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# EXTENDED FUZZY $b$-METRIC SPACES

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Abstract. In this paper, we introduce the notion of extended fuzzy $b$-metric space and prove a Banach-type fixed point theorem, in the setting of this more general class of fuzzy metric spaces. Our result and notions extend and generalize the existing results in literature.

## 1. Introduction and Preliminaries

After the celebrated Banach contraction principle was established, may authors extended this outstanding result to diverse directions [3, 6, 7, 25]. The work of Bakhtin [2], Bourbaki [5], and particularly Czerwik [10] motivated many researchers to expand the theory of fixed points for $b$-metric spaces. In this regard, Czerwik [10] introduced a weaker form of the triangle inequality and formally presented a generalization of metric space by defining a $b$-metric space and established a generalization of the famous Banach contraction principle in $b$-metric spaces. Some applications of relaxed triangle inequality can be seen in the works presented in [8], [9], and [12].

Consider a non-empty set $X$ and a real number $s \geq 1$. A function $d_b\colon X \times X \to [0, \infty)$ is called a $b$-metric [2, 10] on $X$ if for all $x, y, z \in X$, we have:

1) $d_b(x, y) = 0$ if and only if $x = y$;
2) $d_b(x, y) = d_b(y, x)$;
2) $d_b(x, z) \leq s[d_b(x, y) + d_b(y, z)]$.

The triple $(X, d_b, s)$ is called $b$-metric space, with coefficient $s \geq 1$.

Clearly the notion of a $b$-metric space is the generalization of the idea of a metric space. More precisely, if we take $s = 1$ in the above definition of $b$-metric space then it becomes the definition of a metric space.

Examples and fixed point theorems in $b$-metric spaces can be seen in [1, 4, 18, 22, 23].

Recently, Kamran *et al.* [17] introduced the notion of an extended $b$-metric space and established fixed point theorems in extended $b$-metric spaces.

The concept of a fuzzy set was introduced by L. A. Zadeh [26] in 1965 to address the unclear or inexplicit situations in everyday life. By a fuzzy set $F$ in $X$ we mean a function with a domain $X$ and values in the interval $[0, 1]$. Using the concept of

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fuzzy set theory, Kramosil and Michálek [19] introduced the concept of fuzzy metric space which was slightly modified in 1994 by George and Veeramani [13] to obtain a Hausdorff topology on these spaces. A useful theory of fixed points in fuzzy metric spaces is established by Grabiec [14], Dey and Saha [11]. Recently, the notion of fuzzy $b$-metric spaces is investigated in [16] and [20].

Recall that a commutative and associative binary operation $*\colon [0,1]\times[0,1]\to[0,1]$ is called a triangular norm ($t$-norm) [21] if $a*1=a$ and if $a\le c$ and $b\le d$ then $a*b\le c*d$ for all $a,b,c,d\in[0,1]$. The operations $\wedge$, $\cdot$ and $*_L$ are some well-known examples of continuous $t$-norms that are respectively given by $a\wedge b=\min\{a,b\}$, $a\cdot b=ab$ and $a*_L b=\max\{a+b-1,0\}$.

In this paper, we aim to introduce the notion of an extended fuzzy $b$-metric space and establish fuzzy version of Banach fixed point theorem for contraction mappings in this more general class of fuzzy metric spaces. We start by the following George and Veeramani [13] definition of a fuzzy metric space.

**Definition 1.1** ([13])**.** A 3-tuple $(X,M,*)$ is said to be a fuzzy metric space if $X$ is an arbitrary set, $*$ is a continuous $t$-norm and $M$ is a fuzzy set on $X\times X\times[0,\infty)$ satisfying the following conditions for all $x,y,z\in X$ and $t,s>0$:

FM1: $M(x,y,t)>0$;
FM2: $M(x,y,t)=1$ if and only if $x=y$;
FM3: $M(x,y,t)=M(y,x,t)$;
FM4: $M(x,z,t+s)\ge M(x,y,t)*M(y,z,s)$;
FM5: $M(x,y,\cdot)\colon[0,\infty)\to[0,1]$ is continuous.

In [20], Nădăban introduced the idea of a fuzzy $b$-metric space to generalize the notion of a fuzzy metric spaces introduced by Kramosil and Michálek [19].

**Definition 1.2** ([20])**.** Let $X$ be a non empty set and $b\ge 1$ be a given real number and $*$ be a continuous $t$-norm. A fuzzy set $M$ in $X\times X\times[0,\infty)$ is called fuzzy $b$-metric on $X$ if for all $x,y,z\in X$, the following conditions hold.

FbM1: $M(x,y,0)=0$;
FbM2: $M(x,y,t)=1,\forall\, t>0$ if and only if $x=y$;
FbM3: $M(x,y,t)=M(y,x,t)$;
FbM4: $M(x,z,b(t+s))\ge M(x,y,t)*M(y,z,s)\ \forall\, t,s\ge 0$;
FbM5: If $M(x,y,.)\colon(0,\infty)\to[0,1]$ is left continuous, and $\lim_{t\to\infty}M(x,y,t)=1$.

The quadruple $(X,M,*,b)$ is called a fuzzy $b$-metric space.

**Example 1.1.** Let $(X,d_b)$ be a $b$-metric space. Define a function by

$$M\colon X^2\times[0,\infty)\to[0,1],\quad M(x_1,x_2,t)=\exp\Big(-\frac{d_b(x_1,x_2)}{t}\Big)$$

and take the continuous $t$-norm "$*$" as $t_1*t_2=t_1t_2$. Then it is easy to see that $(X,M,*,b)$ is a fuzzy $b$-metric space.

In [14], Grabiec introduced the notion of a Cauchy sequence and that of a complete fuzzy metric space. Both of these definition given by Grabiec are weaker than that given by George and Veeramani [13]. In this article, following Gregori and Sapena [15], we use these notions in the sense of Grabiec [14] and respectively call them a $G$-Cauchy sequence and a $G$-complete fuzzy metric space.

Recently, Kamran *et. al* [17] introduced the following notion of extended $b$-metric space:

**Definition 1.3** ([17]). Let $X$ be a non empty set and $\theta\colon X \times X \to [1, \infty)$. A function $d_\theta\colon X \times X \to [0, \infty)$ is called an extended $b$-metric if for all $x, y, z \in X$, it satisfies:

$[d_\theta 1]$ $d_\theta(x, y) = 0 \Leftrightarrow x = y$;
$[d_\theta 2]$ $d_\theta(x, y) = d_\theta(y, x)$;
$[d_\theta 3]$ $d_\theta(x, z) \leq \theta(x, z)[d_\theta(x, y) + d_\theta(y, z)]$.
The pair $(X, d_\theta)$ is called an extended $b$-metric space.

Note that by setting $\theta(x, y) = s$ for $s \geq 1$, the above definition becomes the definition of a $b$-metric space [10].

## 2. Main Results

Inspired by the work presented in [17], we now introduce the notion of an extended fuzzy $b$-metric space following the approach of Grabiec [14].

**Definition 2.1.** Let $X$ be a non empty set, $\alpha\colon X \times X \to [1, \infty)$ and $*$ be a continuous $t$-norm. A fuzzy set $M_\alpha$ in $X \times X \times [0, \infty)$ is called extended fuzzy $b$-metric on $X$ if for all $x, y, z \in X$, the following conditions hold.

$[FbM_\alpha 1]$ : $M_\alpha(x, y, 0) = 0$;
$[FbM_\alpha 2]$ : $M_\alpha(x, y, t) = 1, \forall\, t > 0$ if and only if $x = y$;
$[FbM_\alpha 3]$ : $M_\alpha(x, y, t) = M_\alpha(y, x, t)$;
$[FbM_\alpha 4]$ : $M_\alpha(x, z, \alpha(x, z)(t + s)) \geq M_\alpha(x, y, t) * M_\alpha(y, z, s)\ \forall\, t, s \geq 0$;
$[FbM_\alpha 5]$ : $M_\alpha(x, y, .)\colon (0, \infty) \to [0, 1]$ is left continuous.
Then $(X, M_\alpha, *, \alpha(x, y))$ is an extended fuzzy $b$-metric space.

Setting $\alpha(x, y) = b$ for some $b \geq 1$ then Definition 1.2 of a fuzzy $b$-metric space becomes a special case of the above definition of extended fuzzy $b$-metric space.

We now illustrate Definition 2.1 by the following example.

**Example 2.1.** Let $X = \{1, 2, 3\}$ and define $d_b\colon X \times X \to \mathbb{R}$ by $d(x, y) = (x - y)^2$. Then it is easy to see that $(X, d_b)$ is a $b$-metric space. Define the mapping

$$\alpha\colon X \times X \to [1, \infty), \quad \alpha(x, y) = 1 + x + y.$$

Let $M_\alpha\colon X \times X \times [0, \infty) \to [0, 1]$ be given by the rule

$$M_\alpha(x, y, t) = \begin{cases} \dfrac{t}{t + d_b(x, y)}, & \text{if } t > 0 \\ 0, & \text{if } t = 0, \end{cases}$$

and take the continuous $t$-norm $* = \wedge$, that is, $t_1 * t_2 = t_1 \wedge t_2 = \min\{t_1, t_2\}$.

We now show that $(X, M_\alpha, \wedge)$ is a fuzzy extended $b$-metric space.

Note that

$$d_b(1, 1) = d_b(2, 2) = d_b(3, 3) = 0,$$
$$d_b(1, 2) = d_b(2, 1) = d_b(2, 3) = d(3, 2) = 1, \quad d_b(1, 3) = d_b(3, 1) = 4.$$

Also,

$$\alpha(1, 1) = 3,\ \alpha(2, 2) = 5,\ \alpha(3, 3) = 7,$$
$$\alpha(1, 2) = \alpha(2, 1) = 4, \quad \alpha(2, 3) = \alpha(3, 2) = 6, \quad \alpha(1, 3) = \alpha(3, 1) = 5.$$

To show that $(X, M_\alpha, \wedge)$ is an extended fuzzy $b$-metric space, we remark that the conditions $[FbM_\alpha 1]$, $[FbM_\alpha 2]$, $[FbM_\alpha 3]$ and $[FbM_\alpha 5]$ of Definition 2.1 are trivially true.

To prove the property $[FbM_\alpha 4]$ for all $x, y \in X$, first note that

$$M_\alpha(x, z, \alpha(x, z)(t+s)) = \frac{\alpha(x, z)(t+s)}{\alpha(x, z)(t+s) + d(x, z)}.$$

For $x = 1$, $z = 2$,

$$M_\alpha(1, 2, \alpha(1, 2)(t+s)) = \frac{\alpha(1, 2)(t+s)}{\alpha(1, 2)(t+s) + d(1, 2)} = \frac{4(t+s)}{4(t+s)+1} = 1 - \frac{1}{4(t+s)+1},$$

$$M_\alpha(1, 3, t) = \frac{t}{t + d(1, 3)} = \frac{t}{t+4} = 1 - \frac{4}{t+4},$$

and

$$M_\alpha(3, 2, s) = \frac{s}{s + d(3, 2)} = \frac{s}{s+1} = 1 - \frac{1}{s+1}.$$

Since, for all $t, s > 0$, starting with the above value of $M_\alpha(1, 2, \alpha(1, 2)(t+s))$, we see that

$$\begin{aligned} 1 - \frac{1}{4(t+s)+1} &= 1 - \frac{4}{16t + 16s + 4} \\ &> 1 - \frac{4}{16t+4} > 1 - \frac{4}{t+4}. \end{aligned}$$

This shows that

$$M_\alpha(1, 2, \alpha(1, 2)(t+s)) > M_\alpha(1, 3, t).$$

Similarly, it can be shown that

$$M_\alpha(1, 2, \alpha(1, 2)(t+s)) > M_\alpha(3, 2, s).$$

It, therefore, follows that

$$M_\alpha(1, 2, \alpha(1, 2)(t+s)) \geq \min\{M_\alpha(1, 3, t), M_\alpha(3, 2, s)\},$$

hence

$$M_\alpha(1, 2, \alpha(1, 2)(t+s)) \geq M_\alpha(1, 3, t) * M_\alpha(3, 2, s).$$

Similarly, one can show that

$$M_\alpha(1, 3, \alpha(1, 3)(t+s)) \geq M_\alpha(1, 2, t) * M_\alpha(2, 3, s),$$

$$M_\alpha(2, 3, \alpha(2, 3)(t+s)) \geq M_\alpha(2, 1, t) * M_\alpha(1, 3, s).$$

Hence for all $x, y, z \in X$

$$M_\alpha(x, z, \alpha(x, z)(t+s)) \geq M_\alpha(x, y, t) * M_\alpha(y, z, s).$$

Therefore $(X, M_\alpha, *)$ is an extended fuzzy $b$-metric space.

The notions of $G$-convergent sequence, $G$-Cauchy sequence and $G$-completeness in extended fuzzy $b$-metric spaces can be generalized naturally as follows:

**Definition 2.2.** Let $(X, M_\alpha, *)$ be an extended fuzzy $b$-metric space.

(1) A sequence $\{x_n\}$ in $X$ is said to be $G$-convergent if there exits $x \in X$ such that

$$\lim_{n\to\infty} M_\alpha(x_n, x, t) = 1, \ \forall\, t > 0.$$

(2) A sequence $\{x_n\}$ in $X$ is said to be a $G$-Cauchy sequence if $\forall\ r \in (0, 1)$, and $\forall\, t > 0$, then there exists $n_0 \in \mathbb{N}$ such that

$$M_\alpha(x_n, x_{n+q}, t) > 1 - r, \ \forall\, n \geq n_0, \ \ q > 0.$$

Or, equivalently, we have $\lim_{n\to\infty} M_\alpha(x_n, x_{n+q}, t) = 1$ for $t > 0$ and $q > 0$ [14].

(3) An extended fuzzy $b$-metric space in which every $G$-Cauchy sequence is convergent is called a $G$-complete fuzzy extended $b$-metric space.

Now we prove the Banach Contraction Theorem in the setting of extended fuzzy $b$-metric space

**Theorem 2.1** (Banach Contraction Theorem in fuzzy extended $b$-metric spaces)**.** *Let $(X, M_\alpha, *)$ be a $G$-complete extended fuzzy $b$-metric space with the mapping $\alpha\colon X \times X \to [1, \infty)$ such that*

$$\lim_{t\to\infty} M_\alpha(x, y, t) = 1. \tag{2.1}$$

*Let $g\colon X \to X$ be a mapping which satisfies the condition*

$$M_\alpha(gx, gy, kt) \geq M_\alpha(x, y, t) \quad \forall\, x, y \in X, \tag{2.2}$$

*where $k \in (0, 1)$. Further, suppose that for an arbitrary $a_0 \in X$, and $n, q \in \mathbb{N}$, we have*

$$\alpha(a_n, a_{n+q}) < \frac{1}{k},$$

*where $a_n = g^n a_0$. Then $g$ has a unique fixed point.*

*Proof.* We start by an arbitrary $a_0 \in X$ and generate a sequence $\{a_n\}$ by the iterative process $a_n = g^n a_0$, $n \in \mathbb{N}$. First, note that for all $n, t > 0$, by successive application of the contractive condition (2.2), we have

$$M_\alpha(a_n, a_{n+1}, kt) = M_\alpha(ga_{n-1}, ga_n, kt) \geq M_\alpha(a_{n-1}, a_n, t) \geq M_\alpha\left(a_{n-2}, a_{n-1}, \frac{t}{k}\right) \geq M_\alpha\left(a_{n-3}, a_{n-2}, \frac{t}{k^2}\right) \geq \ldots \geq M_\alpha\left(a_0, a_1, \frac{t}{k^{n-1}}\right).$$

So, we have

$$M_\alpha(a_n, a_{n+1}, kt) \geq M_\alpha\left(a_0, a_1, \frac{t}{k^{n-1}}\right) \tag{2.3}$$

For any $q \in \mathbb{N}$, writing $t = \frac{qt}{q} = \frac{t}{q} + \cdots + \frac{t}{q}$ and using $[FbM_\alpha 4]$ repeatedly,

$$\begin{aligned} M_\alpha(a_n, a_{n+q}, t) \geq{}& M_\alpha\left(a_n, a_{n+1}, \frac{t}{q\alpha(a_n, a_{n+q})}\right) * M_\alpha\left(a_{n+1}, a_{n+2}, \frac{t}{q\alpha(a_n, a_{n+q})\alpha(a_{n+1}, a_{n+q})}\right) \\ & * M_\alpha\left(a_{n+2}, a_{n+3}, \frac{t}{q\alpha(a_n, a_{n+q})\alpha(a_{n+1}, a_{n+q})\alpha(a_{n+2}, a_{n+q})}\right) * \ldots * \\ & M_\alpha\left(a_{n+q-1}, a_{n+q}, \frac{t}{q\alpha(a_n, a_{n+q})\alpha(a_{n+1}, a_{n+q})\alpha(a_{n+2}, a_{n+q}) \ldots \alpha(a_{n+q-1}, a_{n+q})}\right). \end{aligned}$$

Using (2.3), and $[FbM_\alpha 4]$ we obtain

$$\begin{aligned} M_\alpha(a_n, a_{n+q}, t) \geq{}& M_\alpha\left(a_0, a_1, \frac{t}{q\alpha(a_n, a_{n+q})k^n}\right) * M_\alpha\left(a_0, a_1, \frac{t}{q\alpha(a_n, a_{n+q})\alpha(a_{n+1}, a_{n+q})k^{n+1}}\right) \\ & * M_\alpha\left(a_0, a_1, \frac{t}{q\alpha(a_n, a_{n+q})\alpha(a_{n+1}, a_{n+q})\alpha(a_{n+2}, a_{n+q})k^{n+3}}\right) * \ldots * \\ & M_\alpha\left(a_0, a_1, \frac{t}{q\alpha(a_n, a_{n+q})\alpha(a_{n+1}, a_{n+q})\alpha(a_{n+2}, a_{n+q}) \ldots \alpha(a_{n+q-1}, a_{n+q})k^{n+q-1}}\right). \end{aligned}$$

By the hypothesis of the theorem for all $n, q \in \mathbb{N}$ we have $\alpha(a_n, a_{n+q})k < 1$, and $k \in (0,1)$. Therefore, together with (2.1) and letting $n \to \infty$, we get

$$\lim_{n\to\infty} M_\alpha(a_n, a_{n+q}, t) = 1 * 1 * \ldots * 1 = 1.$$

Hence $\{a_n\}$ is Cauchy sequence.

Since $(X, M_\alpha, *))$ is a complete fuzzy extended $b$-metric space, there exists $a \in X$ such that

$$\lim_{n\to\infty} a_n = a.$$

We want to show that $a$ is a fixed point of $g$.

Using $[FbM_\alpha 4]$ and (2.1) we get

$$\begin{aligned} M_\alpha(ga, a, t) &\geq M_\alpha\left(ga, ga_n, \frac{t}{2\alpha(ga,a)}\right) * M_\alpha\left(ga_n, a, \frac{t}{2\alpha(ga,a)}\right) \\ &\geq M_\alpha\left(a, a_n, \frac{t}{2\alpha(ga,a)k}\right) * M_\alpha\left(a_{n+1}, a_n, \frac{t}{2\alpha(ga,a)}\right) \\ &\longrightarrow 1 * 1 = 1, \ \text{ as } n \to \infty, \ \text{ from } [FbM_\alpha 2]. \end{aligned}$$

This shows that $ga = a$, that is, $a$ is a fixed point of $g$.

To prove the uniqueness, assume $gb = b$ for some $b \in X$, then

$$\begin{aligned} M_\alpha(b, a, t) &= M_\alpha(gb, ga, t) \\ &\geq M_\alpha\left(b, a, \frac{t}{k}\right) = M_\alpha\left(gb, ga, \frac{t}{k}\right) \\ &\geq M_\alpha\left(b, a, \frac{t}{k^2}\right) \geq \ldots \geq M_\alpha\left(b, a, \frac{t}{k^n}\right) \\ &\longrightarrow 1 \quad \text{as} \quad n \to \infty. \end{aligned}$$

Thus $a = b$ and this completes the proof. □

Notice that if we take $\alpha(x, y) = 1$, we get the Banach contraction principle in fuzzy metric spaces [14] and the special case of [Theorem 3.5 [11]].

Now we give an example to illustrate Theorem 2.1.

**Example 2.2.** Let $X = [0,1]$ and $M_\alpha(x, y, t) = \left(\frac{1}{t}\right)^{(x-y)^2}$. It is easy to verify that $(X, M_\alpha, *)$ is a $G$-complete fuzzy extended $b$-metric space. Let $g\colon X \to X$ such that $g(x) = 1 - x$. For all $t > 0$ and $k \in (0,1)$, we have

$$\begin{aligned} M_\alpha(gx, gy, kt) = M_\alpha(1-x, 1-y, kt) &= \left(\frac{1}{kt}\right)^{(1-x-1+y)^2} \\ &= \left(\frac{1}{kt}\right)^{(y-x)^2} = \left(\frac{1}{kt}\right)^{(x-y)^2}. \end{aligned}$$

Since

$$\begin{aligned} k < 1 \qquad &\Rightarrow \quad kt < t \quad \text{and} \quad \frac{1}{kt} > \frac{1}{t} \\ &\Rightarrow \left(\frac{1}{kt}\right)^{(x-y)^2} > \left(\frac{1}{t}\right)^{(x-y)^2}, \end{aligned}$$

we obtain

$$M_\alpha(gx, gy, kt) > M_\alpha(x, y, t).$$

Also $x = \dfrac{1}{2}$ is a unique fixed point of $g$ and $\dfrac{1}{2} \in [0, 1]$.

## 3. Conclusion

Herein, we introduced the notion of extended fuzzy $b$-metric space and a Banach-type of fixed point theorem in this new setting. Since our framework is more general than the class of fuzzy metric spaces, our result and notions extend and generalize several existing results in the literature.

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## References

[1] M. U. Ali, T. Kamran, M. Postolache, Solution of Volterra integral inclusion in $b$-metric spaces via new fixed point theorem, Nonlinear Anal. Modelling Control, 22(2017), No. 1, 17-30.
[2] I. A. Bakhtin, The contraction mapping principle in quasi metric spaces, Funct. Anal. Unianowsk Gos. Ped. Inst., 30(1989), 26-37.
[3] G. Bhaskar, V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications, Nonlinear Anal., 65(2006), No. 7, 1379-1393
[4] M. Boriceanu, A. Petrusel, I. A. Rus, Fixed point theorems for some multivalued generalized contraction in $b$-metric spaces, Int. J. Math. Statistics, 6(2010), 65-76.
[5] N. Bourbaki, Topologie Generale, Herman, Paris, 1974.
[6] B. S. Choudhury, N. Metiya, M. Postolache, A generalized weak contraction principle with applications to coupled coincidence point problems, Fixed Point Theory Appl. 2013, Art. No. 152 (2013).
[7] L. B. Ciric, A generalization of Banach's contraction principle, Proc. Amer. Math. Soc., 45(1974), 267-273.
[8] R. McConnell, R. Kwok, J. Curlander, W. Kober, S. Pang, Y-S correlation and dynamic time warping: Two methods for tracking ice floes, IEEE Trans. Geosci. Remote Sens., 29(1991), 1004-1012.
[9] G. Cortelazzo, G. Mian, G. Vezzi, P. Zamperoni, Trademark shapes description by string matching techniques, Pattern Recognit., 27(1994), 1005-1018.
[10] S. Czerwik, Contraction mappings in $b$-metric space, Acta Math. Inf. Univ. Ostraviensis, 1(1993), 5-11.
[11] D. Dey, M. Saha, An extension of Banach fixed point theorem in fuzzy metric space, Bol. Soc. Paran. Mat., 32(2014), No. 1, 299-304.
[12] R. Fagin, L. Stockmeyer, Relaxing the triangle inequality in pattern matching, Int. J. Comput. Vis., 30(1998), 219-231.
[13] A. George, P. Veeramani, On some results in fuzzy metric spaces, Fuzzy Sets Syst., 64(1994), 395-399.
[14] M. Grabiec, Fixed points in fuzzy metric spaces, Fuzzy Sets Syst., 27(1988), 385-389.
[15] V. Gregori, A. Sapena, On fixed-point theorems in fuzzy metric spaces, Fuzzy Sets Syst., 125(2002) 245-252.
[16] N. Hussain, P. Salimi, V. Parvaneh, Fixed point results for various contractions in parametric and fuzzy b-metric spaces, J. Nonlinear Sci. Appl., 8(2015), 719-739.
[17] T. Kamran, M. Samreen, Q. ul Ain, A generalization of $b$-metric space and some fixed point theorems, Mathematics, 5(2017), No. 2, Art. No. 19.
[18] T. Kamran, M. Postolache, M.U. Ali, Q. Kiran, Feng and Liu type F-contraction in b-metric spaces with application to integral equations, J. Math. Anal. 7(2016), No. 5, 18-27.
[19] I. Kramosil, J. Michálek, Fuzzy metric and statistical metric spaces, Kybernetica, 11(1975), 326-334.

[20] S. Nădăban, Fuzzy $b$-metric spaces, Int. J. Comput. Commun. Control, 11(2016), No. 2, 273-281.
[21] B. Schweizer, A. Sklar, Statistical metric spaces, Pacific J. Math., 10(1960), 314-334.
[22] M. H. Shah, N. Hussain, Nonlinear contraction in partially ordered quasi $b$-metric spaces, Commun. Korean Math. Soc. 27(2012), No. 1, 117-128.
[23] W. Shatanawi, A. Pitea, R. Lazovic, Contraction conditions using comparison functions on b-metric spaces, Fixed Point Theory Appl., 2014, Art. No. 135
[24] G. Song, Comments on 'A common fixed point theorem in a fuzzy metric space', Fuzzy Sets and Systems, 135(2003), 409-413.
[25] T. Suzuki, A generalized Banach contraction principle that characterizes metric completeness, Proc. Amer. Math. Soc., 136(2008), No. 5, 1861-1869.
[26] L. A. Zadeh, Fuzzy sets, Information and Control, 8(1965), 338-353.

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