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ON THE INCLUSIONS OF SOME LORENTZ MIXED NORMED SPACES AND WIENER-DITKIN SETS

AYŞE SANDIKÇI

Abstract. In this paper, we consider sufficient and necessary condition for the inclusion between Lorentz spaces with mixed norms for two different product measures on $(X \times Y, \mathcal{A} \times \mathcal{B})$. Later, we discuss the Wiener-Ditkin sets of Lorentz spaces with mixed norms and Lorentz spaces.

1. Introduction

Let $(X, \mathcal{A}, \mu)$ be a measure space such that μ is a nonnegative measure. For a measurable function f on X, the distribution function is

$$\lambda_f(y) = \mu\left(\{x \in X \mid |f(x)| > y\}\right), \qquad (y > 0).$$

Its rearrangement is defined by

$$f^*(t) = \inf\{y > 0 : \lambda_f(y) \leq t\}, \qquad (t > 0),$$

and its average function is given by

$$f^{**}(x) = \frac{1}{x}\int_0^x f^*(t)dt, \qquad (x > 0).$$

Note that λ_f, f^* and f^{**} are nonincreasing and right continuous functions on $(0, \infty)$. If $\lambda_f(y)$ is continuous and strictly decreasing, then f^* is the inverse function of λ_f. The most important property of f^* is that it has the same distribution function as f. It follows that

$$\left(\int_X |f(x)|^p d\mu(x)\right)^{\frac{1}{p}} = \left(\int_0^\infty [f^*(t)]^p dt\right)^{\frac{1}{p}}. \tag{1.1}$$

The Lorentz space denoted by $L(p,q)(X,\mu)$ (shortly $L(p,q)$) is defined to be vector space of all (equivalence classes) of measurable functions f such that $\| f \|^*_{pq} < \infty$, where

$$\| f \|^*_{pq} = \begin{cases} \left(\frac{q}{p}\int_0^\infty t^{\frac{q}{p}-1}[f^*(t)]^q dt\right)^{\frac{1}{q}}, & 0 < p, q < \infty \\ \sup_{t>0} t^{\frac{1}{p}} f^*(t), & 0 < p \leq q = \infty. \end{cases}$$

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By (1.1), it follows that $\| f \|_{pp}^{*} = \| f \|_{p}$ and so $L(p,p) = L^{p}$. Also, $L(p,q)(X,\mu)$ is a normed space with the norm

$$\| f \|_{pq} = \begin{cases} \left(\frac{q}{p} \int_0^{\infty} t^{\frac{q}{p}-1} [f^{**}(t)]^{q} dt \right)^{\frac{1}{q}}, & 0 < p, q < \infty \\ \sup_{t>0} t^{\frac{1}{p}} f^{**}(t), & 0 < p \leq q = \infty. \end{cases}$$

For any one of the cases $p = q = 1$, $p = q = \infty$ or $1 < p < \infty$ and $1 \leq q \leq \infty$, then the Lorentz space $L(p,q)(X,\mu)$ is a Banach space with respect to the norm $\| \cdot \|_{pq}$. It is also known that if $1 < p < \infty$, $1 \leq q \leq \infty$ we have

$$\| \cdot \|_{pq}^{*} \leq \| \cdot \|_{pq} \leq \frac{p}{p-1} \| \cdot \|_{pq}^{*}.$$

Namely, $\|.\|_{pq}$ is equivalent to $\|.\|_{pq}^{*}$ (see [3, 8, 11]). $L(p,q)$ spaces are function spaces which are closely related to L^{p} spaces.

Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be σ-finite measure spaces, f be a complex valued measurable function on the product space $(X \times Y, \mathcal{A} \times \mathcal{B}, \mu \times \nu)$, $1 < P = (p_1, p_2) < \infty$ and $1 \leq Q = (q_1, q_2) \leq \infty$, that is p_i and q_i, $i = 1, 2$, are between 1 and ∞. The Lorentz mixed norm space $L(P,Q) = L(P,Q)(X \times Y)$ is defined by

$$L(P,Q) = L(p_2,q_2)[L(p_1,q_1)] = \left\{ f : \| f \|_{PQ} = \| f \|_{L(p_2,q_2)(L(p_1,q_1))} = \left\| \| f \|_{p_1 q_1} \right\|_{p_2 q_2} < \infty \right\}.$$

So, $L(P,Q)$ occurs by taking an $L(p_1,q_1)$-norm with respect to first variable and an $L(p_2,q_2)$-norm with respect to second variable. The $L(P,Q)$ space is a Banach space under the norm $\| \cdot \|_{PQ}$ (see [2, 5, 10]).

In section 2, we prove that the inclusion $L(P_1,Q_1)(X \times Y, \mu_1 \times \nu_1) \subseteq L(P_2,Q_2)(X \times Y, \mu_2 \times \nu_2)$ will hold under which condition for two different product measures. The first results in this theory are due to Miamee (see [9]) for Lebesgue spaces with two different measures. In [6], Gürkanlı extended the results of Miamee to the Lorentz spaces. Our theorems generalize previous results from [9] and [6]. In the third section, we investigate the Wiener-Ditkin sets of Lorentz spaces with mixed norms and Lorentz spaces.

In this paper, we denote the Lebesgue space with mixed norm by $L^{p,q}$; the Lorentz space by $L(p,q)$; the Lorentz mixed norm space $L(P,Q)$ or sometimes $L(p_2,q_2)[L(p_1,q_1)]$, where $P = (p_1,p_2)$ and $Q = (q_1,q_2)$.

2. The inclusions of some Lorentz mixed normed spaces

Throughout the paper, the letters P and Q will denote 2-tuples $P = (p_1,p_2)$ and $Q = (q_1,q_2)$, where p_i and q_i, $i = 1, 2$, are between 1 and ∞. When $i = 1$, we shall write $P = p$ and $Q = q$. Moreover, $P \leq Q$ will mean $p_i \leq q_i$ for $i = 1, 2$. Further, in this section, the symbol $\mu \ll \nu$ is used to indicate the absolute continuity of μ with respect to ν.

Lemma 2.1. *Let $0 < P_1, P_2 < \infty$ and $0 < Q_1, Q_2 \leq \infty$, where $P_i = \left(p_i^1, p_i^2\right)$, $Q_i = \left(q_i^1, q_i^2\right)$, $i = 1, 2$. Then the inclusion $L(P_1,Q_1)(X \times Y, \mu_1 \times \nu_1) \subseteq L(P_2,Q_2)(X \times Y, \mu_2 \times \nu_2)$ holds in the sense of equivalence classes if and only if $\mu_1 \times \nu_1$ and $\mu_2 \times \nu_2$ are*

absolute continuous with respect to each other and $L(P_1,Q_1)(X\times Y,\mu_1\times\nu_1)\subseteq L(P_2,Q_2)(X\times Y,\mu_2\times\nu_2)$ *in the sense of individual functions.*

Proof. The 'if' part is easy. To prove the 'only if' part, let us take any $E\in\mathcal{A}\times\mathcal{B}$ with $(\mu_1\times\nu_1)(E)=0$. Then we have $\mu_1(E_y)=0$ for $\nu_1-a.e.\ y\in Y$ and $\nu_1(E_x)=0$ for $\mu_1-a.e.\ x\in X$, where E_y and E_x are the $x-section$ and the $y-section$, respectively, of E. Hence $\chi_{E_y}=0$ $\nu_1-a.e.$ and the rearrangement of χ_{E_y} is

$$\chi^*_{E_y}(s)=\begin{cases}1, & 0<s<\mu_1(E_y)\\ 0, & s\geq\mu_1(E_y)\end{cases}.$$

If $0<P_1,Q_1<\infty$, then we have

$$\begin{aligned}\|\chi_E\|^*_{P_1Q_1}&=\left\|\|\chi_{E_y}\|^*_{p_1^1q_1^1}\right\|^*_{p_1^2q_1^2}=\left\|\left(\frac{q_1^1}{p_1^1}\int_0^\infty s^{\frac{q_1^1}{p_1^1}-1}[\chi^*_{E_y}(s)]^{q_1^1}ds\right)^{\frac{1}{q_1^1}}\right\|^*_{p_1^2q_1^2}\\&=\left\|\left(\frac{q_1^1}{p_1^1}\int_0^{\mu_1(E_y)}s^{\frac{q_1^1}{p_1^1}-1}ds\right)^{\frac{1}{q_1^1}}\right\|^*_{p_1^2q_1^2}\\&=\left\|(\mu_1(E_y))^{\frac{1}{p_1^1}}\right\|^*_{p_1^2q_1^2}=0\end{aligned}\tag{2.1}$$

since $\mu_1(E_y)=0$ *a.e.* with respect to ν_1. Similarly, if $0<P_1<\infty$ and $Q_1=(\infty,\infty)$, we obtain

$$\|\chi_E\|^*_{P_1Q_1}=\left\|\|\chi_{E_y}\|^*_{p_1^1\infty}\right\|^*_{p_1^2\infty}=\left\|(\mu_1(E_y))^{\frac{1}{p_1^1}}\right\|^*_{p_1^2\infty}=0.$$

So, we get $\chi_E\in L(P_1,Q_1)(X\times Y,\mu_1\times\nu_1)$ for $0<P_1<\infty$ and $0<Q_1\leq\infty$. Then χ_E is in the equivalence class 0 of $L(P_1,Q_1)(X\times Y,\mu_1\times\nu_1)$, and so, by the hypothesis, this equivalence class 0 is in the space $L(P_2,Q_2)(X\times Y,\mu_2\times\nu_2)$. Hence χ_E is in the equivalence class 0 of $L(P_2,Q_2)(X\times Y,\mu_2\times\nu_2)$. Thus we write

$$\|\chi_E\|^*_{P_2Q_2}=\left\|(\mu_2(E_y))^{\frac{1}{p_2^1}}\right\|^*_{p_2^2q_2^2}=0.$$

It follows from here $\mu_2(E_y)=0$ *a.e.* with respect to ν_2. That means $(\mu_2\times\nu_2)(E)=\int\mu_2(E_y)\,d\nu_2=0$. Thus the absolute continuity of $\mu_2\times\nu_2$ follows. Similarly, $\mu_1\times\nu_1$ is absolute continuous with respect to $\mu_2\times\nu_2$. □

We shall need the following lemma for the next theorem. In this lemma, we will state the existence of *a.e.* convergent subsequence of a convergent sequence (f_n) in $L(P,Q)(X\times Y,\mu\times\nu)$. This mode of convergence is true for Banach function spaces [1]. Since $L(P,Q)(X\times Y,\mu\times\nu)$ is a Banach function space [2], it is also true for $L(P,Q)(X\times Y,\mu\times\nu)$. But let us provide the details anyway for the sake of completeness.

Lemma 2.2. *Let* $1<P=(p_1,p_2)<\infty$, $1\leq Q=(q_1,q_2)\leq\infty$. *If* (f_n) *is a sequence in* $L(P,Q)(X\times Y,\mu\times\nu)$ *and* $\|f_n-f\|_{PQ}\to 0$, *where* $f\in L(P,Q)(X\times Y,\mu\times\nu)$, *then* (f_n) *contains a subsequence that converges almost everywhere to the limit function.*

Proof. Suppose $\|f_n - f\|_{PQ} \to 0$ as $n \to \infty$. Then $\|f_n - f\|_{p_1 q_1} \to 0$ as $n \to \infty$. By using the inequality $\|f\|^*_{p_1 \infty} \leq \|f\|^*_{p_1 q_1} \leq \|f\|_{p_1 q_1}$, we get $\|f_n - f\|^*_{p_1 \infty} \to 0$ as $n \to \infty$. Thus we write

$$\begin{aligned} \|f_n - f\|^*_{p_1 \infty} &= \sup_{s>0} s^{\frac{1}{p_1}} (f_n - f)^*_y (s) = \sup_{\lambda > 0} \lambda \left(\mu_{(f_n - f)_y} (\lambda) \right)^{\frac{1}{p_1}} \\ &= \sup_{\lambda > 0} \lambda \left(\mu \left\{ x \in X : \left| (f_n - f)_y \right| > \lambda \right\} \right)^{\frac{1}{p_1}} \end{aligned}$$

and so $\mu \left\{ x \in X : \left| (f_n - f)_y \right| > \lambda \right\} \to 0$ as n tends to infinity. This implies $(f_n)_y$ converges to f_y in measure with respect to μ. Hence there is a subsequence $(f_{n_k})_y \subset (f_n)_y$ such that $(f_{n_k})_y$ converges $\mu - a.e.$ to f_y. Finally, by holding x fixed and repeating the same procedure with respect to second variable y, we obtain a subsequence (f_{n_k}) of (f_n) converges almost everywhere to the limit function f. □

Theorem 2.3. *Let $1 < P_1, P_2 < \infty$ and $1 \leq Q_1, Q_2 \leq \infty$, where $P_i = \left(p_i^1, p_i^2\right)$, $Q_i = \left(q_i^1, q_i^2\right)$, $i = 1, 2$. Then $L(P_1, Q_1)(X \times Y, \mu_1 \times \nu_1) \subset L(P_2, Q_2)(X \times Y, \mu_2 \times \nu_2)$ if and only if $\mu_1 \times \nu_1$ is absolute continuous with respect to $\mu_2 \times \nu_2$ and there exists a constant $C > 0$ such that*

$$\|f\|_{P_2 Q_2, \mu_2 \times \nu_2} \leq C \|f\|_{P_1 Q_1, \mu_1 \times \nu_1} \tag{2.2}$$

for all $f \in L(P_1, Q_1)(X \times Y, \mu_1 \times \nu_1)$.

Proof. Suppose that $L(P_1, Q_1)(X \times Y, \mu_1 \times \nu_1) \subset L(P_2, Q_2)(X \times Y, \mu_2 \times \nu_2)$. Then $\mu_1 \times \nu_1$ and $\mu_2 \times \nu_2$ are absolute continuous with respect to each other by Lemma 2.1. In order to show (2.2), define the linear operator $I(f) = f$ from $L(P_1, Q_1)(X \times Y, \mu_1 \times \nu_1)$ into $L(P_2, Q_2)(X \times Y, \mu_2 \times \nu_2)$. Then the graph of I is

$$G(I) = \{(f, I(f)) : f \in L(P_1, Q_1)(X \times Y, \mu_1 \times \nu_1)\}$$

and it is a vector subspace of $L(P_1, Q_1)(X \times Y, \mu_1 \times \nu_1) \times L(P_2, Q_2)(X \times Y, \mu_2 \times \nu_2)$ since I is linear. Now we will show that $G(I)$ is close. Assume that $(f_n, I(f_n)) \subset G(I)$ converges to (f, g), where $f \in L(P_1, Q_1)(X \times Y, \mu_1 \times \nu_1)$, $g \in L(P_2, Q_2)(X \times Y, \mu_2 \times \nu_2)$. Then, for any $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\|(f_n, I(f_n)) - (f, g)\| = \|f_n - f\|_{P_1 Q_1, \mu_1 \times \nu_1} + \|I(f_n) - g\|_{P_2 Q_2, \mu_2 \times \nu_2} < \epsilon$$

for all $n \geq n_0$. Thus we get $\|f_n - f\|_{P_1 Q_1, \mu_1 \times \nu_1} \to 0$ and $\|I(f_n) - g\|_{P_2 Q_2, \mu_2 \times \nu_2} \to 0$. Then by Lemma 2.2, there exists a subsequence (f_{n_k}) of (f_n) such that

$$f_{n_k} \to f, \quad \mu_1 \times \nu_1 - a.e. \qquad \text{and} \qquad I(f_{n_k}) = f_{n_k} \to g, \quad \mu_2 \times \nu_2 - a.e.$$

Since $\mu_2 \times \nu_2 \ll \mu_1 \times \nu_1$, we have

$$f_{n_k} \to f, \quad \mu_2 \times \nu_2 - a.e. \qquad \text{and} \qquad f_{n_k} \to g, \quad \mu_2 \times \nu_2 - a.e.$$

Thus $f = g$, $\mu_2 \times \nu_2 - a.e.$ and so $(f, g) \in G(I)$. That means $G(I)$ is close. Hence the linear operator I is bounded by closed graph theorem. Then there exists $C > 0$ such that

$$\|f\|_{P_2 Q_2, \mu_2 \times \nu_2} \leq C \|f\|_{P_1 Q_1, \mu_1 \times \nu_1}$$

for all $f \in L(P_1, Q_1)(X \times Y, \mu_1 \times \nu_1)$.
The if-part of this theorem is obvious. □

Proposition 2.4. *Let $0 < P = (p_1, p_2) < \infty$, $0 < Q = (q_1, q_2) \leq \infty$ and $f \in L(P,Q)(X \times Y, \mu_1 \times \nu_1)$ be a real valued measurable function. If there exist constants $m, n > 0$ such that $\mu_2(A) \leq m\mu_1(A)$ for all $A \in \mathcal{A}$ and $\nu_2(B) \leq n\nu_1(B)$ for all $B \in \mathcal{B}$, then we have the inequality*

$$\|f\|^*_{PQ,\mu_2\times\nu_2} \leq m^{\frac{1}{p_1}} n^{\frac{1}{p_2}} \|f\|^*_{PQ,\mu_1\times\nu_1}.$$

Moreover, there is the inclusion $L(P,Q)(X \times Y, \mu_1 \times \nu_1) \subseteq L(P,Q)(X \times Y, \mu_2 \times \nu_2)$.

Proof. Let $f \in L(P,Q)(X \times Y, \mu_1 \times \nu_1)$. Then $f_y \in L(p_1,q_1)(X,\mu_1)$ and $\|f_y\|_{p_1q_1,\mu_1} \in L(p_2,q_2)(Y,\nu_1)$. Since $f_y \in L(p_1,q_1)(X,\mu_1)$ is a measurable real valued function, then

$$E_\sigma = \{x \in X : f_y(x) > \sigma\} \in \mathcal{A}$$

for all real number σ. Also let M and N such that

$$M = \left\{\sigma > 0 : \lambda^{\mu_2}_{f_y}(\sigma) \leq t\right\} \qquad \text{and} \qquad N = \left\{\sigma > 0 : \lambda^{m\mu_1}_{f_y}(\sigma) \leq t\right\}$$

for all $t > 0$, where $\lambda^{\mu_2}_{f_y}(\sigma) = \mu_2(E_\sigma)$ and $\lambda^{m\mu_1}_{f_y}(\sigma) = (m\mu_1)(E_\sigma)$. Since $\mu_2(A) \leq m\mu_1(A)$, we get $\lambda^{\mu_2}_{f_y}(\sigma) \leq \lambda^{m\mu_1}_{f_y}(\sigma)$ and so $N \subset M$. Thus we can write

$$f^{*,\mu_2}_y(t) = \inf_\sigma M \leq \inf_\sigma N = f^{*,m\mu_1}_y(t). \tag{2.3}$$

Moreover, since

$$\begin{aligned}\left\{\sigma > 0 : \lambda^{m\mu_1}_{f_y}(\sigma) \leq t\right\} &= \{\sigma > 0 : (m\mu_1)(E_\sigma) \leq t\} \\ &= \left\{\sigma > 0 : (\mu_1)(E_\sigma) \leq \frac{t}{m}\right\},\end{aligned} \tag{2.4}$$

we have by (2.3) and (2.4)

$$f^{*,\mu_2}_y(t) \leq f^{*,m\mu_1}_y(t) = f^{*,\mu_1}_y\left(\frac{t}{m}\right).$$

Thus we get

$$\begin{aligned}\| f_y \|^*_{p_1q_1,\mu_2} &= \left(\frac{q_1}{p_1}\int_0^\infty t^{\frac{q_1}{p_1}-1}[f^{*,\mu_2}_y(t)]^{q_1}dt\right)^{\frac{1}{q_1}} \\ &\leq \left(\frac{q_1}{p_1}\int_0^\infty t^{\frac{q_1}{p_1}-1}\left(f^{*,\mu_1}_y(\frac{t}{m})\right)^{q_1}dt\right)^{\frac{1}{q_1}} \\ &= \left(\frac{q_1}{p_1}\int_0^\infty (mu)^{\frac{q_1}{p_1}-1}\left(f^{*,\mu_1}_y(u)\right)^{q_1}mdu\right)^{\frac{1}{q_1}} \\ &= m^{\frac{1}{p_1}} \| f_y \|^*_{p_1q_1,\mu_1}.\end{aligned} \tag{2.5}$$

Also since $\|f_y\|_{p_1q_1,\mu_1} \in L(p_2,q_2)(Y,\nu_1)$ and $\nu_2(B) \leq n\nu_1(B)$ for all $B \in \mathcal{B}$, by applying the same procedure to second variable, we obtain

$$\left\|\|f_y\|^*_{p_1q_1,\mu_1}\right\|^*_{p_2q_2,v_2} \leq n^{\frac{1}{p_2}} \left\|\|f_y\|^*_{p_1q_1,\mu_1}\right\|^*_{p_2q_2,v_1}. \tag{2.6}$$

Hence we have $\|f_y\|_{p_1q_1,\mu_1} \in L(p_2,q_2)(Y,\nu_2)$. Since $L(p_2,q_2)(Y,\nu_2)$ is a solid space, we obtain from (2.5) and (2.6)

$$\begin{aligned} \| f \|^*_{PQ,\mu_2\times\nu_2} &= \left\| \|f_y\|^*_{p_1q_1,\mu_2} \right\|^*_{p_2q_2,v_2} \leq m^{\frac{1}{p_1}} \left\| \|f_y\|^*_{p_1q_1,\mu_1} \right\|^*_{p_2q_2,v_2} \\ &\leq m^{\frac{1}{p_1}} n^{\frac{1}{p_2}} \left\| \|f_y\|^*_{p_1q_1,\mu_1} \right\|^*_{p_2q_2,v_1} = m^{\frac{1}{p_1}} n^{\frac{1}{p_2}} \| f \|^*_{PQ,\mu_1\times\nu_1} . \end{aligned} \tag{2.7}$$

Let S denote the set of simple functions. By Proposition 2.2 in [5], $\overline{S} = L(P,Q)(X\times Y,\mu_1\times\nu_1)$. Let I be a unit function from S into $L(P,Q)(X\times Y,\mu_2\times\nu_2)$. By using (2.7), we have the inequality

$$\|f\|_{PQ,\mu_2\times\nu_2} \leq C \|f\|_{PQ,\mu_1\times\nu_1}$$

for all $f \in S$, where $C = m^{\frac{1}{p_1}} n^{\frac{1}{p_2}}$. That means I is continuous from S into $L(P,Q)(X\times Y,\mu_2\times\nu_2)$. Then I is continuously extended to the space $L(P,Q)(X\times Y,\mu_1\times\nu_1)$. Thus we get the inequality

$$\|f\|_{PQ,\mu_2\times\nu_2} \leq C \|f\|_{PQ,\mu_1\times\nu_1}$$

for all $f \in L(P,Q)(X\times Y,\mu_1\times\nu_1)$. Hence we have the inclusion $L(P,Q)(X\times Y,\mu_1\times\nu_1) \subseteq L(P,Q)(X\times Y,\mu_2\times\nu_2)$. □

In Proposition 4, if there exists the inclusion which is mentioned, we could not obtain the converse of this proposition for $P=(p_1,p_2)$. But we can state it when $P=(p,p)$ as in the following.

Proposition 2.5. *Let $0<P=(p,p)<\infty$, $0<Q=(q_1,q_2)\leq\infty$. If $L(P,Q)(X\times Y,\mu_1\times\nu_1) \subseteq L(P,Q)(X\times Y,\mu_2\times\nu_2)$, then there exists $K>0$ such that $(\mu_2\times\nu_2)(E) \leq K(\mu_1\times\nu_1)(E)$ for all $E\in\mathcal{M}$.*

Proof. Let $L(P,Q)(X\times Y,\mu_1\times\nu_1) \subseteq L(P,Q)(X\times Y,\mu_2\times\nu_2)$. By Theorem 2.3 there exists $C>0$ such that

$$\|f\|_{PQ,\mu_2\times\nu_2} \leq C \|f\|_{PQ,\mu_1\times\nu_1}$$

for all $f\in L(P,Q)(X\times Y,\mu_1\times\nu_1)$. If it is taken $f=\chi_E$, we have from (2.1)

$$\mu_2(E_y)^{\frac{1}{p}} \nu_2(E_x)^{\frac{1}{p}} \leq C\mu_1(E_y)^{\frac{1}{p}} \nu_1(E_x)^{\frac{1}{p}},$$

where E_y and E_x are the $x-$*section* and the $y-$*section* of E, respectively. Hence we obtain

$$(\mu_2(E_y)\nu_2(E_x))^{\frac{1}{p}} \leq C(\mu_1(E_y)\nu_1(E_x))^{\frac{1}{p}}$$

and

$$(\mu_2\times\nu_2)(E) \leq K(\mu_1\times\nu_1)(E),$$

where $K=C^{\frac{1}{p}}$. □

Proposition 2.6. *Let $P_1=(p_1,p_1)$, $P_2=(p_2,p_2)$, $Q_i=\left(q_i^1,q_i^2\right)$, $i=1,2$, and $0<Q_1\leq P_1<P_2\leq Q_2<\infty$. Then there is the inclusion $L(P_1,Q_1)(X\times Y,\mu\times\nu)\subseteq L(P_2,Q_2)(X\times Y,\mu\times\nu)$ if and only if there exists a constant $k>0$ such that $(\mu\times\nu)(E)\geq k$ for every $\mu\times\nu-$non-null set $E\in\mathcal{M}$.*

Proof. Assume that $L(P_1,Q_1)(X\times Y,\mu\times\nu) \subseteq L(P_2,Q_2)(X\times Y,\mu\times\nu)$. Then there exists $C>0$ such that

$$\|f\|_{P_2Q_2,\mu\times\nu} \leq C \|f\|_{P_1Q_1,\mu\times\nu} \tag{2.8}$$

for all f in $L(P_1,Q_1)(X\times Y,\mu\times\nu)$. Let $E\in\mathcal{M}$ be a $\mu\times\nu-$non-null set and $(\mu\times\nu)(E)<\infty$. Then by taking χ_E instead of f in (2.8), we get

$$\begin{aligned}\mu(E_y)^{\frac{1}{p_2}}\nu(E_x)^{\frac{1}{p_2}} &\leq C\mu(E_y)^{\frac{1}{p_1}}\nu(E_x)^{\frac{1}{p_1}}\\ \frac{1}{C} &\leq \mu(E_y)^{\frac{1}{p_1}-\frac{1}{p_2}}\nu(E_x)^{\frac{1}{p_1}-\frac{1}{p_2}}=(\mu(E_y)\nu(E_x))^{\frac{1}{p_1}-\frac{1}{p_2}}\\ k &\leq (\mu\times\nu)(E),\end{aligned}$$

where $k=C^{\frac{p_1p_2}{p_1-p_2}}$.
Conversely, let $f\in L(P_1,Q_1)(X\times Y,\mu\times\nu)$ and

$$E_n=\{(x,y)\,||f(x,y)|>n\}$$

for all $n\in\mathbb{N}$. Since $Q_1\leq P_1$, we have

$$L(P_1,Q_1)(X\times Y,\mu\times\nu)\subset L(P_1,P_1)(X\times Y,\mu\times\nu)=L^{p_1,p_1}(X\times Y,\mu\times\nu),$$

where the space $L^{p_1,p_1}(X\times Y,\mu\times\nu)$ is the Lebesgue space with mixed norm $L^{p_1}(X\times Y,\mu\times\nu)$. Then there exists $C>0$ such that

$$\|f\|_{p_1,\mu\times\nu}\leq C\|f\|_{P_1Q_1,\mu\times\nu}$$

for all $f\in L(P_1,Q_1)(X\times Y,\mu\times\nu)$. From the definition of E_n, we write

$$n^{p_1}\mu\left((E_n)_y\right)\leq\int\limits_{(E_n)_y}|f_y|^{p_1}\,d\mu\leq\int\limits_X|f_y|^{p_1}\,d\mu.$$

Applying the same procedure to second variable, we have

$$n^{p_1}\mu\left((E_n)_y\right)\nu((E_n)_x)\leq\int\limits_Y\int\limits_X|f_y|^{p_1}\,d\mu d\nu$$

and

$$n^{p_1}(\mu\times\nu)(E_n)\leq\|f\|^{p_1}_{p_1,\mu\times\nu}\leq\left(C\|f\|_{P_1Q_1,\mu\times\nu}\right)^{p_1}<\infty$$

for every $n\in\mathbb{N}$. Thus by hypothesis, either $(\mu\times\nu)(E_n)=0$ or $(\mu\times\nu)(E_n)\geq k$. Moreover, since $E_{n+1}\subset E_n$, $(\mu\times\nu)(E_1)<\infty$ and $\bigcap\limits_{n=1}^{\infty}E_n=\varnothing$, we get $(\mu\times\nu)(E_n)\to 0$. Thus there exists $n_0\in\mathbb{N}$ such that $|f(x,y)|\leq n_0$, $\mu\times\nu$ *a.e.* for all $(x,y)\in X\times Y$. Then we write

$$\int\limits_{X\times Y}|f|^{p_2}\,d(\mu\times\nu)\leq n_0^{p_2-p_1}\int\limits_{X\times Y}|f|^{p_1}\,d(\mu\times\nu).$$

So $f\in L^{p_2}(X\times Y,\mu\times\nu)=L(P_2,P_2)(X\times Y,\mu\times\nu)$ and from here $L(P_1,Q_1)(X\times Y,\mu\times\nu)\subset L(P_2,P_2)(X\times Y,\mu\times\nu)$. Also since $P_2\leq Q_2$, we obtain the desired result. □

3. Wiener-Ditkin sets for Lorentz spaces with mixed norms

In this section we investigate the Wiener-Ditkin sets for Lorentz spaces with mixed norms and Lorentz spaces. Let X and Y be locally compact Abelian groups. In the spirit of [7] a closed set A in $(X\times Y)^{\wedge}$ is said to be a Wiener-Ditkin set for $L(P,Q)(X\times Y)$ if each $f\in L(P,Q)(X\times Y)$ such that $\hat{f}$ vanishes on A can be approximated in $L(P,Q)(X\times Y)$ with functions $f*F$ such that $\hat{F}$ vanishes in some neighbourhood on A.

For the next theorem, we need the following remark.

Remark. *A Banach space X is called a (left) Banach module over a Banach algebra A if it is a (left) module over A in the algebraic sense and satisfies $\|ax\|_X \leq \|a\|_A \|x\|_X$ for all $a \in A$, $x \in X$. The closed linear span of $AX = \{ax : a \in A, x \in X\}$ in X is called essential part X_e of X. X is called essential if $X_e = X$, [4]. $L(P,Q)(X \times Y)$ is a Banach module over $L^1(X \times Y)$ for $1 < P < \infty$, $1 \leq Q < \infty$ by Proposition 4 in [12]. Again, it can be seen that the mapping $z \to T_z f$ from $X \times Y$ into $L(P,Q)(X \times Y)$ is continuous for $1 < P < \infty$, $1 \leq Q < \infty$ as in the proof of Theorem 10 in [12], where T_z is translation operator. So it is easy two show that $L(P,Q)(X \times Y)$ is an essential Banach module over $L^1(X \times Y)$ by module factorization theorem.*

Theorem 3.1. *Let $1 < P < \infty$, $1 \leq Q < \infty$. If a closed set A in $(X \times Y)^\wedge$ is a Wiener-Ditkin set for $L^1(X \times Y)$, then it is a Wiener-Ditkin set for $L(P,Q)(X \times Y)$.*

Proof. Suppose that $A \subset (X \times Y)^\wedge$ is a Wiener-Ditkin set for $L^1(X \times Y)$. Let $f \in L(P,Q)(X \times Y)$ such that $\hat{f}$ vanishes on A. Since $L(P,Q)(X \times Y)$ is an essential Banach module over $L^1(X \times Y)$ by above remark, then there exists $\alpha \in I$ such that

$$\|f - f * e_\alpha\|_{PQ} < \frac{\epsilon}{2}$$

for any $\epsilon > 0$. Since $A \subset (X \times Y)^\wedge$ is a Wiener-Ditkin set for $L^1(X \times Y)$, then there exists $F_1 \in L^1(X \times Y)$ such that $\overset{\wedge}{F_1}$ vanishes on a neighbourhood of A. Again since $L(P,Q)(X \times Y)$ is an essential Banach module over $L^1(X \times Y)$, for the same $\epsilon > 0$, we write

$$\|f - f * F_1\|_{PQ} < \frac{\epsilon}{2\|e_\alpha\|_1}.$$

Let us take $F = F_1 * e_\alpha$. Then $\hat{F} = \overset{\wedge}{F_1}\overset{\wedge}{e_\alpha}$ and so $\hat{F}$ vanishes on a neighbourhood of A. Thus we obtain

$$\begin{aligned} \|f - f * F\|_{PQ} &= \|f - f * e_\alpha + f * e_\alpha - f * e_\alpha * F_1\|_{PQ} \\ &\leq \|f - f * e_\alpha\|_{PQ} + \|e_\alpha * (f - f * F_1)\|_{PQ} \\ &\leq \|f - f * e_\alpha\|_{PQ} + \|e_\alpha\|_1 \|f - f * F_1\|_{PQ} \\ &< \frac{\epsilon}{2} + \|e_\alpha\|_1 \frac{\epsilon}{2\|e_\alpha\|_1} = \epsilon. \end{aligned}$$

Hence A is a Wiener-Ditkin set for $L(P,Q)(X \times Y)$. □

Since $L(P,Q) = L(p,q)$ when $P = p$ and $Q = q$, Theorem 3.1 can be stated for Lorentz space $L(p,q)$ as in the following.

Corollary 3.2. *Let $1 < p < \infty$, $1 \leq q < \infty$. If a closed set $A \subset \hat{X}$ is a Wiener-Ditkin set for $L^1(X)$, then it is a Wiener-Ditkin set for $L(p,q)(X)$.*

It is known that closed subgroups of $\mathbb{R}^d$ are Wiener-Ditkin sets for weighted space $L^1_\alpha(\mathbb{R}^d)$ $(0 \leq \alpha < 1)$ from the Theorem in [13]. If $\alpha = 0$, then it is true for $L^1(\mathbb{R}^d)$. Namely, closed subgroups of $\mathbb{R}^d$ are Wiener-Ditkin sets for $L^1(\mathbb{R}^d)$. So, we have the following corollaries by using Theorem 3.1 and Corollary 3.2, respectively.

Corollary 3.3. *Closed subgroups of $\mathbb{R}^{2d}$ are Wiener-Ditkin sets for $L(P,Q)(\mathbb{R}^{2d})$.*

Corollary 3.4. *Closed subgroups of $\mathbb{R}^d$ are Wiener-Ditkin sets for $L(p,q)(\mathbb{R}^d)$.*

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Ayşe Sandikçi

Department of Mathematics, Faculty of Arts and Sciences, Ondokuz Mayis University, Samsun, Turkey

Email address: ayses@omu.edu.tr