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EXISTENCE THEORY OF HIV-1 INFECTION MODEL BY USING ARBITRARY ORDER DERIVATIVE OF WITHOUT SINGULAR KERNEL TYPE

SAMIA BUSHNAQ, SAJJAD ALI KHAN, KAMAL SHAH, GUL ZAMAN

Abstract. This manuscript is devoted to establish the existence theory of solutions to HIV -1 infection model of CD4+T-cells with fractional order derivative. The corresponding fractional order derivative is considered in Caputo-Fabrizio sense, which possesses more important characteristics in mathematical modeling. With the help of Sumudu transform and Picard successive approximation techniques, some interesting results are obtained. Existence and uniqueness for the equilibrium solutions are discussed. Some numerical simulations are provided to show the effectiveness of the theoretical results.

1. Introduction

Human immune deficiency virus (HIV) is a lenti virus that causes acquired immune deficiency syndrome ($AIDS$). This serious disease destroys the immune system of human being which produce life-threatening opportunistic infections in the body. HIV infects primary cell in human immune system such as helper macrophages, dendritic cells and T-cell. Cell-mediated immunity is lost and the body become increasingly susceptible to opportunistic infections and hence the numbers of CD4+T-cell fall below the critical condition. HIV is an epidemic disease which spreading continuously all over the world. It is a visible improvement in our knowledge about the molecular biology of the virus and its effect on a human body. HIV viruses transfer from a calamitous illness into a chronic conditions. HIV led to dramatic change in mobility and mortality from illness. Furthermore, despite these improvement on the biomedical front, the spreading of this epidemic continue and treatment remains unavailable to the overwhelming majority of those who require it. It causes expenditure of a very large amount of money in health care and research and destroy millions of peoples. In the medical field there have been many achievement for health care, but there is still no vaccine available for HIV. For the determination of the transmission dynamics of HIV-1 disease, a mathematical model is a useful tool which also provide technique to control the spread of these disease [11]. In [1], a simple mathematical model introduced by

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Perelson for the primary infection with HIV. In the field of mathematical modeling for HIV infection this model has great importance and also many other models with HIV infection have been proposed, which consider this model as their inspiration. Perelson extended this model by considering four categories of this model such as, uninfected CD4+T-cell, latently infected CD4+T-cell, productively infected CD4+T-cell and virus population. Rong et al. [3], made modification in this model to study the evaluation of drug resistance. Zebai et al. also present a mathematical model of HIV-1 infected T-cell and evaluate stability of the said model. Mathematical models are powerful tools in this approach which help us to optimize the use of finite sources or simply to goal (the incidence of infection) control measures more impressively. The original integer order model which we refer in [2] and is given by

$$\begin{cases} \dfrac{dT(t)}{dt} = \beta - kVT - dT + bT', \\ \dfrac{dT'(t)}{dt} = kVT - (b+s)T', \\ \dfrac{dV(t)}{dt} = \delta T' - cV. \end{cases} \tag{1.1}$$

Where T, T' and V represents the uninfected CD4+T cells, infected CD4+T-cells and free HIV virus particles in the human blood respectively, the natural death rate is denoted by d, rate of infection T-cells represented by k, δ represents death rate of infected T-cells while b represent rate of those infected cells which return to uninfected class and c represent death rate of virus.

Recently, more attentions have been given in studying fractional-order derivatives in describing the memory effects in the dynamical systems. Because, it has been found that the area involving fractional order differential equations have significant applications in various disciplines of science and technology, we refer few of them in [4, 5, 6, 7, 8]. In recent years, the fractional order models are given much attentions, because the biological models that involved fractional order derivative are more realistic and accurate as compared to the classical order models, for detail see[9, 10]. Further, the hereditary characteristic and properties of various material and process can be excellently describe by using fractional order derivatives and integrals. By adapting fractional order derivative in mathematical modeling is a global operator as compared to classical derivative which is local. Treatment of real-world problems via the use of derivative with fractional order proves to be more accurate and efficient, specially when dealing with a model where memory or hereditary property characteristics play a fundamental role.

Variety of definitions have been given for fractional order derivative by the researchers including Riemann-Liouville, Caputo, Hadmard, etc. Since the old version contains a kernel with singularity and hence can not explain the characteristic of memory and hereditary process in physical problems accurately. To avoid singular kernel, Caputo and Fabrizio have recently considered a new fractional derivative and named it Caputo-Fabrizio. They established with the concept of convolution, however this time the convolute filter is the exponential function, which helps to reduce the risk of singularity. This type of derivatives have attracted the attentions of many researchers, for some detail see [12, 13, 15]. Inspired from the aforementioned work, in this manuscript we consider the model discussed in (1.1), by taking the fractional order derivative in Caputo-Fabrizio sense as

$$\begin{cases} {}_0^{CF}D_t^{\beta}T(t) = \beta - kVT - dT + bT', \\ {}_0^{CF}D_t^{\beta}T'(t) = kVT - (b+s)T', \\ {}_0^{CF}D_t^{\beta}V(t) = \delta T' - cV. \end{cases} \tag{1.2}$$

Where $\beta \in (0, 1]$. With the help of Sumudu transform, we transfer system (1.2) to a system of algebraic equations, for which we investigate existence uniqueness, stability results. Also numerical results are provided to demonstrate the solutions graphically. The existence theory of the proposed model provides information about the well-posedness and ill-posedness of the system (1.2).

This paper is managed as follows: In Section 2, we provide some preliminaries results which are needed in this paper. In Section 3, we derive special solutions for the proposed model. In section 4, we have given the stability results of the equilibrium solutions for the proposed model. In section 5, uniqueness of the special solution has been proved. In section 6, we have given a brief conclusion.

2. Preliminaries

We first give the definitions of Caputo-Fabrizio derivative of fractional order. Caputo-Fabrizio derivative with fractional order has been considered with no singular kernel in last few years, for detail see [4, 14].

Definition 2.1. *Let $\phi \in H^1(a, b)$, $b > a$, $\beta \in (0, 1)$, then the new fractional order in Caputo derivative sense is recalled as*

$${}^{C}D_t^{\beta}(\phi(t)) = \frac{M(\beta)}{1-\beta}\int_a^t \phi'(x)\exp[-\beta\frac{t-x}{1-\beta}]dx,$$

where the normalization function is denoted by $M(\beta)$ with $M(0) = M(1) = 1$. But, if the function does not belong to $H^1(a, b)$, then the derivative can be reformulated as

$${}^{C}D_t^{\beta}(\phi(t)) = \frac{\beta M(\beta)}{1-\beta}\int_a^t (\phi(t)-\phi(x))\exp[-\beta\frac{t-x}{1-\beta}]dx.$$

Remark. *If, we take $\sigma = \frac{1-\beta}{\beta} \in [0, \infty)$, $\beta = \frac{1}{1+\beta} \in [0, 1]$, then we will get the new Caputo derivative having fractional order as*

$${}^{C}D_t^{\beta}(\phi(t)) = \frac{M(\sigma)}{\sigma}\int_a^t \phi'(x)\exp[-\frac{t-x}{\sigma}]dx, \quad M(0) = M(\infty) = 1.$$

In addition,

$$\lim_{\sigma\to 0}\frac{1}{\sigma}\exp[-\frac{t-x}{\sigma}] = \delta(x-t).$$

The connected anti-derivative turns out to be imperative at this instant following to the preface of the novel derivative, which was proposed by Nieto and Losada [14].

Definition 2.2. *Let $\beta \in (0, 1)$. Then the fractional integral of order β of a function f is defined by*

$${}^{CF}I_t^{\beta}(\phi(t)) = \frac{2(1-\beta)}{(2-\beta)M(\beta)} + \frac{2\beta}{(2-\beta)M(\beta)}\int_0^t \phi(s)ds, \quad t \geq 0.$$

Remark. *According to above definition, it has to be noted that the fractional integral of Caputo type of a function of order $0 < \beta \leq 1$ is an average between function ϕ and its integral of order one. This therefore imposes*

$$\frac{2(1-\beta)}{(2-\beta)M(\beta)} + \frac{2\beta}{(2-\beta)M(\beta)} = 1.$$

The above equation generate an explicit formula for

$$M(\beta) = \frac{2}{(2-\beta)}, \quad 0 \leq \beta \leq 1$$

Keep in view the above, Nieto and Losada reformulated the new Caputo derivative of order $0 < \beta < 1$ as

$$^{C}D_t^{\beta}(\phi(t)) = \frac{1}{1-\beta}\int_a^t \phi'(x)\exp[-\beta\frac{t-x}{1-\beta}]dx.$$

Definition 2.3. *For any function $\phi(t)$ over a set, the Sumudu transform will be given as*

$$\mathbf{A} = \{\phi(t) : \textit{there exist } \Lambda, \tau_1, \tau_2 > 0, \ |\phi(t)| < \Lambda \exp(\frac{|t|}{\tau_i}, \ \textit{if } t \in (-1)^j \times [0,\infty)\}$$

is defined by

$$F(u) = \mathcal{S}[\phi(t)] = \int_0^{\infty} \exp(-t)\phi(ut)dt, \ u \in (-\tau_1, \tau_2).$$

Definition 2.4. *The Sumudu transform of ordinary Caputo fractional order derivative of a function $\phi(t)$ is given by*

$$\mathcal{S}[^{C}D^{\beta}\phi(t)] = u^{-\beta}\left[F(u) - \sum_{i=0}^{n} u^{\beta-i}[^{C}D^{\beta-i}(\phi(t))]_{t=0}\right], \ n-1 < \beta \leq n.$$

In view of Definition 2.4, we recall the Sumudu transform of a function $\phi(t)$ with Caputo-Fabrizio fractional derivative as bellow:

Definition 2.5. [16]*Let $\phi(t)$ be a function for which the Caputo-Fabrizio exists, then the Sumudu transform of the Caputo-Fabrizio fractional derivative of $f(t)$ is given as*

$$\mathcal{S}(^{CF}_{0}D_t^{\beta})(\phi(t)) = M(\beta)\frac{\mathcal{S}(\phi(t)) - \phi(0)}{1-\beta+\beta u}.$$

3. Derivation of the special solution

The aim of this section is to provide a special solution of the system (1.2) by applying the Sumudu transform on both sides of all equations of (1.2) together with an iterative method. To get this we proceed as:
Applying Sumudu transform on both sides of proposed model (1.2), we obtain

$$\begin{aligned}
M(\beta)\frac{\mathcal{S}(T(t)) - T(0)}{1-\beta+\beta u} &= \mathcal{S}[\beta - kVT - dT + bT'],\\
M(\beta)\frac{\mathcal{S}(T'(t)) - T'(0)}{1-\beta+\beta u} &= \mathcal{S}[kVT - (b+s)T'],\\
M(\beta)\frac{\mathcal{S}(V(t)) - V(0)}{1-\beta+\beta u} &= \mathcal{S}[\delta T' - cV].
\end{aligned}$$

Rearranging, we obtain

$$\begin{aligned}
\mathcal{S}(T(t)) &= T(0) + \frac{(1-\beta+\beta u)}{M(\beta)}\mathcal{S}[\beta - kVT - dT + bT'],\\
\mathcal{S}(T'(t)) &= T'(0) + \frac{(1-\beta+\beta s)}{M(\beta)}\mathcal{S}[kVT - (b+s)T'],\\
\mathcal{S}(V(t)) &= V(0) + \frac{(1-\beta+\beta u)}{M(\beta)}\mathcal{S}[\delta T' - cV].
\end{aligned} \tag{3.1}$$

If we apply the inverse Sumudu transform on both sides of equation (3.1), we obtain

$$\begin{aligned}
T(t) &= T(0) + \mathcal{S}^{-1}\left[\frac{(1-\beta+\beta u)}{M(\beta)}\mathcal{S}[\beta - kVT - dT + bT']\right],\\
T'(t) &= T'(0) + \mathcal{S}^{-1}\left[\frac{(1-\beta+\beta u)}{M(\beta)}\mathcal{S}[kVT - (b+s)T']\right],\\
V(t) &= V(0) + \mathcal{S}^{-1}\left[\frac{(1-\beta+\beta u)}{M(\beta)}\mathcal{S}[\delta T' - cV]\right].
\end{aligned} \tag{3.2}$$

We set the recursive formula from (3.2) as

$$\begin{aligned}
T_{n+1}(t) &= T_n(0) + \mathcal{S}^{-1}\left[\frac{(1-\beta+\beta u)}{M(\beta)}\mathcal{S}[\beta - kV_nT_n - dT_n + bT'_n]\right],\\
T'_{n+1}(t) &= T'_n(0) + \mathcal{S}^{-1}\left[\frac{(1-\beta+\beta u)}{M(\beta)}\mathcal{S}[kV_nT_n - (b+s)T'_n]\right],\\
V_{n+1}(t) &= V_n(0) + \mathcal{S}^{-1}\left[\frac{(1-\beta+\beta u)}{M(\beta)}\mathcal{S}[\delta T'_n - cV_n]\right].
\end{aligned} \tag{3.3}$$

In this way, the solution of (3.3) is provided by

$$\begin{aligned}
T(t) &= \lim_{n\to\infty} T_n(t),\\
T'(t) &= \lim_{n\to\infty} T'_n(t),\\
V(t) &= \lim_{n\to\infty} V_n(t).
\end{aligned}$$

By assigning some random values to $T(0) = 10, T'(0) = 5, V(0) = 20$. To the parameters we assign the values $c = 0.6, \gamma = 0.4, \alpha = 0.2, b = 0.3, \delta = 0.5$. Inserting these values in System (3.3), the plots of the approximate solutions against different fractional order β are provided as in Figure 1, Figure 2 and Figure 3 respectively.

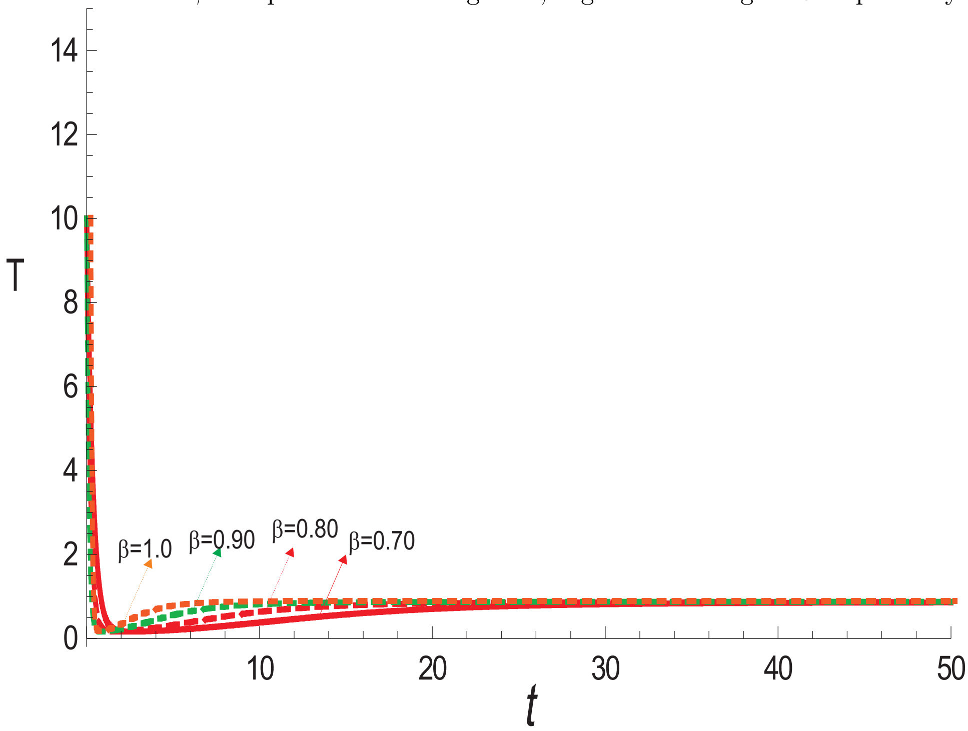

Figure 1. Plots of approximate solutions for the compartment T at different fractional order β .

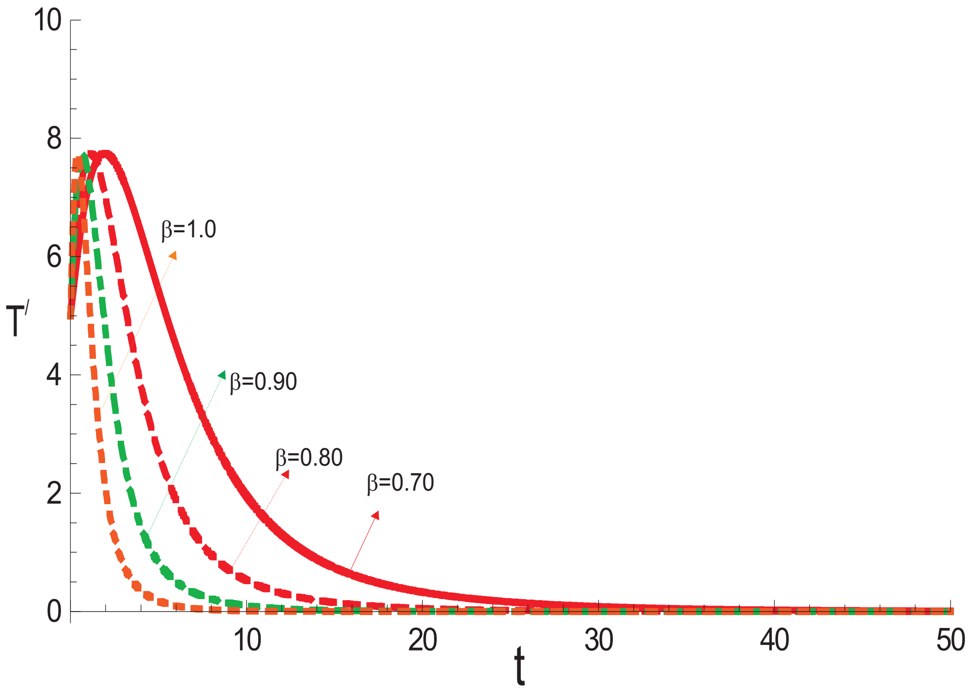

Figure 2. Plots of approximate solutions for the compartment T' at different fractional order β.

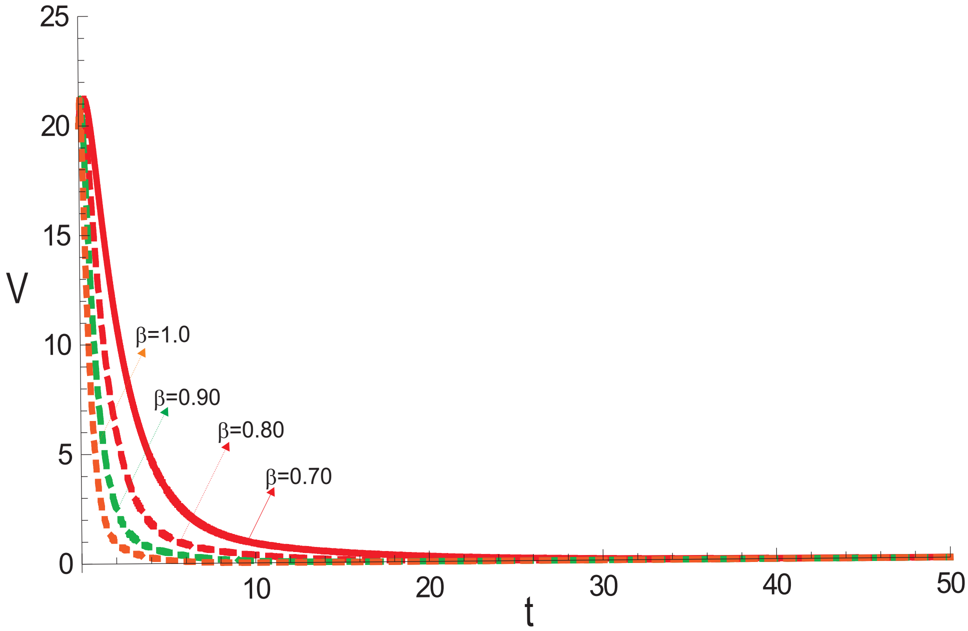

Figure 3. Plots of approximate solutions for the compartment V at different fractional order β.

4. Application of fixed-point theorem for stability analysis of iteration method

Let us suppose $(X_1, \|\cdot\|)$ as a Banach space and P as a self-map of X_1. Let $y_{n+1} = g(P, y_n)$ be particular recursive procedure. Suppose that, $F(P)$ the fixed-point set of P has at least one element and that y_n converges to a point $p \in F(P)$. Let $\{x_n \subseteq X_1\}$ and define $e_n = \|x_{n+1} - g(P, x_n)\|$. If $\lim_{n\to\infty} e^n = 0$ implies that $lim_{n\to\infty} x^n = p$, then the iteration method $y_{n+1} = g(P, y_n)$ is said to be P-stable. Analogously, we therefore consider that, our sequence $\{x_n\}$ has an upper bound, otherwise there is no possibility of convergence. The iteration will be P-stable, if all these conditions are satisfied for $y_{n+1} = Py_n$ which is also known as Picard's iteration as given in [17]. We shall then state the following theorem.

Theorem 4.1. *Let $(X_1, \|\cdot\|)$ be a Banach space and P be a self-map of X_1 satisfying*

$$\|P_x - P_y\| \leq C\|x - P_x\| + c\|x - y\|$$

for all $x, y \in X_1$ where $0 \leq C, 0 \leq c < 1$. Suppose that P is Picard $P-$stable . Let us take into account the following recursive formula from (3.3) connected to (1.2).

$$T_{n+1}(t) = T_n(0) + \mathcal{S}^{-1}\left[\frac{(1-\beta+\beta u)}{M(\beta)}\mathcal{S}[\beta - kV_nT_n - dT_n + bT'_n]\right],$$

$$T'_{n+1}(t) = T'_n(0) + SL^{-1}\left[\frac{(1-\beta+\beta u)}{M(\beta)}\mathcal{S}[kV_nT_n - (b+s)T'_n]\right],$$

$$V_{n+1}(t) = V_n(0) + \mathcal{S}^{-1}\left[\frac{(1-\beta+\beta u)}{M(\beta)}\mathcal{S}[\delta T'_n - cV_n]\right].$$

where $\frac{(1-\beta+\beta u)}{M(\beta)}$ *is the fractional Lagrange multiplier.*

Theorem 4.2. *Let us defined a self-map* P *as*

$$\begin{aligned}
P(T_n(t)) &= T_{n+1}(t) = T_n(0) + \mathcal{S}^{-1}\left[\frac{(1-\beta+\beta u)}{M(\beta)}\mathcal{S}[\beta - kV_nT_n - dT_n + bT_n']\right],\\
P(T_n'(t)) &= T_{n+1}'(t) = T_n'(0) + \mathcal{S}^{-1}\left[\frac{(1-\beta+\beta u)}{M(\beta)}\mathcal{S}[kV_nT_n - (b+s)T_n']\right],\\
P(V_n(t)) &= V_{(n+1)}(t) = V_n(0) + \mathcal{S}^{-1}\left[\frac{(1-\beta+\beta u)}{M(\beta)}\mathcal{S}[\delta T_n' - cV_n]\right].
\end{aligned}$$

is $P-$*stable in* $L^1(a,b)$ *if*

$$\begin{cases} 1-(d-b)f(\gamma) - k(M+L)g(\gamma)\} < 1,\\ 1-(b+s)f_1(\gamma) + k(M_1+L_1)g_1(\gamma)\} < 1,\\ 1+(\delta-c)f_2(\gamma)\} < 1. \end{cases} \tag{4.1}$$

Proof. In the first step of the proof we will show that P has a fixed point. For this, we evaluate the followings for all $(m,n) \in N \times N$.

$$\begin{aligned}
P(T_n(t)) - P(T_m(t)) &= T_n(t) - T_m(t)\\
&+ \mathcal{S}^{-1}\left[\frac{(1-\beta+\beta u)}{M(\beta)}\mathcal{S}[\beta - kV_nT_n - dT_n + bT_n']\right]\\
&- \mathcal{S}^{-1}\left[\frac{(1-\beta+\beta u)}{M(\beta)}\mathcal{S}[\beta - kV_mT_m - dT_m + bT_m']\right],\\
P(T_n'(t)) - P(T_m'(t)) &= T_n'(t) - T_m'(t)\\
&+ \mathcal{S}^{-1}\left[\frac{(1-\beta+\beta u)}{M(\beta)}\mathcal{S}[kV_nT_n - (b+s)T_n']\right]\\
&- \mathcal{S}^{-1}\left[\frac{(1-\beta+\beta u)}{M(\beta)}\mathcal{S}[kV_mT_m - (b+s)T_m']\right],\\
P(V_n(t)) - P(V_m(t)) &= V_n(t) - V_m(t)\\
&+ \mathcal{S}^{-1}\left[\frac{(1-\beta+\beta u)}{M(\beta)}\mathcal{S}[\delta T_n' - cV_n]\right]\\
&- \mathcal{S}^{-1}\left[\frac{(1-\beta+\beta u)}{M(\beta)}\mathcal{S}[\delta T_m' - cV_m]\right].
\end{aligned} \tag{4.2}$$

Let consider the first equation of (4.2) and taking norm of both hand sides, then without loss of generality, we have

$$\begin{aligned}
\|P(T_n(t)) - P(T_m(t))\| = \Bigg\| & T_n(t) - T_m(t)\\
&+ \mathcal{S}^{-1}\left[\frac{(1-\beta+\beta u)}{M(\beta)}\mathcal{S}[\beta - kV_nT_n - dT_n + bT_n']\right]\\
&- \mathcal{S}^{-1}\left[\frac{(1-\beta+\beta u)}{M(\beta)}\mathcal{S}[\beta - kV_mT_m - dT_m + bT_m']\right]\Bigg\|
\end{aligned} \tag{4.3}$$

Thank to the triangular inequality, the right hand side of the equation (4.3) becomes

$$\begin{aligned}\|P(T_n(t)) - P(T_m(t))\| &\leq \|T_n(t) - T_m(t)\| \\ &+ \left\| \mathcal{S}^{-1}\left[\frac{(1-\beta+\beta u)}{M(\beta)}\mathcal{S}[\beta - kV_nT_n - dT_n + bT'_n]\right]\right. \\ &- \left.\mathcal{S}^{-1}\left[\frac{(1-\beta+\beta u)}{M(\beta)}\mathcal{S}[\beta - kV_mT_m - dT_m + bT'_m]\right]\right\|.\end{aligned} \tag{4.4}$$

Upon further simplification, (4.4) yields that

$$\begin{aligned}&\|P(T_n(t)) - P(T_m(t))\| \leq \|T_n(t) - T_m(t)\| \\ &+ \mathcal{S}^{-1}\left[\mathcal{S}\frac{(1-\beta+\beta u)}{M(\beta)}[\|-kV_n(T_n - T_m)\| + \|-kT_m(V_n - V_m)\| + \|-d(T_n - T_m)\| + \|b(T'_n - T'_m)\|]\right].\end{aligned} \tag{4.5}$$

Since both the solutions play the same role, we shall assume in this case that

$$\begin{aligned}\|T_n(t) - T_m(t)\| &\cong \|T'_n(t) - T'_m(t)\| \\ \|T_n(t) - T_m(t)\| &\cong \|V_n(t) - V_m(t)\|.\end{aligned}$$

Replacing this in equation (4.5), we obtain the following relation

$$\begin{aligned}&\|P(T_n(t)) - P(T_m(t))\| \leq \|T_n(t) - T_m(t)\| \\ &+ \mathcal{S}^{-1}\left[\mathcal{S}\frac{(1-\beta+\beta u)}{M(\beta)}\Big(\|-kV_n(T_n - T_m)\| + \|-kT_m(T_n - T_m)\| + \|-d(T_n - T_m)\| + \|b(T'_n - T'_m)\|\Big)\right].\end{aligned} \tag{4.6}$$

Since V_n, T_m,V_m and T_n are bounded as they are convergent sequence, therefore, we can find four different positive constants, M, L, M_1 and L_1 for all t such that

$$\|V_n\| < M,\ \|T_m\| < L,\ \|V_m\| < M_1,\ \|T_n\| < L_1,\ (m,n) \in \mathbb{N}\times\mathbb{N}. \tag{4.7}$$

Now considering equation (4.6) with (4.7), we obtain the following

$$\|P(T_n(t)) - P(T_m(t))\| \leq \Big[1 - (d-b)f(\gamma) - k(M+L)g(\gamma)\Big]\|(T_n - T_m)\|. \tag{4.8}$$

Where f, g are functions from $\mathcal{S}^{-1}\left[\mathcal{S}(\frac{1-\beta+\beta u}{M(\beta)})\right]$.

In the same way, we will get

$$\|P(T'_n(t)) - P(T'_m(t))\| \leq \{1 - (b+s)f_1(\gamma) + k(M_1 + L_1)g_1(\gamma)\}\|(T'_n(t)) - (T'_m(t))\|, \tag{4.9}$$

$$\|P(V_n(t)) - P(V_m(t))\| \leq \{1 + (\delta - c)f_2(\gamma)\}\|(V_n(t)) - (V_m(t))\|. \tag{4.10}$$

Where

$$\begin{aligned}&\{1 - (d-b)f(\gamma) - k(M+L)g(\gamma)\} < 1, \\ &\{1 - (b+s)f_1(\gamma) + k(M_1 + L_1)g_1(\gamma)\} < 1, \\ &\{1 + (\delta - c)f_2(\gamma)\} < 1.\end{aligned}$$

Thus the nonlinear P-self mapping has a fixed point. We next show that, P satisfies the conditions in Theorem 4.1. Let (4.8), (4.9)and (4.10) hold and therefore using

$$c = (0,0,0),\quad C = \begin{cases}\{1 - (d-b)f(\gamma) - k(M+L)g(\gamma)\}, \\ \{1 - (b+s)f_1(\gamma) + k(M_1 + L_1)g_1(\gamma)\}, \\ \{1 + (\delta - c)f_2(\gamma)\}.\end{cases}$$

Then the above shows that condition of Theorem 4.1 exist for the nonlinear mapping P. Thus all the conditions in Theorem 4.1 are satisfied for the defined non-linear mapping P. Hence P is Picard P-stable. □

5. Uniqueness of the special solution

In this section, we show that the special solution of equation (1.2) using the iteration method is unique. We shall first assume that, equation (1.2) has an exact solution via which, the special solution converges for a large number m. We consider the following Hilbert space $H = L((a,b)\times(0,T))$ which can be defined as

$$y : (a,b)\times(0,T) \longrightarrow \mathbb{R}, \text{ such that } \int\int uydudy < \infty.$$

We now, consider the following operator

$$P(T,T',V) = \begin{cases} \beta - kVT - dT + bT', \\ kVT - (b+s)T', \\ \delta T' - cV. \end{cases}$$

The aim of this part is to prove that the inner product of

$$P\bigg((X_{11}-X_{12}, X_{21}-X_{22}, X_{31}-X_{32}), (w_1,w_2,w_3)\bigg),$$

where $(X_{11}-X_{12})$, $(X_{21}-X_{22})$and $(X_{31}-X_{32})$, are special solution of system. However,

$$\begin{aligned} &P\bigg((X_{11}-X_{12}, X_{21}-X_{22}, X_{31}-X_{32}), (w_1,w_2,w_3)\bigg) \\ &= \begin{cases} (-k(X_{31}-X_{32})(X_{11}-X_{12}) - d(X_{11}-X_{12}) + b(X_{21}-X_{22}), w_1), \\ (k(X_{31}-X_{32})(X_{11}-X_{12}) - (b+s)(X_{21}-X_{22}), w_2) \\ (\delta(X_{21}-X_{22}) - c(X_{31}-X_{32}), w_3). \end{cases} \end{aligned} \tag{5.1}$$

We shall evaluate the first equation in the system without loss of generality

$$\begin{aligned} &(-k(X_{31}-X_{32})(X_{11}-X_{12}) - d(X_{11}-X_{12}) + b(X_{21}-X_{22}), w_1) \\ &\cong (-k(X_{31}-X_{32})(X_{11}-X_{12}), w_1) + (-d(X_{11}-X_{12}), w_1) + (b(X_{21}-X_{22}), w_1). \end{aligned} \tag{5.2}$$

Since both solutions play almost the same role, we can assume that,

$$(X_{11}-X_{12}) \cong (X_{21}-X_{22}) \cong (X_{31}-X_{32}).$$

Then the equation (5.2) becomes

$$(-k(X_{11}-X_{12})^2 - d(X_{11}-X_{12}) + b(X_{11}-X_{12}), w_1)$$

Thanking to the relationship between norm and the inner product, we obtain the following

$$\begin{aligned} &(-k(X_{11}-X_{12})^2 - d(X_{11}-X_{12}) + b(X_{11}-X_{12}), w_1) \\ &\cong (-k(X_{11}-X_{12})^2, w_1) + (-d(X_{11}-X_{12}), w_1) + (b(X_{11}-X_{12}), w_1) \\ &\le k\|(X_{11}-X_{12})^2\|\|w_1\| + d\|(X_{11}-X_{12})\|\|w_1\| + b\|(X_{11}-X_{12})\|\|w_1\| \\ &= (k\overline{\omega_1} + d + b)\|(X_{11}-X_{12})\|\|w_1\|. \end{aligned} \tag{5.3}$$

Repeating the same fashion, the second equation of the system(5.1) yields

$$\begin{aligned}&(k(X_{31}-X_{32})(X_{11}-X_{12})-(b+s)(X_{21}-X_{22}),w_2)\\&\leq k\|(X_{21}-X_{22})^2\|\|w_2\|+(b+s)\|(X_{21}-X_{22})\|,\\&=(k\overline{\omega_2}+b+s)\|(X_{21}-X_{22})\|\|w_2\|.\end{aligned}\tag{5.4}$$

In the same way, third equation of the system (5.1) yields that

$$(\delta(X_{21}-X_{22})-c(X_{31}-X_{32}),w_3)\leq(\delta+c)\|(X_{31}-X_{32})\|\|w_3\|.\tag{5.5}$$

Upon putting equations (5.3), (5.4)and (5.5) in equation (5.1), we get

$$\begin{aligned}&P\Big((X_{11}-X_{12},X_{21}-X_{22},X_{31}-X_{32}),(w_1,w_2,w_3)\Big)\\&\leq\begin{cases}(k\overline{\omega_1}+d+b)\|(X_{11}-X_{12})\|\|w_1\|\\(k\overline{\omega_2}+b+s)\|(X_{21}-X_{22})\|\|w_2\|\\(\delta+c)\|(X_{31}-X_{32})\|\|w_3\|.\end{cases}\end{aligned}\tag{5.6}$$

But, for sufficiently large values of m_i, with $i=1,2,3$ both the solutions converge to the exact solution, using the topological concept, there exist three very small positive parameters $l_{m_1},l_{m_2}$and l_{m_3} such that

$$\begin{aligned}&\|T-X_{11}\|,\|T-X_{12}\|<\frac{l_{m_1}}{3(k\overline{\omega_1}+d+b)\|w_1\|},\\&\|T'-X_{21}\|,\|T'-X_{22}\|<\frac{l_{m_2}}{3(k\overline{\omega_2}+b+s)\|w_2\|},\\&\|V-X_{31}\|,\|V-X_{32}\|<\frac{l_{m_3}}{(\delta+c)\|w_3\|}.\end{aligned}$$

Thus plugging the exact solution in the right hand side of equation (5.6) and applying the triangular inequality with using taking $M=\max(m_1,m_2,m_3),l=\max(l_{m_1},l_{m_2},l_{m_3})$. We obtain

$$\begin{cases}(k\overline{\omega_1}+d+b)\|(X_{11}-X_{12})\|\|w_1\|\\(k\overline{\omega_2}+b+s)\|(X_{21}-X_{22})\|\|w_2\|\\(\delta+c)\|(X_{31}-X_{32})\|\|w_3\|\end{cases}<\begin{cases}l,\\l,\\l.\end{cases}$$

As l is very very small positive parameter, therefore, on the basis of topological idea, we have

$$\begin{cases}(k\overline{\omega_1}+d+b)\|(X_{11}-X_{12})\|\|w_1\|\\(k\overline{\omega_2}+b+s)\|(X_{21}-X_{22})\|\|w_2\|\\(\delta+c)\|(X_{31}-X_{32})\|\|w_3\|.\end{cases}<\begin{cases}0,\\0,\\0.\end{cases}$$

But, it is obvious that

$$(k\overline{\omega_1}+d+b)\neq0,\ (k\overline{\omega_2}+b+s)\neq0,(\delta+c)|\neq0.$$

Therefore, we have

$$\|X_{11}-X_{12}\|=0,\ \|X_{21}-X_{22}\|=0,\ \|X_{31}-X_{32}\|=0.$$

Which yields that

$$X_{11}=X_{12},\ X_{21}=X_{22},\ X_{31}=X_{32}.$$

This completes the proof of uniqueness.

6. Conclusion

In this article, we have established some adequate conditions for existence and uniqueness of equilibrium solutions to HIV-1 infection model with non-integer fractional order derivative. The corresponding derivative has been taken in the Caputo-Febrizo sense. With the help of Picard successive approximation technique, we have shown that the equilibrium solutions of the proposed model (1.2) is P-stable and the equilibrium (exact) solution is unique. By the developed conditions, we have provided a new approach to investigate biological model in another way. With the help of Mathematica, we have also presented numerical simulations to the approximate solutions. From simulation, we concluded that fractional order systems reveal richer dynamics than the classical integer order counterpart. Hence we can say that fractional dynamics leads to time responses with super-fast transients and super-slow evolutions towards the steady-state, effects not easily captured by the ordinary order models. In future, the above analysis can be extended to more complicated models of physics.

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Samia Bushnaq
Department of Basic Sciences, Princess Sumaya University for Technology, Amman 11941, Jordan
E-mail address: s.bushnaq@psut.edu.jo

Sajjad Ali Khan, Kamal Shah, Gul Zaman
Department of Mathematics, University of Malakand, Dir(L), Khyber Pakhtunkhwa, Pakistan
E-mail address: sajjadkks@gmail.com
E-mail address: kamalshah408@gmail.com
E-mail address: talash74@yahoo.com