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SECOND ORDER COHOMOLOGY OF MATRIX BANACH ALGEBRAS WITH RESPECT TO CHARACTERS

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Abstract. In this paper, we investigate the second order cohomology of Banach algebras $\mathfrak{E}_p(I)$ $(1 \leq p < \infty)$ with coefficients corresponding to characters. Consequently, the necessary and sufficient conditions are established for contractibility of these Banach algebras. As an application, we investigate weak character amenability of $A(G)$ and contractibility of $L^2(G)$, when in either case G is a compact group.

1. Introduction

The Banach algebras $\mathfrak{E}_p(I)$, where $p \in [1, \infty] \cup \{0\}$, have been introduced and extensively studied in Section 28 of [8]. Amenability, weak amenability and approximate weak amenability of these Banach algebras have been studied by H. Samea in [15](see also [11]). The concept of φ-amenability was also first introduced by Kaniuth, Lau, and Pym (see [9], [10]), and simultaneously by Monfared (see [12])and weak φ-amenability was recently studied for a Banach algebra A by R. Nasr-Isfahani, S. Shahmoradi and S. Soltani Renani([13]). In this paper, we investigate weak character amenability for the subalgebras of $\mathfrak{E}(I)$ for which their character space is non-empty, together with their applications to a number of convolution Banach algebras on compact groups. The paper concluded by studying contractibility of Banach algebras $\mathfrak{E}_p(I)$ $(1 \leq p < \infty)$.

Let I be an arbitrary index set. Suppose that $\{H_i;\ i \in I\}$ be a family of Hilbert spaces H_i of dimension d_i $(i \in I)$ and consider $\mathcal{B}(H_i)$ is the space of all bounded linear operators on H_i. Clearly we can identify $\mathcal{B}(H_i)$ with $\mathbb{M}_{d_i}(\mathbb{C})$ (the space of all $d_i \times d_i$-matrices on $\mathbb{C}$). The set $\mathfrak{E}(I) := \prod_{i \in I} \mathcal{B}(H_i)$ is a $*$-algebra, the scaler multiplication, addition, multiplication, and the adjoint of an element, are defined coordinatewise. In $\mathfrak{E}(I)$ one introduces various norms:
for $E := (E_i)_{i \in I} \in \mathfrak{E}(I)$ and $(a_i)_{i \in I}$ with $a_i \geq 1$ for all $i \in I$;

$$\|E\|_p := \begin{cases} \left(\sum_{i \in I} a_i \|E_i\|_{\varphi_p}^p\right)^{\frac{1}{p}} & \text{if } 1 \leq p < \infty; \\ \sup_{i \in I}\ \|E_i\|_{\varphi_\infty} & \text{if } p = \infty, \end{cases}$$

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where

$$\|E_i\|_{\varphi_p} = \left(\sum_{i=1}^{d_i} |\lambda_i|^p\right)^{\frac{1}{p}} (1 \le p < \infty),\ \|E_i\|_{\varphi_\infty} = \max_{1\le i\le d_i} |\lambda_i|,$$

and $(\lambda_1, \ldots, \lambda_{d_i})$ is the sequence of eigenvalues of the positive-definite operator $|E_i| := (E_i\tilde{E}_i)^{\frac{1}{2}}$ (for $E \in \mathbb{M}_{d_i}(\mathbb{C})$, $\tilde{E} \in \mathbb{M}_{d_i}(\mathbb{C})$ is defined by $(\tilde{E})_{ij} = \overline{E_{ji}}$ $(1 \le i, j \le d_i)$). For more details see Definition D.37 and Theorem D.40 in [8]. Let for $1 \le p \le \infty$,

$$\mathfrak{E}_p(I) := \{E \in \mathfrak{E}(I);\ \|E\|_p < \infty\}.$$

By Theorems 28.25 and 28.32(v) of [8], $(\mathfrak{E}_p(I), \|\cdot\|_p)(1 \le p \le \infty)$ are Banach algebras.

2. General results

In this paper, we determine the conditions under which the Banach algebras $\prod_{i\in I} \mathbb{M}^{UT}_{d_i}(\mathbb{C})$, $\mathbb{M}^{UT}_{d_i}(\mathbb{C})$ is the space of all upper triangular $d_i \times d_i$-matrices , and $\prod_{i\in I} \mathbb{M}^{d}_{d_i}(\mathbb{C})$, $\mathbb{M}^{d}_{d_i}(\mathbb{C})$ is the space of all diagonal $d_i \times d_i$-matrices whose entries are in $\mathbb{C}$, are weak character amenable.
Throughout this paper the character space of A is denoted by $\Delta(A)$, that is, all non-zero multiplicative linear functionals on A.
Note, in general, $\Delta(\mathfrak{E}_p(I)) = \emptyset$, because for the case that I is singleton, $\Delta(\mathfrak{E}_p(I)) = \Delta(\mathbb{M}_{d_i}(\mathbb{C})) = \emptyset$.
By the fact that $\Delta(\mathbb{M}^{UT}_{d_i}(\mathbb{C})) = \{\varphi_m;\ 1 \le m \le d_i\}$ such that for $M = [\lambda_{ij}] \in \mathbb{M}^{UT}_{d_i}(\mathbb{C})$, $\varphi_m(M) = \lambda_{mm}$, we introduce

$$\Delta(\prod_{i\in I} \mathbb{M}^{UT}_{d_i}(\mathbb{C})) = \{\varphi_{m_0} \circ \pi_{i_0};\ \varphi_{m_0} \in \Delta(\mathbb{M}^{UT}_{d_i}(\mathbb{C}))\},$$

where for $i_0 \in I$, π_{i_0} is the projection map from $\prod_{i\in I} \mathbb{M}^{UT}_{d_i}(\mathbb{C})$ onto $\mathbb{M}^{UT}_{d_{i_0}}(\mathbb{C})$ and $1 \le m_0 \le d_{i_0}$. It is easy to see that $\varphi_{m_0} \circ \pi_{i_0}$ $(1 \le m_0 \le d_{i_0})$ is a character on $\prod_{i\in I} \mathbb{M}^{UT}_{d_i}(\mathbb{C})$. Now, we show that any non-zero character $\varphi \in \Delta(\prod_{i\in I} \mathbb{M}^{UT}_{d_i}(\mathbb{C}))$ is of the form $\varphi_{m_0} \circ \pi_{i_0}$.
First, note for $j, k \in \{1, \ldots, d_i\}$, $\mathcal{E}_{jk}$ is a matrix that its jk-th entry is 1 and all the other entries are zero and

$$\mathcal{E}_{mj}\mathcal{E}_{kn} = \begin{cases} \mathcal{E}_{mn}, & j = k; \\ 0, & \text{otherwise} \end{cases}.$$

Each basis element for $\mathfrak{E}(I)$, has the form

$$(\mathcal{E}^{\ell}_{mn})_i = \begin{cases} \mathcal{E}_{mn}, & \text{if } \ell = i; \\ 0, & \text{otherwise,} \end{cases}$$

also

$$(\mathcal{E}^{i}_{mj})_i(\mathcal{E}^{i}_{kn})_i = \begin{cases} (\mathcal{E}^{i}_{mn})_i, & j = k \\ 0, & \text{otherwise} \end{cases}; \tag{2.1}$$

where $1 \le m, n, j, k \le d_i$ and $i, \ell \in I$.

Let $E \in \prod_{i\in I} \mathbb{M}^{UT}_{d_i}(\mathbb{C})$ with $E = \sum_{i\in I}\sum_{j=1}^{d_i}\sum_{k=1}^{d_i} \lambda_{jk}(\mathcal{E}^i_{jk})$ and $\varphi \in \Delta(\prod_{i\in I} \mathbb{M}^{UT}_{d_i}(\mathbb{C}))$, then

$$\varphi(E) = \sum_{i\in I}\sum_{j=1}^{d_i}\sum_{k=1}^{d_i} \lambda_{jk}\varphi((\mathcal{E}^i_{jk})).$$

By 2.1, if $j \neq k$, then $\varphi((\mathcal{E}^i_{jk})) = 0$. But one of $\varphi((\mathcal{E}^i_{11})), \cdots, \varphi((\mathcal{E}^i_{d_i d_i}))$ is nonzero, otherwise $\varphi = 0$ and this is a contradiction. Let $\varphi((\mathcal{E}^{i_0}_{11})) \neq 0$ and $j \neq 1$ or $i \neq i_0$. Since φ is multiplicative, $\varphi((\mathcal{E}^{i_0}_{11}))\varphi((\mathcal{E}^i_{11})) = \varphi((\mathcal{E}^{i_0}_{11}\mathcal{E}^i_{jj})) = 0$ and then $\varphi((\mathcal{E}^i_{jj})) = 0$. On the other hand since $\varphi((\mathcal{E}^{i_0}_{11}))\varphi((\mathcal{E}^{i_0}_{11})) = \varphi((\mathcal{E}^{i_0}_{11}))$, then $\varphi((\mathcal{E}^{i_0}_{11})) = 1$. This implies that $\varphi(E) = \lambda^{i_0}_{11}$, so we have $\varphi = \varphi_1 \circ \pi_{i_0}$. Similarly if $\varphi((\mathcal{E}^i_{mm})) \neq 0$, we have $\varphi = \varphi_m \circ \pi_i$; $(1 \leq m \leq d_i,\ i \in I)$.

In fact, only $\prod_{i\in I} \mathbb{M}^{UT}_{d_i}(\mathbb{C})$ or $\prod_{i\in I} \mathbb{M}^{LT}_{d_i}(\mathbb{C})$ (the product space of Lower triangular matrices) has non-zero characters.

Notation 2.1. *Let $\mathbb{C}$ be the Banach space of complex numbers. We denote by ${}_0\mathbb{C}_\varphi$, the Banach A-bimodule endowed with the following operations*

$$a \cdot \lambda = 0,\ \ \lambda \cdot a = \varphi(a)\lambda$$

for all $a \in A, \lambda \in \mathbb{C}$ and $\varphi \in \Delta(A) \cup \{0\}$.

In this section, we state the conditions under which $\mathcal{H}^2(\mathfrak{E}_p(I), {}_0\mathbb{C}_\varphi)$ equals to zero, for $\varphi \in \Delta(\mathfrak{E}_p(I))$ and $1 \leq p < \infty$. For this purpose, we require essential results mentioned in [14].

Definition 2.2. *Let A be a Banach algebra, and E be a Banach A-bimodule. For $n \in \mathbb{N}_0$, let*

$$\mathcal{H}^n(A, E) := \mathcal{Z}^n(A, E)/\mathcal{N}^n(A, E)$$

is defined n-th Hochschild cohomology group of A with coefficients in E, where $\mathcal{Z}^n(A, E)$ is the group of n-cocycles *and $\mathcal{N}^n(A, E)$ is the group of* n-coboundaries.

Proposition 2.3. *Let A be an algebra and E be a A-bimodule. Then*

$$\mathcal{H}^{k+p}(A, E) \simeq \mathcal{H}^k(A, (\mathcal{L}^p(A, E), \star));\ \ k, p \in \mathbb{N}$$

for $a, b \in A$ and $T \in \mathcal{L}(A, E)$,

$$(a \star T)(b) = a \cdot Tb,\ \ (T \star a)(b) = T(ab) - Ta \cdot b,$$

where $\mathcal{L}^n(A, E) := \{T : A^n \longrightarrow E;\ T \text{ is bounded and n-linear}\}$, the elements of $\mathcal{L}^n(A, E)$ are called *n-cochains*.

From now on, we're going to identify for which subalgebras A of $\mathfrak{E}(I)$ and under what conditions, $\mathcal{H}^2(A, {}_0\mathbb{C}_\varphi) = \{0\}$.
For the subalgebra $\mathfrak{E}_p(I)$ of $\mathfrak{E}(I)$, $\mathcal{H}^2(\mathfrak{E}_p(I), {}_0\mathbb{C}_\varphi) = \mathcal{H}^1(\mathfrak{E}_p(I), \mathcal{L}^1(\mathfrak{E}_p(I), {}_0\mathbb{C}_\varphi))$, that by Lemma 28.28 of [8], $\mathcal{L}^1(\mathfrak{E}_p(I), {}_0\mathbb{C}_\varphi))$ is the dual Banach $\mathfrak{E}_p(I)$-bimodule $(\mathfrak{E}_p(I))^* = \mathfrak{E}_q(I)$ with the following module operations:

$$E \cdot T_F = 0, \qquad T_F \cdot E = T_F E - T_F(E)\varphi \tag{2.2}$$

where $(T_F E)(G) = T_F(EG)$ for $E \in \mathfrak{E}_p(I),\ T_F \in \mathfrak{E}_q(I)$.
The following results are frequently used in the rest of the paper.
Suppose that D be a derivation on $\mathfrak{E}_p(I)$. For basis elements $(\mathcal{E}^i_{mn})_i \in \mathfrak{E}_p(I)$ and fixed $i \in I$, we have:

$$\begin{aligned} D\big((\mathcal{E}^i_{mn})(\mathcal{E}^i_{jk})\big) &= (\mathcal{E}^i_{mn})D\big((\mathcal{E}^i_{jk})\big) + D\big((\mathcal{E}^i_{mn})\big)(\mathcal{E}^i_{jk}) \\ &= 0 + D\big((\mathcal{E}^i_{mn})\big)(\mathcal{E}^i_{jk}) - \langle D\big((\mathcal{E}^i_{mn})\big), (\mathcal{E}^i_{jk})\rangle\varphi, \end{aligned}$$

therefore if $n \neq j$,

$$\begin{aligned} 0 &= D(0) = D((\mathcal{E}^i_{mn})(\mathcal{E}^i_{jk})) \\ &= D((\mathcal{E}^i_{mn}))(\mathcal{E}^i_{jk}) - \langle D((\mathcal{E}^i_{mn})), (\mathcal{E}^i_{jk})\rangle\varphi \end{aligned}$$

this follows that

$$\langle D((\mathcal{E}^i_{mn})), (\mathcal{E}^i_{jk})\rangle = 0 \tag{2.3}$$

and if $j \neq k$,

$$\langle D((\mathcal{E}^i_{jk})), (\mathcal{E}^i_{jk})\rangle = 0. \tag{2.4}$$

Similarly for $j, k \neq m_0$,

$$0 = D((\mathcal{E}^i_{m_0m_0}))(\mathcal{E}^i_{jk}) - \langle D((\mathcal{E}^i_{m_0m_0})), (\mathcal{E}^i_{jk})\rangle\varphi.$$

So

$$\langle D((\mathcal{E}^i_{m_0m_0})), (\mathcal{E}^i_{jk})\rangle = 0. \tag{2.5}$$

But if $m = n = j = k$,

$$\begin{array}{rcl} D((\mathcal{E}^i_{mn})) & = & D((\mathcal{E}^i_{mn})(\mathcal{E}^i_{mn})) \\ & = & D((\mathcal{E}^i_{mn}))(\mathcal{E}^i_{mn}) - \langle D((\mathcal{E}^i_{mn})), (\mathcal{E}^i_{mn})\rangle\varphi, \end{array}$$

thus

$$\langle D((\mathcal{E}^i_{mm})), (\mathcal{E}^i_{m_0})\rangle = 0 - \langle D((\mathcal{E}^i_{mm})), (\mathcal{E}^i_{mm})\rangle = -\langle D((\mathcal{E}^i_{mm})), (\mathcal{E}^i_{mm})\rangle. \tag{2.6}$$

According to the Lemma 28.28 of [8], the effect of $D((\mathcal{E}^i_{mn}))$, as an element of $\mathfrak{E}_q(I)$, on the basic element $(\mathcal{E}^i_{mn})$ is determined as follows.

$$\begin{array}{rcll} \langle D((\mathcal{E}^i_{mn})), (\mathcal{E}^i_{mn})\rangle & = & \sum_{i\in I} a_i tr(\mathcal{E}_{mn}(D((\mathcal{E}^i_{mn})))_i) & \\ & = & a_i tr(\mathcal{E}_{mn}(D((\mathcal{E}^i_{mn})))_i) & \text{(because of the nonzero component)} \\ & = & a_i((D((\mathcal{E}^i_{mn})))_i)_{mn}. & \end{array} \tag{2.7}$$

Let $\mathfrak{T}$ be an ideal in $\mathfrak{E}_p(I)$. Define

$$\mathfrak{T}_i = \{A \in \mathbb{M}_{d_i}(\mathbb{C}) :\ \exists \tilde{A} \in \mathfrak{T}\ s.t\ \tilde{A}_i = A\}.$$

Proposition 2.4. *Each ideal of* $\mathfrak{E}_p(I)$ $(1 \leq p < \infty)$ *is of the form* $\mathfrak{E}_p(I')$*, where* $I' = \{i \in I;\ \mathfrak{T}_i = \mathbb{M}_{d_i}(\mathbb{C})\}$.

Remark. *It is not hard to show that the set* $\big(\prod_{i\in I} \mathbb{M}^d_{d_i}(\mathbb{C}), \|\cdot\|_p\big)$ *is an ideal in* $\mathfrak{E}_p(I)$*. Therefore by Proposition 2.4, there exist a subset* $I' \subset I$ *such that*

$$\big(\prod_{i\in I} \mathbb{M}^d_{d_i}(\mathbb{C}), \|\cdot\|_p\big) = \mathfrak{E}_p(I').$$

3. Main results

Hereafter, we denote $\big(\prod_{i\in I} \mathbb{M}^d_{d_i}(\mathbb{C}), \|\cdot\|_p\big)$ by $\mathfrak{E}_p(I')$ and its dual by $\mathfrak{E}_q(I')$, as a Banach $\mathfrak{E}_p(I')$-bimodule with the same module operations 2.2.
We are thus led to the following result:

Theorem 3.1. *suppose* $\varphi \in \Delta(\mathfrak{E}_1(I')) \cup \{0\}$*. Then* $\mathcal{H}^2(\mathfrak{E}_1(I'), {}_0\mathbb{C}_\varphi) = \{0\}$ *if* $\sup_{i\in I'} a_i < \infty$.

Proof. Let $\sup_{i\in I'} a_i < \infty$ and $D : \mathfrak{E}_1(I') \longrightarrow \mathfrak{E}_\infty(I')$ be a derivation. Then for $E, G \in \mathfrak{E}_1(I')$;

$$E = \sum_{i\in I'} \sum_{m=1}^{d_i} \lambda^i_{mm}(\mathcal{E}^i_{mm}),\ G = \sum_{i\in I'} \sum_{j=1}^{d_i} \beta^i_{jj}(\mathcal{E}^i_{jj})$$

by 2.3, 2.5 and 2.6, we have

$$\begin{aligned}
\langle D(E), G\rangle &= \langle D(\sum_{i\in I'}\sum_{m=1}^{d_i}\lambda^i_{mm}(\mathcal{E}^i_{mm})), \sum_{i\in I'}\sum_{j=1}^{d_i}\beta^i_{jj}(\mathcal{E}^i_{jj})\rangle \\
&= \sum_{i\in I'}\sum_{m=1}^{d_i}\lambda^i_{mm}\langle D((\mathcal{E}^i_{mm})), \sum_{i\in I'}\sum_{j=1}^{d_i}\beta^i_{jj}(\mathcal{E}^i_{jj})\rangle \\
&= \sum_{i\in I'}\sum_{j=1}^{d_i}\lambda^i_{jj}\beta^i_{jj}\langle D((\mathcal{E}^i_{jj})), (\mathcal{E}^i_{jj})\rangle \\
&+ \sum_{i\in I'}\sum_{j=1}^{d_i}\beta^{i_0}_{m_0}\lambda^i_{jj}\langle D((\mathcal{E}^i_{jj})), (\mathcal{E}^i_{m_0})\rangle \\
&= \sum_{i\in I'}\sum_{j=1}^{d_i}\lambda^i_{jj}\beta^i_{jj}\langle D((\mathcal{E}^i_{jj})), (\mathcal{E}^i_{jj})\rangle \\
&- \sum_{i\in I'}\sum_{j=1}^{d_i}\beta^{i_0}_{m_0}\lambda^i_{jj}\langle D((\mathcal{E}^i_{jj})), (\mathcal{E}^i_{jj})\rangle \qquad (3.1)
\end{aligned}$$

In the following, we denote by $a_{i,m}$ the coefficients a_i for $\langle D((\mathcal{E}^i_{mm})), (\mathcal{E}^i_{mm})\rangle$ in 2.7. Define $F = (F_i)_{i\in I'}$ by

$$F_i = -\frac{1}{a_i}\sum_{m=1}^{d_i}\big(a_{i,m}((D((\mathcal{E}^i_{mm})))_i)_{mm}\big)\mathcal{E}^i_{mm}$$

where

$$(F_i)_{mm} = -\frac{1}{a_i}a_{i,m}((D((\mathcal{E}^i_{mm})))_i)_{mm}.$$

Therefore $F = -\sum_{i\in I'}\frac{1}{a_i}\sum_{j=1}^{d_i}a_{i,m}((D((\mathcal{E}^i_{mm})))_i)_{mm}(\mathcal{E}^i_{mm})$.

On the one hand, for F defined above, and elements $E, G \in \mathfrak{E}_1(I')$, we have

$$\begin{aligned}
(E\cdot T_F - T_F\cdot E)(G) &= 0 - T_F(EG) + \varphi(G)T_F(E) \\
&= -\sum_{i\in I'}a_i tr(E_iG_i\tilde{F}_i) + \beta^{i_0}_{m_0}\sum_{i\in I'}a_i tr(E_i\tilde{F}_i) \\
&= \sum_{i\in I'}a_i\sum_{j=1}^{d_i}\lambda^i_{jj}\beta^i_{jj}\frac{1}{a_i}a_{i,j}((D((\mathcal{E}^i_{jj})))_i)_{jj} \\
&- \beta^{i_0}_{m_0}\sum_{i\in I'}a_i\sum_{j=1}^{d_i}\lambda^i_{jj}\frac{1}{a_i}a_{i,j}((D((\mathcal{E}^i_{jj})))_i)_{jj} \\
&= \sum_{i\in I'}\sum_{j=1}^{d_i}\lambda^i_{jj}\beta^i_{jj}\langle D((\mathcal{E}^i_{jj})), (\mathcal{E}^i_{jj})\rangle \qquad \text{by (2.7)} \\
&- \sum_{i\in I'}\sum_{j=1}^{d_i}\beta^{i_0}_{m_0}\lambda^i_{jj}\langle D((\mathcal{E}^i_{jj})), (\mathcal{E}^i_{jj})\rangle. \qquad (3.2)
\end{aligned}$$

Thus $\langle D(E), G\rangle = (E\cdot T_F - T_F\cdot E)(G)$.

On the other hand, for each F_i let $(\lambda_{1,F_i}, \ldots, \lambda_{d_i,F_i})$ be the sequence of eigenvalues of the operator $|F_i|$, written in any order. Therefore

$$\lambda_{1,F_i}=\frac{1}{a_i}a_{i,1}\lambda_{1,(D((\mathcal{E}^i_{11})))_i},\ldots,\lambda_{d_i,F_i}=\frac{1}{a_i}a_{i,d_i}\lambda_{d_i,(D((\mathcal{E}^i_{d_id_i})))_i}$$

and

$$\begin{aligned}\|F_i\|_{\varphi_\infty} &= \max\{|\lambda_{1,F_i}|,\ldots,|\lambda_{d_i,F_i}|\}\\ &= \max\{\frac{1}{a_i}a_{i,1}|\lambda_{1,(D((\mathcal{E}^i_{11})))_i}|,\ldots,\frac{1}{a_i}a_{i,d_i}|\lambda_{d_i,(D((\mathcal{E}^i_{d_id_i})))_i}|\}\\ &\le \max\{a_{i,1}\cdot\|(D((\mathcal{E}^i_{11})))_i\|_{\varphi_\infty},\ldots,a_{i,d_i}\cdot\|(D((\mathcal{E}^i_{d_id_i})))_i\|_{\varphi_\infty}\}\\ &\le \max\{a_{i,1}\cdot\|D((\mathcal{E}^i_{11}))\|_\infty,\ldots,a_{i,d_i}\cdot\|D((\mathcal{E}^i_{d_id_i}))\|_\infty\}\\ &\le \|D\|\max\{a_{i,1},\ldots,a_{i,d_i}\}\\ &= \|D\|\cdot a_{i,k_i},\qquad k_i\in\{1,\ldots,d_i\}.\end{aligned}$$

where $a_{i,k_i}=\max\{a_{i,1},\ldots,a_{i,d_i}\}$ and $\lambda_{j,(D((\mathcal{E}^i_{jj})))_i}$ is $j-th$ eigenvalue of the operator $|(D((\mathcal{E}^i_{jj})))_i|$. Therefore

$$\begin{aligned}\|F\|_\infty &= \sup\{\|F_i\|_{\varphi_\infty}\}\\ &\le \sup\{\|D\|a_{i,k_i}\}\\ &\le \|D\|\cdot\sup\{a_{i,k_i}\},\qquad i\in I'.\end{aligned}$$

This means that $F\in\mathfrak{E}_\infty(I')$, because $\sup_{i\in I'} a_i<\infty$. It can be concluded that if $\sup_{i\in I'} a_i<\infty$, then $\mathcal{H}^2(\mathfrak{E}_1(I'),{}_0\mathbb{C}_\varphi)=\{0\}$ for each $\varphi\in\Delta(\mathfrak{E}_1(I'))\cup\{0\}$. □

Set $\mathfrak{A}_p=(\prod_{i\in I}\mathbb{M}^{UT}_{d_i}(\mathbb{C}),\|\cdot\|_p)$. We have

$$\mathcal{H}^2(\mathfrak{A}_p(I),{}_0\mathbb{C}_\varphi)=\mathcal{H}^1(\mathfrak{A}_p(I),\mathcal{L}^1(\mathfrak{A}_p(I),{}_0\mathbb{C}_\varphi)),$$

$\mathcal{L}^1(\mathfrak{A}_p(I),{}_0\mathbb{C}_\varphi))$ is the dual Banach $\mathfrak{A}_p(I)$-bimodule $\mathfrak{A}_q(I)=(\mathfrak{A}_p(I))^*\subset\mathfrak{E}_q(I)$ with the same module operations 2.2.
By applying 2.4 we have

$$\langle D((\mathcal{E}^i_{jk})),(\mathcal{E}^i_{jk})\rangle=0\ (k\ne j).$$

Hence

$$\langle D(E),G\rangle=\sum_{i\in I}\sum_{j=1}^{d_i}\lambda^i_{jj}\beta^i_{jj}\langle D((\mathcal{E}^i_{jj})),(\mathcal{E}^i_{jj})\rangle-\sum_{i\in I}\sum_{j=1}^{d_i}\beta^{i_0}_{m_0}\lambda^i_{jj}\langle D((\mathcal{E}^i_{jj})),(\mathcal{E}^i_{jj})\rangle.$$

Using the proof of the previous theorem, we conclude that if $\sup_{i\in I} a_i<\infty$:
$1-\ \mathcal{H}^2\big(\mathfrak{A}_1(I),{}_0\mathbb{C}_\varphi\big)=\{0\}$,
$2-\ \mathcal{H}^2(\mathfrak{E}_1(I),{}_0\mathbb{C}_0)=\{0\}$.

Now, Let A be a Banach algebra and $\varphi\in\Delta(A)\cup\{0\}$. ${}_\varphi A$ is a Banach A-bimodule with the left and right actions

$$a\cdot x=\varphi(a)x,\ x\cdot a=xa$$

for all $a,x\in A$, respectively. Moreover, $({}_\varphi A)^*$ is a Banach A-bimodule with the left and right actions defined by

$$a\cdot f=af,\ f\cdot a=\varphi(a)f$$

for all $a\in A$ and $f\in A^*$, respectively; here $af\in A^*$ is defined by $(af)(b)=f(ba)$ for all $b\in A$.

Definition 3.2. *Let A be a Banach algebra and $\varphi \in \Delta(A) \cup \{0\}$. We say that A is weakly φ-amenable if $H^1(A, ({}_{\varphi}A)^*) = \{0\}$.*
We also say that A is weakly character amenable if A is weakly φ-amenable for all $\varphi \in \Delta(A) \cup \{0\}$.

In the next lemma, the relationship between weak character amenability of Banach algebra A and $\mathcal{H}^2\big(A, {}_0\mathbb{C}_{\varphi}\big)$ is discussed.

Lemma 3.3. *Let A be a Banach algebra and $\varphi \in \Delta(A) \cup \{0\}$. Consider the following statements:*
(a) A has a bounded approximate identity.
(b) $\mathcal{H}^2(A, {}_0\mathbb{C}_{\varphi}) = \{0\}$.
(c) A is weak character amenable.

Then we have $(a) \Longrightarrow (b) \Longrightarrow (c)$.
By Lemma 3.3 and Theorem 3.1, the following result is obtained.

Corollary 3.4. *$\mathfrak{E}_1(I)$ is weak character amenable if $\sup_{i\in I} a_i < \infty$.*

The proof of the following theorem is trivial.

Theorem 3.5. *$\mathcal{H}^2\big(\mathfrak{E}_p(I'), {}_0\mathbb{C}_{\varphi}\big) = \{0\}$ $(1 < p < \infty)$ if I' is finite .*

By above theorem one can see that $\mathfrak{E}_p(I')$ $(1 < p < \infty)$ is weak character amenable if I' is finite.

Theorem 3.6. *$\mathcal{H}^2(\mathfrak{E}_p(I), {}_0\mathbb{C}_0) = \{0\}$ $(1 < p \leq 2)$ if and only if I is finite.*

Proof. "the necessary condition" immediately follows from Lemma 3.3.
To prove the converse, suppose that I is infinite. Then the identity map $Id : \mathfrak{E}_p(I) \longrightarrow \mathfrak{E}_q(I)$ is a derivation, because for $1 < p \leq 2$ we have $2 \leq q < \infty$ and by Theorem 28.32 of [8], $\mathfrak{E}_p(I) \subseteq \mathfrak{E}_q(I)$. Then $E = Id(E) = E \cdot F - F \cdot E$. Therefore $E = FE$ which means that F is an identity element in $\mathfrak{E}_p(I)$. But since I is infinite, $\mathfrak{E}_p(I)$ has no identity. □

For our next theorem in the case $p > 1$, we need the following results:
We recall that a Banach A-bimodule X is said to be $right - annihilator$ $(left - annihilator)$ whenever $x \cdot a = 0$ $(a \cdot x = 0)$, and $annihilator$ if $x \cdot a = 0$ and $a \cdot x = 0$ $(x \in X, a \in A)$.

Proposition 3.7. *Let A have either a left or a right approximate identity. Then for any annihilator X, $\mathcal{Z}^2(A, X) = \tilde{\mathcal{N}}^2(A, X)$; in other words, every annihilator extension of the algebra A algebraically splits.*

Let A be a Banach algebra and S_A be the unit ball in A. Set $S = S_A \cap A^{[2]}$ and suppose that T is the convex hull of the set $\{ab : a, b \in S_A\}$. Besides the original norm $\|\cdot\|$, we also consider in $A^{[2]}$ the norm $\|\cdot\|_{\pi}$ - the Minkowski gauge function of the set T. We note that the new norm majorizes the original norm and can be expressed as $\|a\|_{\pi} = \inf \sum_{k=1}^{n} \|b_k\|\|c_k\|$, where the lower bound is taken over all possible representations of $a \in A^{[2]}$ in the form $a = \sum_{k=1}^{n} b_k c_k$ ([7]).
A normed algebra $(A, \|\cdot\|)$ has *S-property* if there is a constant $C > 0$ such that

$$\|a\|_{\pi} \leq C\|a\| \ (a \in A^{[2]}).$$

Proposition 3.8. *Let A be a Banach algebra. Then the followings are equivalent:*
(a) $\mathcal{N}^2(A,E) = \tilde{\mathcal{N}}^2(A,E)$ for each finite-dimensional, annihilator Banach A-bimodule E;
(b) A has the S-property.

Lemma 3.9. *Let $p > 2$. Then $\mathfrak{E}_p(I)$ has the S-property if and only if I is finite.*

We now result the last theorem of this section as follows

Theorem 3.10. *$\mathcal{H}^2(\mathfrak{E}_p(I), {}_0\mathbb{C}_0) = \{0\}$ $(p > 2)$ if and only if I is finite.*

Proof. The 'if' part by Lemma 3.3 is obvious.
Conversely, let $\mathcal{H}^2(\mathfrak{E}_p(I), {}_0\mathbb{C}_0) = \{0\}$ $(p > 2)$. Suppose to the contrary that I is infinite. So by lemma 3.9, $\mathfrak{E}_p(I)$ hasn't S-property and using equivalence statements from proposition 3.8, we have $\mathcal{N}^2(\mathfrak{E}_p(I), {}_0\mathbb{C}_0) \neq \tilde{\mathcal{N}}^2(\mathfrak{E}_p(I), {}_0\mathbb{C}_0)$ and also by Proposition 3.7, $\mathcal{Z}^2(\mathfrak{E}_p(I), {}_0\mathbb{C}_0) = \tilde{\mathcal{N}}^2(\mathfrak{E}_p(I), {}_0\mathbb{C}_0)$. This is clearly a contradiction. □

4. Application

Let I be an arbitrary set. For $1 \leq p < \infty$, $\ell^p(I)$ with pointwise multiplication is equal to $\mathfrak{E}_p(I)$ whenever for each $i \in I$ we take $H_i = \mathbb{C}$ and $a_i = 1$.
However, the following two results have already been obtained in [13], but these are immediate consequence of the above, Theorem 3.4 and Theorem 3.5.

Corollary 4.1. *$\ell^1(I)$ is weak character amenable.*

Corollary 4.2. *$\ell^p(I)$; $p > 1$ is weak character amenable if I is finite.*

Let $\mathbb{N}$ denote the set of natural numbers. As shown in example 2.3 of [13], the Banach algebra $\ell^2(\mathbb{N})$ of all sequences $a := (a(n))$ of complex numbers with $\|a\|_p := \sum_{n=1}^{\infty} |a(n)|^2 < \infty$ endowed with the pointwise product, is a contradiction for converse of the Corollary 4.2.

Let G be a compact group with dual $\hat{G}$ (the set of all irreducible representations of G). For each $\sigma \in \hat{G}$, let H_σ be the representation space of σ . The algebras $\mathfrak{E}_p(\hat{G})$ for $p \in [1, \infty)$, are defined as introduction with each a_σ equal to the dimension d_σ of $\sigma \in \hat{G}$ [5, Definition 28.34]. For a locally compact group G and a function $f : G \to \mathbb{C}$, $\check{f}$ is defined by $\check{f}(x) = f(x^{-1})$ $(x \in G)$. Let $A(G)$ (or $\mathfrak{K}(G)$, defined in 35.16 of [8]) consists of all functions h in $C_0(G)$ that can be written in at least one form as $\sum_{n=1}^{\infty} f_n * \check{g_n}$, where $f_n, g_n \in L^2(G)$, and $\sum_{n=1}^{\infty} \|f_n\|_2 \|g_n\|_2 < \infty$. For $h \in A(G)$, define

$$\|h\|_{A(G)} = \inf \left\{ \sum_{n=1}^{\infty} \|f_n\|_2 \|g_n\|_2 : h = \sum_{n=1}^{\infty} f_n * \check{g_n} \right\}.$$

With this norm $A(G)$ is a Banach space. For more details see 35.16 of [8].

Observing that, $A(G)$ under convolution product and the norm $\|\cdot\|_{A(G)}$ defines a Banach algebra which is isometrically algebra isomorphic with $\mathfrak{E}_1(\hat{G})$ ([8], Theorem 34.32]).
According to the above discussion and Corollary 3.4, we have the following result.

Corollary 4.3. *$A(G)$ is weak character amenable if $\sup_{\sigma \in \hat{G}} d_\sigma < \infty$.*

The next result is a special case of the above consequence.

Corollary 4.4. *$A(G)$ is weak character amenable if G is abelian and compact.*

Proof. $\hat{G}$ is the set of all one-dimensional representations of G over $\mathbb{C}$ if and only if G is abelian [2]. By above observation and Corollary 4.3, whenever G is abelian and compact, $A(G)$ will be weak character amenable. $\square$

For the rest, we need some preliminary facts which are themselves important. Let X be a left Banach A-module and Y be a right Banach A-module. Then the quotient of the projective tensor product $X\hat{\otimes}Y$ by the closed linear span of the set of elements of the form $\{ya \otimes x - y \otimes ax\}$ where $y \in Y,\ x \in X$ and $a \in A$ is called the tensor product of Y and X over A and is written $Y\underset{A}{\hat{\otimes}}X$.

Definition 4.5. *A Banach algebra A is called* self-induced *if multiplication*

$$A\underset{A}{\hat{\otimes}}A \xrightarrow{\Pi} A;\ \ a \underset{A}{\otimes} b \mapsto ab$$

is a bimodule isomorphism.

Proposition 4.6. *Banach algebra A is self-induced if and only if $\mathcal{H}^1(A, X) = \mathcal{H}^2(A, X) = 0$ for any annihilator bimodule X.*

A Banach algebra A is *biprojective* if the multiplication map $\Pi : A\hat{\otimes}A \longrightarrow A;\ a \otimes b \mapsto ab$ has a bounded right inverse which is an A-bimodule homomorphism and it is *biflat* if $\Pi^* : A^* \longrightarrow (A\hat{\otimes}A)^*$ has a bounded left inverse which is an A-bimodule homomorphism.

Proposition 4.7. *Banach algebra A is biflat if and only if it is self-induced and $\mathcal{H}^1(A, X^*) = \{0\}$ for all induced modules X.*

We say that A is *contractible* if $\mathcal{H}^1(A, X) = \{0\}$ for every Banach A-bimodule X, or equivalently, each continuous derivation from A into X is inner.

Proposition 4.8. *Let A be a non-zero Banach algebra. Then the following statements are equivalent:*
(a) *A is contractible;*
(b) *A is unital and biprojective;*

As a consequence, we show $\mathfrak{E}_p(I);\ p > 1$, is contractible if and only if I is finite.

Theorem 4.9. *$\mathfrak{E}_p(I)$ $(p > 1)$ is contractible if and only if I is finite.*

Proof. Let I be infinite. Using Theorem 3.6 and Theorem 3.10,
$\mathcal{H}^2\big(\mathfrak{E}_p(I), {}_0\mathbb{C}_0\big) \neq \{0\}$. So by Propositions 4.6 and 4.7 , $\mathfrak{E}_p(I)$ $(p > 1)$ is not self-induced and so not biflat. Hence it is not biprojective. Consequently $\mathfrak{E}_p(I)$ $(p > 1)$ is not contractible.
Conversely, if $|I| = n < \infty$, then $\mathfrak{E}_p(I)$ has the identity. Moreover, we will need to prove $\mathfrak{E}_p(I)$ is biprojective. It's clear that $(\mathbb{M}_{d_i}(\mathbb{C}), \|\cdot\|_{\varphi_p})$ is biprojective because there exist morphism $\rho_i : \mathbb{M}_{d_i}(\mathbb{C}) \longrightarrow \mathbb{M}_{d_i}(\mathbb{C})\hat{\otimes}\mathbb{M}_{d_i}(\mathbb{C})$ by $\rho_i(\mathcal{E}_{jk}) = \mathcal{E}_{jn} \otimes \mathcal{E}_{nk}$ for basis elements $\mathcal{E}_{jk} \in \mathbb{M}_{d_i}(\mathbb{C})$ such that ρ_i is a bounded $\mathbb{M}_{d_i}(\mathbb{C})$-bimodule morphism and also ρ_i is a right inverse for the multiplication map from $\mathbb{M}_{d_i}(\mathbb{C})\hat{\otimes}\mathbb{M}_{d_i}(\mathbb{C})$ into $\mathbb{M}_{d_i}(\mathbb{C})$.

Now we show that $\mathfrak{E}_p(I)$ $(p > 1)$ is biprojective if I is finite. First, define monomorphism $\tau_i : \mathbb{M}_{d_i}(\mathbb{C}) \longrightarrow \mathfrak{E}_p(I)$ by $\tau_i(A) = (A_i)_j = \begin{cases} A & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$. By considering

$$\mathfrak{E}_p(I) \xrightarrow{\pi_i} \mathbb{M}_{d_i}(\mathbb{C}) \xrightarrow{\rho_i} \mathbb{M}_{d_i}(\mathbb{C})\hat{\otimes}\mathbb{M}_{d_i}(\mathbb{C}) \xrightarrow{\tau_i\otimes\tau_i} \mathfrak{E}_p(I)\hat{\otimes}\mathfrak{E}_p(I),$$

we set $\rho := \sum_{i=1}^{n}(\tau_i \otimes \tau_i)(\rho_i \circ \pi_i)$. It will then follows from linearity of π_i (as projection map), ρ_i and τ_i that ρ is a linear map. Also, it is not hard to prove that morphism ρ is bounded because

$$\|\rho(E)\| \leq \sum_{i=1}^{n} \|\tau_i \otimes \tau_i\| \|\rho_i\| \|\pi_i\| \|E\|_p \leq \max\{\|\tau_i \otimes \tau_i\| \|\rho_i\| \|\pi_i\|\} \sum_{i=1}^{n} \|E\|_p.$$

So $\|\rho\| \leq \max\{\|\tau_i \otimes \tau_i\| \|\rho_i\| \|\pi_i\|\} n < \infty$. In addition, for $E, G \in \mathfrak{E}_p(I)$

$$\begin{aligned}
(\tau_i \otimes \tau_i)(\rho_i \circ \pi_i(E))G &= (\tau_i \otimes \tau_i)(\rho_i \circ \pi_i((E_i)_i))(G_i)_i \\
&= (\tau_i \otimes \tau_i)(\rho_i(E_i))(G_i)_i \\
&= (\tau_i \otimes \tau_i)(\sum_{j,k=1}^{d_i} \lambda_{jk}\mathcal{E}_{jn} \otimes \mathcal{E}_{nk})(G_i)_i \\
&= \sum_{j,k=1}^{d_i} \tau_i(\lambda_{jk}\mathcal{E}_{jn}) \otimes \tau_i(\mathcal{E}_{nk})(G_i)_i \\
&= \sum_{j,k=1}^{d_i} \tau_i(\lambda_{jk}\mathcal{E}_{jn}) \otimes \tau_i(\mathcal{E}_{nk}G_i) \qquad \text{(by definition of } \tau_i) \\
&= (\tau_i \otimes \tau_i)(\sum_{j,k=1}^{d_i} \lambda_{jk}\mathcal{E}_{jn} \otimes \mathcal{E}_{nk}G_i) \\
&= (\tau_i \otimes \tau_i)(\rho_i(E_i)G_i) \\
&= (\tau_i \otimes \tau_i)(\rho_i(E_iG_i)) \\
&= (\tau_i \otimes \tau_i)(\rho_i \circ \pi_i)(EG).
\end{aligned}$$

So $(\tau_i \otimes \tau_i)(\rho_i \circ \pi_i(E))G = (\tau_i \otimes \tau_i)(\rho_i \circ \pi_i(EG))$. Therefore $\rho(E)G = \rho(EG)$ and similarly $\rho(EG) = E\rho(G)$. Finally, ρ is a right inverse for the multiplication map. In fact $\Pi \circ \rho = I_{\mathfrak{E}_p(I)}$ because

$$\begin{aligned}
\Pi(\rho(E)) &= \sum_{i\in I} \Pi(\tau_i \otimes \tau_i)(\rho_i \circ \pi_i(E)) \\
&= \sum_{i\in I} \Pi(\sum_{j,k=1}^{d_i} (\tau_i(\lambda_{jk}\mathcal{E}_{jn}) \otimes \tau_i(\mathcal{E}_{nk})) \\
&= \sum_{i\in I} \sum_{j,k=1}^{d_i} \tau_i(\lambda_{jk}\mathcal{E}_{jn})\tau_i(\mathcal{E}_{nk}) \\
&= \sum_{i\in I} \sum_{j,k=1}^{d_i} (\lambda_{jk}\mathcal{E}_{jn}\mathcal{E}_{nk})_i \\
&= \sum_{i\in I} \sum_{j,k=1}^{d_i} \lambda_{jk}(\mathcal{E}^i_{jk})_i = E.
\end{aligned}$$

□

Corollary 4.10. *$L^2(G)$ is contractible if and only if G is finite.*

Proof. Note that by Theorem 28.43 of [8], the Banach algebra $L^2(G)$ is isometrically algebra isomorphic with $\mathfrak{E}_2(\hat{G})$. Now by the previous theorem, $\mathfrak{E}_2(\hat{G})$ is contractible if and only if $\hat{G}$ is finite. $\square$

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