

Then from Theorem 4.9, M_{BT} is diffeomorphic to a cuspidal edge at $p = (s_0, \tau_\gamma(s_0)/\kappa_\gamma(s_0))$ if and only if $(\tau_\gamma/\kappa_\gamma)' \neq 0$ at s_0, which is equivalent to $\tau'_\gamma \neq 0$ at s_0. □

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