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FIXED POINTS ON QUASI-METRIC SPACES VIA SIMULATION FUNCTIONS AND CONSEQUENCES

HASSEN AYDI∗, ABDELBASSET FELHI, ERDAL KARAPINAR, FATIMAH A. ALOJAIL

Abstract. In this paper, we provide some fixed points for triangular α-admissible contraction mappings via simulations functions in the class of quasi-metric spaces. Some consequences are presented. We also give some illustrated examples.

1. Introduction

The concept of a quasi metric space is one of the interesting generalization of the metric space which was obtained by lifting the symmetry condition. Among any other generalization of the metric space, the most natural one is quasi metric space which can be matched in real-world problem easily. The most trivial example of quasi metric can be considered as a distance taking by a worker between house and workplace in a city that is settled by one-way streets and two-way roads. For more and concrete examples of quasi-metric with some interesting fixed point results on this setting, we refer the readers to [3, 4, 13, 15, 17, 19, 18, 20].

In the last decades, a number of fixed point theory papers have released most of which generalizes the well-known fixed point results in different aspects: by changing the abstract space, by replacing the contraction condition with a weaker one, and so on. Accordingly, the following natural question has been asked: Whether the existing results can be combined in a simple way? Several answers have been given and among them one of the response, simulation function, takes our interest so much. The notion of simulation function is introduced by Khojasteh *et al.* [24] to unify the most of the single valued of fixed point by using an auxiliary function, called a simulation function in the setting of a standard metric space.

In this manuscript, we shall investigate the answer of the question: How we can combine existing fixed point theorems in context of a quasi-metric space by using the simulation function? For giving the most successive answer, we shall also use another auxiliary function, admissible mapping. It is very interesting that admissible mapping has a power to combine the fixed point theorems on a

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metric space equipped with a partially ordered set and the corresponding fixed point theorems that are obtained by cyclic contractions or standard contractions, for more details, see e.g. [14, 16, 22, 30, 31, 32]. Consequently, we unified several fixed point results in the set-up of a quasi-metric space by the help of both simulation functions and admissible mappings. We present some examples to illustrate the contribution of our results.

2. Preliminaries

First, we recall some basic concepts and notations.

Definition 2.1. *Let X be a non-empty and $s \geq 1$. Let $d : X \times X \to [0, \infty)$ be a function which satisfies:*
$(d1) d(x, y) = 0$ if and only if $x = y$,
$(d2) d(x, y) \leq d(x, z) + d(z, y)$. Then, d is called a quasi-metric and the pair (X, d) is called a quasi-metric space.

Example 1. *Let $X = l_1$ be defined by*

$$l_1 = \{\{x_n\}_{n\geq 1} \subset \mathbb{R}, \ \sum_{n=1}^{\infty} |x_n| < \infty\}.$$

Consider $d : X \times X \to [0, \infty)$ such that

$$d(x, y) = \begin{cases} 0 & \text{if } x \preceq y, \\ \sum_{n=1}^{\infty} |x_n| & \text{if } x \succeq y, \end{cases}$$

d is a quasi-metric. Mention that $x \succeq y$ if $x_n \geq y_n$ for all n, where $x = \{x_n\}$ and $y = \{y_n\}$ are in X.

Now, we give convergence and completeness on quasi-metric spaces.

Definition 2.2. *Let (X, d) be a quasi-metric space, $\{x_n\}$ be a sequence in X and $x \in X$. The sequence $\{x_n\}$ converges to x if and only if*

$$\lim_{n\to\infty} d(x_n, x) = \lim_{n\to\infty} d(x, x_n) = 0. \tag{2.1}$$

Remark. *In a quasi-metric space (X, d), the limit for a convergent sequence is unique. Also, if $x_n \to x$, we have for all $y \in X$*

$$\lim_{n\to\infty} d(x_n, y) = d(x, y) \quad \text{and} \quad \lim_{n\to\infty} d(y, x_n) = d(y, x).$$

Definition 2.3. *Let (X, d) be a quasi-metric space and $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is left-Cauchy if and only if for every $\varepsilon > 0$, there exists a positive integer $N = N(\varepsilon)$ such that $d(x_n, x_m) < \varepsilon$ for all $n \geq m > N$.*

Definition 2.4. *Let (X, d) be a quasi-metric space and $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is right-Cauchy if and only if for every $\varepsilon > 0$, there exists a positive integer $N = N(\varepsilon)$ such that $d(x_n, x_m) < \varepsilon$ for all $m \geq n > N$.*

Definition 2.5. *Let (X, d) be a quasi-metric space and $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is Cauchy if and only if for every $\varepsilon > 0$, there exists a positive integer $N = N(\varepsilon)$ such that $d(x_n, x_m) < \varepsilon$ for all $m, n > N$.*

Remark. *A sequence $\{x_n\}$ in a quasi-metric space is Cauchy if and only if it is left-Cauchy and right-Cauchy.*

Definition 2.6. *Let (X,d) be a quasi-metric space.*
(1)(X,d) *is said left-complete if and only if each left-Cauchy sequence in X is convergent.*
(2)(X,d) *is said right-complete if and only if each right-Cauchy sequence in X is convergent.*
(3)(X,d) *is said complete if and only if each Cauchy sequence in X is convergent.*

Lemma 2.7. *Let (X,d) be a quasi-metric space and $T: X \to X$ be a given mapping. Suppose that T is continuous at $u \in X$. Then for all sequence $\{x_n\}$ in X such that $x_n \to u$, we have $Tx_n \to Tu$, that is,*

$$\lim_{n\to\infty} d(Tx_n, Tu) = \lim_{n\to\infty} d(Tx, Tx_n) = 0.$$

In 2012, Samet et al. [29] introduced the concept of α-admissible mappings as follows.

Definition 2.8. [29] *For a nonempty set X, let $T: X \to X$ and $\alpha: X\times X \to [0,\infty)$ be given mappings. We say that T is $\alpha-$admissible if for all $x,y \in X$, we have*

$$\alpha(x,y) \geq 1 \Longrightarrow \alpha(Tx,Ty) \geq 1. \tag{2.2}$$

The concept of α-admissible mappings is used frequently in several papers, see [7]-[12]. Later, Karapinar et al. [23] introduced the notion of triangular α-admissible mappings.

Definition 2.9. [23] *Let $T: X \to X$ and $\alpha: X \times X \to [0,\infty)$ be given mappings. A mapping $T: A \to B$ is called triangular α-admissible if*

(T_1) *T is α-admissible;*
(T_2) *$\alpha(x,y) \geq 1$ and $\alpha(y,z) \geq 1 \Rightarrow \alpha(x,z) \geq 1$, $x,y,z \in X$.*

Very recently, Khojasteh, Shukla and Radenović [24] introduced a new class of mappings called simulation functions. Using the above concept, they [24] proved several fixed point theorems and showed that many known results in literature are simple consequences of their obtained results. Later, Argoubi, Samet and Vetro [6] slightly modified the definition of simulation functions by withdrawing a condition, see also [26].

Let $\mathcal{Z}^*$ be the set of simulation functions in the sense of Argoubi et al. [6].

Definition 2.10. [6] *A simulation function is a mapping $\zeta: [0,\infty) \times [0,\infty) \to \mathbb{R}$ satisfying the following conditions:*

(ζ_1) *$\zeta(t,s) < s - t$ for all $t,s > 0$;*
(ζ_2) *if $\{t_n\}$ and $\{s_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = \ell \in (0,\infty)$, then*

$$\limsup_{n\to\infty} \zeta(t_n, s_n) < 0.$$

Example 2. [6] *Let $\zeta_\lambda: [0,\infty) \times [0,\infty) \to \mathbb{R}$ be the function defined by*

$$\zeta_\lambda(t,s) = \begin{cases} 1 & \text{if } (t,s) = (0,0), \\ \lambda s - t & \text{otherwise}, \end{cases}$$

where $\lambda \in (0,1)$. Then, $\zeta_\lambda \in \mathcal{Z}^*$.

Example 3. *Let $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ be the function defined by $\zeta(t,s) = \psi(s) - \varphi(t)$ for all $t, s \geq 0$, where $\psi : [0,\infty) \to \mathbb{R}$ is an upper semi-continuous function and $\varphi : [0,\infty) \to \mathbb{R}$ is a lower semi-continuous function such that $\psi(t) < t \leq \varphi(t)$ for all $t > 0$. Then $\zeta \in \mathcal{Z}^*$.*

In this paper, we provide some fixed point results for triangular α-admissible contraction mappings using simulation functions in the class of quasi-metric spaces. Some consequences are also derived. Moreover, we present some examples in support of the given results.

3. Main results

Our first main result is

Theorem 3.1. *Let (X, d) be a complete quasi-metric space and $T : X \to X$ be a given mapping. Suppose that there exist $\zeta \in \mathcal{Z}^*$ nonincreasing with respect to the first variable and $\alpha : X \times X \to [0,\infty)$ such that*

$$\zeta\left(\alpha(x,y)d(Tx,Ty), d(x,y)\right) \geq 0, \tag{3.1}$$

for all $x, y \in X$. Assume that

- (i) *T is triangular α-admissible;*
- (ii) *there exists an element $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(Tx_0, x_0) \geq 1$;*
- (iii) *T is continuous.*

Then T has a fixed point, that is, there exists $z \in X$ such that $Tz = z$.

Proof. Define a sequence $\{x_n\}$ by $x_n = T^n x_0$ for all $n \geq 0$. If $x_n = x_{n+1}$ for some n, then $x_n = x_{n+1} = Tx_n$, that is, x_n is a fixed point of T and so the proof is completed. Suppose now that $x_n \neq x_{n+1}$ for all $n = 0, 1, \ldots$ We split the proof into several steps.

Step 1: $\alpha(x_n, x_m) \geq 1$ for all $m, n \geq 0$.

By assumption (ii), $\alpha(x_0, Tx_0) \geq 1$. We have $\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq 1$. Since T is α-admissible, by induction, we have

$$\alpha(x_n, x_{n+1}) \geq 1 \quad \text{for all } n \geq 0.$$

The mapping T is triangular α-admissible, so

$$\alpha(x_n, x_{n+1}) \geq 1 \text{ and } \alpha(x_{n+1}, x_{n+2}) \geq 1 \Rightarrow \alpha(x_n, x_{n+2}) \geq 1.$$

Thus, by induction

$$\alpha(x_n, x_m) \geq 1 \quad \text{for all } m > n \geq 0.$$

Proceeding similarly, in view of $\alpha(Tx_0, x_0) \geq 1$, we have $\alpha(x_n, x_m) \geq 1$ for all $n > m \geq 0$.

Step 2: We shall prove that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = \lim_{n\to\infty} d(x_{n+1}, x_n) = 0. \tag{3.2}$$

By Step 1, we have $\alpha(x_n, x_m) \geq 1$ for all $m > n \geq 0$. Then from (3.1)

$$\zeta\left(\alpha(x_{n-1}, x_n)d(x_n, x_{n+1}), d(x_{n-1}, x_n)\right) = \zeta\left(\alpha(x_{n-1}, x_n)d(Tx_{n-1}, Tx_n), d(x_{n-1}, x_n)\right) \geq 0.$$

From condition (ζ_1), we have

$$0 \leq \zeta\left(\alpha(x_{n-1}, x_n)d(x_n, x_{n+1}), d(x_{n-1}, x_n)\right) < d(x_{n-1}, x_n) - \alpha(x_{n-1}, x_n)d(x_n, x_{n+1}), \quad \text{for all } n \geq 1.$$

Necessarily, we have

$$d(x_n, x_{n+1}) \leq \alpha(x_{n-1}, x_n) d(x_n, x_{n+1}) < d(x_{n-1}, x_n), \quad \text{for all } n \geq 1, \tag{3.3}$$

which implies that $\{d(x_n, x_{n+1})\}$ is a decreasing sequence of positive real numbers, so there exists $t \geq 0$ such that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = t. \tag{3.4}$$

Using the fact that ζ is nonincreasing with respect to the first variable,

$$0 \leq \zeta\left(\alpha(x_{n-1}, x_n) d(x_n, x_{n+1}), d(x_{n-1}, x_n)\right) \leq \zeta\left(d(x_n, x_{n+1}), d(x_{n-1}, x_n)\right).$$

Suppose that $t > 0$. By (3.4) and the condition (ζ_2),

$$0 \leq \limsup_{n\to\infty} \zeta\left(d(x_n, x_{n+1}), d(x_{n-1}, x_n)\right) < 0,$$

which is a contradiction. We conclude that $t = 0$. Similarly, using the fact that $\alpha(x_n, x_m) \geq 1$ for all $n > m \geq 0$, we get

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0.$$

Step 3: We shall prove that

$$\lim_{n,m\to\infty} d(x_n, x_m) = 0. \tag{3.5}$$

First, we claim that $\{x_n\}$ is a right-Cauchy sequence in the quasi-metric space (X, d). Suppose to the contrary. Then there exists $\varepsilon > 0$ for which we can find subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ with $m(k) > n(k) > k$ such that for every k

$$d(x_{n(k)}, x_{m(k)}) \geq \varepsilon. \tag{3.6}$$

Moreover, corresponding to $n(k)$ we can choose $m(k)$ in such a way that it is the smallest integer with $m(k) > n(k)$ and satisfying (3.6). Then

$$d(x_{n(k)}, x_{m(k)-1}) < \varepsilon. \tag{3.7}$$

Using (3.6), (3.7) and the triangular inequality, we get

$$\begin{aligned} \varepsilon &\leq d(x_{n(k)}, x_{m(k)}) \leq d(x_{n(k)}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{m(k)}) \\ &< d(x_{m(k)-1}, x_{m(k)}) + \varepsilon. \end{aligned}$$

By (3.2)

$$\lim_{k\to\infty} d(x_{n(k)}, x_{m(k)}) = \lim_{k\to\infty} d(x_{n(k)}, x_{m(k)-1}) = \varepsilon. \tag{3.8}$$

Moreover,

$$d(x_{n(k)}, x_{m(k)-1}) - d(x_{n(k)}, x_{n(k)-1}) - d(x_{m(k)}, x_{m(k)-1}) \leq d(x_{n(k)-1}, x_{m(k)})$$

and

$$d(x_{n(k)-1}, x_{m(k)}) \leq d(x_{n(k)-1}, x_{n(k)}) + d(x_{n(k)}, x_{m(k)}).$$

Take $k \to \infty$ in the above inequalities. Using (3.2) and (3.8), we obtain

$$\lim_{k\to\infty} d(x_{n(k)-1}, x_{m(k)}) = \varepsilon. \tag{3.9}$$

Again,

$$d(x_{n(k)-1}, x_{m(k)-1}) \leq d(x_{n(k)-1}, x_{m(k)}) + d(x_{m(k)}, x_{m(k)-1}),$$

and

$$d(x_{n(k)-1}, x_{m(k)}) \leq d(x_{n(k)-1}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{m(k)}).$$

Letting $k \to \infty$ in the two above inequalities and using (3.2) and (3.9), we have

$$\lim_{k\to\infty} d(x_{n(k)-1}, x_{m(k)-1}) = \varepsilon. \tag{3.10}$$

By (3.1) and as $\alpha(x_{n(k)-1}, x_{m(k)-1}) \geq 1$ for all $k \geq 1$, we get

$$0 \leq \zeta\left(\alpha(x_{n(k)-1}, x_{m(k)-1}) d(x_{n(k)}, x_{m(k)}), d(x_{n(k)-1}, x_{m(k)-1})\right) \leq \zeta\left(d(x_{n(k)}, x_{m(k)}), d(x_{n(k)-1}, x_{m(k)-1})\right).$$

On the other hand, if $x_n = x_m$ for some $n < m$, then $x_{n+1} = Tx_n = Tx_m = x_{m+1}$. (3.3) leads to

$$0 < d(x_n, x_{n+1}) = d(x_m, x_{m+1}) < d(x_{m-1}, x_m) < \cdots < d(x_n, x_{n+1}),$$

which is a contradiction. Then $x_n \neq x_m$ for all $n < m$. The condition (ζ_2) together with (3.8) and (3.10) imply that

$$0 \leq \limsup_{k\to\infty} \zeta\left(d(x_{n(k)}, x_{m(k)}), d(x_{n(k)-1}, x_{m(k)-1})\right) < 0,$$

which is a contradiction. Thus $\{x_n\}$ is right-Cauchy in (X, d). Similarly, $\{x_n\}$ is left-Cauchy. Consequently, (3.5) holds, that is, $\{x_n\}$ is a Cauchy sequence.
Step 4: We shall show that T has a fixed point.
Since (X, d) is complete, there exists some $z \in X$ such that

$$\lim_{n\to\infty} d(x_n, z) = \lim_{n\to\infty} d(z, x_n) = 0. \tag{3.11}$$

The continuity of T yields that

$$\lim_{n\to\infty} d(Tx_n, Tz) = \lim_{n\to\infty} d(Tz, Tx_n) = 0. \tag{3.12}$$

Therefore, $x_{n+1} \to Tz$ in (X, d), so by uniqueness of limit, we get $Tz = z$. □

We may replace the continuity hypothesis of T in Theorem 3.1 by the following hypothesis:

(H) If $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geq 1$ for all k.

Theorem 3.2. *Let (X, d) be a complete quasi-metric space. Let $T : X \to X$ be a given mapping. Suppose that there exist $\zeta \in \mathcal{Z}^*$ nonincreasing with respect to the first variable and $\alpha : X \times X \to [0, \infty)$ such that*

$$\zeta\left(\alpha(x, y) d(Tx, Ty), d(x, y)\right) \geq 0, \tag{3.13}$$

for all $x, y \in X$. Assume that

- *(i) T is triangular α-admissible;*
- *(ii) there exists an element $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(Tx_0, x_0) \geq 1$;*
- *(iii) (H) holds.*

Then T has a fixed point.

Proof. Following the proof of Theorem 3.1, there exists a sequence $\{x_n\}$ such that $\alpha(x_n, x_m) \geq 1$ for all $m, n \geq 0$. Also $\{x_n\}$ is Cauchy in (X, d) and converges to some $z \in X$. We claim that z is a fixed point of T. If there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} = z$ or $x_{n_k+1} = Tz$ for all k, then $d(z, Tz) = d(z, x_{n_k+1})$ for all k. Letting $k \to \infty$, we get $d(z, Tz) = 0$, that is, $Tz = z$ and the proof is complete. So, without loss of generality, we may suppose that $x_n \neq z$ and $x_n \neq Tz$ for all nonnegative integer n. Suppose that $d(z, Tz) > 0$. By assumption (iii), there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, z) \geq 1$ for all k. By (3.13), for all $k \geq 1$, we get

$$0 \leq \zeta\left(\alpha(x_{n(k)}, z)d(x_{n(k)+1}, Tz), d(x_{n(k)}, z)\right). \tag{3.14}$$

From the condition (ζ_1)

$$0 \leq \zeta\left(\alpha(x_{n(k)}, z)d(x_{n(k)+1}, Tz), d(x_{n(k)}, z)\right) < d(x_{n(k)}, z) - d(x_{n(k)}, z)d(x_{n(k)+1}, Tz).$$

As $\alpha(x_{n(k)}, z) \geq 1$, we deduce

$$d(x_{n(k)+1}, Tz) \leq d(x_{n(k)}, z)d(x_{n(k)+1}, Tz) \leq d(x_{n(k)}, z).$$

By (3.11)

$$\lim_{k\to\infty} d(x_{n(k)+1}, Tz) = d(z, Tz) \leq 0,$$

which is a contradiction and hence $d(z, Tz) = 0$, that is, $Tz = z$ and so z is a fixed point of T. This ends the proof of Theorem 3.2. □

In Theorem 3.1 and Theorem 3.2, the uniqueness of the fixed point is not ensured. In order to get a unique fixed point, we consider the following:

(U) $\alpha(u, v) \geq 1$ for all $u, v \in Fix(T)$, where $Fix(T)$ is the set of fixed points of T.

Theorem 3.3. *Adding (U) to the hypotheses of Theorem 3.1 (resp. Theorem 3.2), T has a unique fixed point.*

Proof. We argue by contradiction, that is, there exist $u, v \in X$ such that $u = Tu$ and $v = Tv$ with $u \neq v$. By (3.1), we get

$$\zeta(\alpha(u, v)d(Tu, Tv), d(u, v)) \leq 0.$$

Applying (U), the nonincreasing property of ζ with respect to the first variable together with (ζ_1) imply that

$$0 \leq \zeta(d(u, v), d(u, v) < d(u, v) - d(u, v) = 0.$$

It is a contradiction. Hence $u = v$. □

4. Consequences

In this section, as consequences of our obtained results, we provide various fixed point results in the literature.

Corollary 4.1. *Let (X, d) be a complete quasi-metric space and $T : X \to X$ be a given mapping. Suppose that there exist $k \in (0, 1)$ and $\alpha : X \times X \to [0, \infty)$ such that*

$$\alpha(x, y)d(Tx, Ty) \leq kd(x, y),$$

for all $x, y \in X$. Assume that

- (i) *T is triangular α-admissible;*
- (ii) *there exists an element $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(Tx_0, x_0) \geq 1$;*
- (iii) *either T is continuous, or (H) holds.*

Then T has a fixed point.

Proof. It suffices to take a simulation function $\zeta(t, s) = ks - t$ for all $s, t \geq 0$ in Theorem 3.1 (resp. Theorem 3.2). □

Corollary 4.2. *Let (X,σ) be a complete quasi-metric space. Let $T : X \to X$ be a given mapping. Suppose that there exist a lower semi-continuous function $\varphi : [0,\infty) \to [0,\infty)$ with $\varphi(t) > 0$ for all $t > 0$ and $\alpha : X \times X \to [0,\infty)$ such that*

$$\alpha(x,y)d(Tx,Ty) \leq d(x,y) - \varphi(d(x,y))$$

for all $x,y \in X$. Assume that

(i) *T is triangular α-admissible;*
(ii) *there exists an element $x_0 \in X$ such that $\alpha(x_0,Tx_0) \geq 1$ and $\alpha(Tx_0,x_0) \geq 1$;*
(iii) *either T is continuous, or (H) holds.*

Then T has a fixed point.

Proof. It suffices to take a simulation function $\zeta(t,s) = s - \varphi(s) - t$ for all $s,t \geq 0$ in Theorem 3.1 (resp. Theorem 3.2). □

Corollary 4.3. *Let (X,σ) be a complete quasi-metric space. Let $T : X \to X$ be a given mapping. Suppose there exist a function $\varphi : [0,\infty) \to [0,1)$ with $\lim_{t\to r^+} \varphi(t) < 1$ for all $r > 0$ and $\alpha : X \times X \to [0,\infty)$ such that*

$$\alpha(x,y)d(Tx,Ty) \leq \varphi(d(x,y))d(x,y),$$

for all $x,y \in X$. Assume that

(i) *T is triangular α-admissible;*
(ii) *there exists an element $x_0 \in X$ such that $\alpha(x_0,Tx_0) \geq 1$ and $\alpha(Tx_0,x_0) \geq 1$;*
(iii) *either T is continuous, or (H) holds.*

Then T has a fixed point.

Proof. It suffices to take a simulation function $\zeta(t,s) = s\varphi(s) - t$ for all $s,t \geq 0$ in Theorem 3.1 (resp. Theorem 3.2). □

Corollary 4.4. *Let (X,σ) be a complete quasi-metric space. Let $T : X \to X$ be a given mapping. Suppose there exist an upper semi-continuous function $\varphi : [0,\infty) \to [0,\infty)$ with $\varphi(t) < t$ for all $t > 0$ and $\alpha : X \times X \to [0,\infty)$ such that*

$$\alpha(x,y)d(Tx,Ty) \leq \varphi(d(x,y)),$$

for all $x,y \in X$. Assume that

(i) *T is triangular α-admissible;*
(ii) *there exists an element $x_0 \in X$ such that $\alpha(x_0,Tx_0) \geq 1$ and $\alpha(Tx_0,x_0) \geq 1$;*
(iii) *either T is continuous, or (H) holds.*

Then T has a fixed point.

Proof. It suffices to take a simulation function $\zeta(t,s) = \varphi(s) - t$ for all $s,t \geq 0$ in Theorem 3.1 (resp. Theorem 3.2). □

Corollary 4.5. *Let (X,σ) be a complete quasi-metric space. Let $T : X \to X$ be a given mapping. Suppose there exists $\zeta \in \mathcal{Z}^*$ nonincreasing with respect to the first variable such that*

$$\zeta\left(\alpha(x,y)d(Tx,Ty), d(x,y)\right) \geq 0,$$

for all $x,y \in X$. Then T has a fixed point.

Proof. It suffices to take $\alpha(x,y)=1$ in Theorem 3.2. $\square$

Now, let $(X,d,\preceq)$ be a partially ordered quasi-metric space. Consider the following hypothesis:

(G) If $\{x_n\}$ is a sequence in X such that $x_n \preceq x_{n+1}$ for all n and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $x_{n(k)} \preceq x$ for all k.

Corollary 4.6. *Let $(X,d,\preceq)$ be a complete partially ordered quasi-metric space. Let $T: X \to X$ be a given mapping. Suppose there exists $\zeta \in \mathcal{Z}^*$ nonincreasing with respect to the first variable such that*

$$\zeta\left(\sigma(Tx,Ty), d(x,y)\right) \geq 0,$$

for all $x,y \in X$ satisfying $x \preceq y$. Assume that

(i) *T is is non-decreasing;*
(ii) *there exists an element $x_0 \in X$ such that $x_0 \preceq Tx_0$ and $Tx_0 \preceq x_0$;*
(iii) *T is continuous, or (G) holds.*

Then T has a fixed point.

Proof. Let $\alpha : X \times X \to X$ be such that

$$\alpha(x,y) = \begin{cases} 1 & \text{if } \ x \preceq y; \\ 0 & \text{otherwise.} \end{cases}$$

Then all hypotheses of Theorem 3.1 (resp. Theorem 3.2) are satisfied and hence T has a fixed point. $\square$

On the other hand, Mustafa and Sims [25] introduced the concept of G-metric spaces. Let (X,G) be a G-metric space. It is known that the function defined by

$$d_G(x,y) = G(x,y,y) + G(y,x,x), \text{ for all } x,y \in X, \tag{4.1}$$

is a metric on X. Furthermore, (X,G) is G-complete if and only if (X,d_G) is complete.

Recently, Jleli and Samet [20] gave the following results.

Theorem 4.7. (See [20]) *Let (X,G) be a G-metric space. Let $d: X \times X \to [0,\infty)$ be the function defined by $d(x,y) = G(x,y,y)$. Then*

(1) *(X,d) is a quasi-metric space;*
(2) *$\{x_n\} \subset X$ is G-convergent to $x \in X$ if and only if $\{x_n\}$ is convergent to x in (X,d);*
(3) *$\{x_n\} \subset X$ is G-Cauchy if and only if $\{x_n\}$ is Cauchy in (X,d);*
(4) *(X,G) is G-complete if and only if (X,d) is complete.*

Every quasi-metric induces a metric, that is, if (X,d) is a quasi-metric space, then the function $\delta : X \times X \to [0,\infty)$ defined by

$$\delta(x,y) = \max\{d(x,y), d(y,x)\} \tag{4.2}$$

is a metric on X [20].

Theorem 4.8. (See [20]) *Let (X,G) be a G-metric space. Let $\delta : X \times X \to [0,\infty)$ be the function defined by $\delta(x,y) = \max\{G(x,y,y), G(y,x,x)\}$. Then*

(1) *(X,δ) is a metric space;*

(2) *$\{x_n\} \subset X$ is G-convergent to $x \in X$ if and only if $\{x_n\}$ is convergent to x in (X, δ);*
(3) *$\{x_n\} \subset X$ is G-Cauchy if and only if $\{x_n\}$ is Cauchy in (X, δ);*
(4) *(X, G) is G-complete if and only if (X, δ) is complete.*

We need the following definition of Alghamdi and Karapınar [1, 2].

Definition 4.9. [1] *For a nonempty set X, let $T : X \to X$ and $\beta : X^3 \to [0, \infty)$ be mappings. We say that the self-mapping T on X is β-admissible if for all $x, y \in X$, we have*

$$\beta(x, y, y) \geq 1 \Longrightarrow \beta(Tx, Ty, Ty) \geq 1. \tag{4.3}$$

We define the concept of triangular β-admissibility as follows.

Definition 4.10. *For a nonempty set X, let $T : X \to X$ and $\beta : X^3 \to [0, \infty)$ be mappings. We say that the self-mapping T on X is triangular β-admissible if T is β-admissible and*

$$\beta(x, y, y) \geq 1 \text{ and } \beta(y, z, z) \geq 1 \Longrightarrow \beta(x, z, z) \geq 1. \tag{4.4}$$

Clearly, we have

Lemma 4.11. *Let X be a non-empty set. The mapping $f : X \to X$ is triangular β-admissible if and only if f is triangular α-admissible.*

Now, we can give the following two corollaries on G-metric spaces.

Corollary 4.12. *Let (X, G) be a complete G-metric space and $T : X \to X$ be a given mapping. Suppose that there exist $\zeta \in \mathcal{Z}^*$ nonincreasing with respect to the first variable and $\beta : X \times X \times X \to [0, \infty)$ such that*

$$\zeta\left(\beta(x, y, y)G(Tx, Ty, Ty), G(x, y, y)\right) \geq 0, \tag{4.5}$$

for all $x, y \in X$. Assume that

(i) *T is triangular β-admissible;*
(ii) *there exists an element $x_0 \in X$ such that $\beta(x_0, Tx_0, Tx_0) \geq 1$ and $\beta(Tx_0, x_0, x_0) \geq 1$;*
(iii) *T is continuous.*

Then T has a fixed point.

Proof. It suffices to take the quasi-metric $d(x, y) = G(x, y, y)$ and $\alpha(x, y) = \beta(x, y, y)$. Due to (4.5), we get (3.1). Then due to Lemma 4.11, the result follows from Theorem 3.1. □

Alghamdi and Karapınar [1, 2] also defined the following hypothesis.

(W)**:** if $\{x_n\}$ is a sequence in X such that $\beta(x_n, x_{n+1}, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\beta(x_{n(k)}, x, x) \geq 1$ for all k.

In view of Theorem 3.2, we have

Corollary 4.13. *Let (X, G) be a complete G-metric space and $T : X \to X$ be a given mapping. Suppose that there exist $\zeta \in \mathcal{Z}^*$ nonincreasing with respect to the first variable and $\beta : X \times X \times X \to [0, \infty)$ such that*

$$\zeta\left(\beta(x, y, y)G(Tx, Ty, Ty), G(x, y, y)\right) \geq 0, \tag{4.6}$$

for all $x, y \in X$. Assume that

(i) T *is triangular* β*-admissible;*
(ii) *there exists an element* $x_0 \in X$ *such that* $\beta(x_0, Tx_0, Tx_0) \geq 1$ *and* $\beta(Tx_0, x_0, x_0) \geq 1$;
(iii) (W) *hods.*

Then T *has a fixed point.*

5. Examples

We provide the following examples.

Example 4. *For* $X = [0, \infty)$*, consider the mapping* $d : X \times X \to [0, \infty)$ *such that*

$$d(x, y) = |x| \text{ if } x \neq y \text{ and } d(x, y) = 0 \text{ if } x = y.$$

It is obvious that (X, d) *is a complete quasi-metric space. Suppose that* $T : X \to X$ *is defined by*

$$Tx = \begin{cases} x^3 - 4x & \text{if } x > 2 \\ \frac{x}{3} & \text{if } x \in [0, 2]. \end{cases}$$

Note that the Banach contraction principle is not applicable for the standard metric $d_0(x, y) = |x - y|$ *for. Indeed, for* $x = 0$ *and* $y = 3$*, we have*

$$d_0(T0, T3) = 15 > 3 = d_0(0, 3).$$

Now, define $\alpha : X \times X \to [0, \infty)$ *as*

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x, y \in [0, 1] \\ 0 & \text{if not.} \end{cases}$$

If $x, y \in [0, 1]$ *and* $x \neq y$*, we have*

$$d(Tx, Ty) = d(Tx, Ty) = |Tx| = \frac{x}{3} = \frac{1}{3} d(x, y).$$

Hence, the condition (3.13) of Theorem 3.2 is satisfied. Indeed,

$$0 \leq \zeta(d(x, y), \alpha(x, y) d(Tx, Ty)) < d(x, y) - \alpha(x, y) d(Tx, Ty) = \frac{2}{3} d(x, y). \quad (5.1)$$

In particular, for Corollary 4.1, the condition 3.13 is fulfilled for $k = \frac{1}{3}$.

Notice also that $0 \leq \zeta(d(x, y), \alpha(x, y) d(Tx, Ty))$ *holds in the cases (*$x, y \in [0, 1]$ *with* $x = y$*) and (*x *or* y *is not in* $[0, 1]$*). Note that there is no need to check the other possibilities due to the fact that* $\alpha(x, y) = 0$ *in these cases.*

Now, we shall prove that the hypothesis (H) *is satisfied. For this, let* $\{x_n\}$ *be a sequence in* X *such that* $\alpha(x_n, x_{n+1}) \geq 1$ *for all* n *and* $x_n \to x \in X$ *as* $n \to \infty$*. Then by definition of* α*, we get*

$$(x_n, x_{n+1}) \in [0, 1] \times [0, 1] \text{ for all } n.$$

Assume that $x > 1$*. Then* $x_n \neq x$ *for all* n*. Since* $x_n \to x \in X$*, so* $d(x, x_n) = |x| \to 0$*, which is a contradiction. Thus,* $x \in [0, 1]$*. We get that*

$$(x_n, x) \in [0, 1] \times [0, 1] \text{ for all } n,$$

that is, $\alpha(x_n, x) = 1$*, i.e,* (H) *is verified. Take* $x_0 = 1$*. We have*

$$\alpha(x_0, Tx_0) = \alpha(1, \frac{1}{3}) = 1 \text{ and } \alpha(Tx_0, x_0) = \alpha(\frac{1}{3}, 1) = 1.$$

The mapping T is α-admissible. In fact, let $x, y \in X$ be such that $\alpha(x,y) \geq 1$, so $x, y \in [0,1]$. Then

$$\alpha(Tx,Ty) = \alpha(\frac{x}{3}, \frac{y}{3}) = 1.$$

All hypotheses of Theorem 3.2 (in particular, Corollary 4.1) hold and the mapping T has a fixed point in X. Note that in this case, we have two fixed points of T which are $u = 0$ and $v = \sqrt{5}$.

Example 5. *Let $X = [0,1] \cup \{2,3,4\}$. Consider $A = [0,1]$ and $B = \{2,3,4\}$. Clearly, $X = A \cup B$. Define*

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 2 & \text{if } x \in A,\ y \in B \\ 1 & \text{otherwise.} \end{cases}$$

Obviously, (X,d) is a complete quasi-metric space. Consider

$$\alpha(x,y) = \begin{cases} 1 & \text{if } x, y \in B, \text{ or } x = y \\ 0 & \text{if not,} \end{cases}$$

and

$$Tx = \begin{cases} 1 - x & \text{if } x \in A \\ 3 & \text{if } x \in B. \end{cases}$$

It is easy that all conditions of Theorem 3.2 hold. Here, T has two fixed points, which are, $u = \frac{1}{2}$ and $v = 3$.

Example 6. *Let $X = [0,\infty)$. Define*

$$d(x,y) = \max\{x - y, 2(y - x)\}$$

Obviously, (X,d) is a complete quasi-metric space. Consider

$$\alpha(x,y) = \begin{cases} 1 & \text{if } x, y \in [0,1] \\ 0 & \text{if not,} \end{cases}$$

and

$$Tx = \begin{cases} \ln(1 + \frac{x}{2}) & \text{if } x \in [0,1] \\ 2(x-1) + \ln\frac{3}{2} & \text{if } x < 1. \end{cases}$$

Let $\zeta(t,s) = \frac{1}{2}s - t$. Clearly, T is continuous. All conditions of Theorem 3.1 hold. Here, T has two fixed points, which are, $u = 0$ and $v = 2 - \ln\frac{3}{2}$.

6. Conclusion

The given consequences results of our main results, in the previous section, is not complete. Indeed, we can able to a very long list of our results either by choosing the auxiliary function α in a suitable way, see e.g. [22] or by using the following examples, see e.g. [5, 24, 28].

Example 7. *Let $\phi_i : [0,\infty) \to [0,\infty)$ be continuous functions verifying: $\phi_i(t) = 0$ if and only if $t = 0$. For $i = 1, 2, 3, 4, 5, 6$, we define the mappings $\zeta_i : [0,\infty) \times [0,\infty) \to \mathbb{R}$, as follows*

- *(i)*: *$\zeta_1(t,s) = \phi_1(s) - \phi_2(t)$ for all $t, s \in [0,\infty)$, where $\phi_1(t) < t \leq \phi_2(t)$ for all $t > 0$.*

(ii): $\zeta_2(t,s) = s - \dfrac{f(t,s)}{g(t,s)}t$ *for all* $t,s \in [0,\infty)$, *where* $f,g : [0,\infty)^2 \to (0,\infty)$ *are two continuous functions with respect to each variable such that* $f(t,s) > g(t,s)$ *for all* $t,s > 0$.

(iii): $\zeta_3(t,s) = s - \phi_3(s) - t$ *for all* $t,s \in [0,\infty)$.

(iv): *If* $\varphi : [0,\infty) \to [0,1)$ *is a function such that* $\limsup_{t\to r^+} \varphi(t) < 1$ *for all* $r > 0$, *and we define*

$$\zeta_4(t,s) = s\,\varphi(s) - t \qquad \text{for all } s,t \in [0,\infty).$$

(v): *If* $\eta : [0,\infty) \to [0,\infty)$ *is an upper semi-continuous mapping such that* $\eta(t) < t$ *for all* $t > 0$ *and* $\eta(0) = 0$, *and we define*

$$\zeta_5(t,s) = \eta(s) - t \qquad \text{for all } s,t \in [0,\infty).$$

(vi): *If* $\phi : [0,\infty) \to [0,\infty)$ *is a function such that* $\int_0^\varepsilon \phi(u)du$ *exists and* $\int_0^\varepsilon \phi(u)du > \varepsilon$, *for each* $\varepsilon > 0$, *and we define*

$$\zeta_6(t,s) = s - \int_0^t \phi(u)du \qquad \text{for all } s,t \in [0,\infty).$$

It is clear that each function ζ_i $(i = 1,2,3,4,5,6)$ *forms a simulation function.*

It is clear that each distinct choice of ζ in Theorem 3.1 and Theorem 3.2 gives some existing results on the topic in the literature.

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Hassen Aydi
Department of Mathematics, College of Education in Jubail, Imam Abdulrahman Bin Faisal University, P.O. 12020, Industrial Jubail 31961, Saudi Arabia.
E-mail address: hmaydi@iau.edu.sa

Abdelbasset Felhi
Université de Carthage, Institut Supérieur aux Etudes d'Ingénieur de Bizerte, Mathematics and Physics, Bizerte, Tunisia.
E-mail address: abdelbassetfelhi@gmail.com

Erdal Karapinar
Department of Mathematics, Atilim University, 06836, İncek, Ankara, Turkey,, Department of Mathematics, King Saud University, Riyadh, Saudi Arabia
E-mail address: erdalkarapinar@yahoo.com

Fatimah A. Alojail
Department of Mathematics, College of Education in Jubail, Imam Abdulrahman Bin Faisal University, P.O. 12020, Industrial Jubail 31961, Saudi Arabia.
E-mail address: falojail@iau.edu.sa