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BEST PROXIMITY POINT THEOREMS FOR IMPLICIT PROXIMAL CONTRACTIONS ON GAUGE SPACES

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Abstract. In this paper, we will prove best proximity point theorems for mappings satisfying implicit proximal contraction conditions on gauge spaces. As consequence of our results we will obtain best proximity point theorems on metric spaces.

1. Introduction

As we know, fixed point theory focuses on two main directions. The first one is finding suitable conditions for the existence and uniqueness of the solutions of non-linear equations of the form $Tx = x$, when T is self mapping defined on a subset of a metric space or some pertinent framework [1–5]. Other one is the numerical calculation of these solutions [6–8]. But it is not necessary that such equations have solution if the mapping T is non-self, that is $T\colon A \to B$, where A and B are nonempty subsets of a metric space X. A best proximity point theorem provides the global minimization of the real valued function $x \to d(x, Tx)$ that is an indicator of the error involved for an approximate solution of the equation $Tx = x$. Because of the fact that for a non-self mapping $T\colon A \to B$, $d(x, Tx)$ is at least $d(A, B)$ for all $x \in A$, a best proximity point theorem ensures global minimum of the error $d(x, Tx)$ by confining an approximate solution x of the equation $Tx = x$ to comply with the condition that $d(x, Tx) = d(A, B)$. Such an approximate solution of the equation $Tx = x$ is said to be best proximity point of the non-self mapping $T\colon A \to B$. A well known best approximation theorem was proved by Fan [9]. Some generalizations and extensions of the theorem have been proved by Prolla [10], Reich [11], Sehgal and Singh [12, 13], Vetrivel *et al.* [14] and some others. Many authors have explored best proximity point theorems of mappings satisfying different contractive conditions, see for example [15–29]. Frigon in [30] proved fixed point results for generalized contractions on gauge spaces. Later on these results have been extended by several authors [31–36]. In this paper we will introduce the notions of implicit generalized proximal contractions and prove some best proximity point theorems for such mappings in the context of gauge spaces.

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2. Preliminaries

First, we give some definitions to illustrate gauge spaces.

Definition 2.1 ([35])**.** Let X be a nonempty set. A function $d\colon X \times X \to [0, \infty)$ is called pseudo metric on X if for each $x, y, z \in X$, the following axioms hold:
(i) $d(x, x) = 0$;
(ii) $d(x, y) = d(y, x)$;
(iii) $d(x, z) \leq d(x, y) + d(y, z)$.

Definition 2.2 ([35])**.** Let X be a nonempty set endowed with the pseudo metric d. The d-ball of radius $\epsilon > 0$ centered at $x \in X$ is the set

$$B(x, d, \epsilon) = \{y \in X : d(x, y) < \epsilon\}.$$

Definition 2.3 ([35])**.** A family $\mathfrak{F} = \{d_v | v \in \mathfrak{A}\}$ of pseudo metrics is said to be separating if for each pair (x, y) with $x \neq y$, there exists $d_v \in \mathfrak{F}$ with $d_v(x, y) \neq 0$.

Definition 2.4 ([35])**.** Let X be a nonempty set and $\mathfrak{F} = \{d_v | v \in \mathfrak{A}\}$ be a family of pseudo metrics on X. The topology $\mathfrak{T}(\mathfrak{F})$ having subbases the family

$$\mathfrak{B}(\mathfrak{F}) = \{B(x, d_v, \epsilon) : x \in X, d_v \in \mathfrak{F} \text{ and } \epsilon > 0\}$$

of balls is called topology induced by the family $\mathfrak{F}$ of pseudo metrics. The pair $(X, \mathfrak{T}(\mathfrak{F}))$ is called a gauge space. Note that $(X, \mathfrak{T}(\mathfrak{F}))$ is Hausdorff if we take $\mathfrak{F}$ as separating.

Definition 2.5 ([35])**.** Let $(X, \mathfrak{T}(\mathfrak{F}))$ be a gauge space with respect to the family $\mathfrak{F} = \{d_v | v \in \mathfrak{A}\}$ of pseudo metrics on X. Let $\{x_n\}$ be a sequence in X and $x \in X$. Then

(i) the sequence $\{x_n\}$ converges to x if for each $v \in \mathfrak{A}$ and $\epsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that $d_v(x_n, x) < \epsilon$ for each $n \geq N_0$. We denote it as $x_n \to^{\mathfrak{F}} x$.

(ii) the sequence $\{x_n\}$ is a Cauchy sequence if for each $v \in \mathfrak{A}$ and $\epsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that $d_v(x_n, x_m) < \epsilon$ for each $n, m \geq N_0$.

(iii) $(X, \mathfrak{T}(\mathfrak{F}))$ is complete if each Cauchy sequence in $(X, \mathfrak{T}(\mathfrak{F}))$ is convergent in X.

(iv) a subset of X is said to be closed if it contains the limit of each convergent sequence of its elements.

We will use the following family of functions in implicit generalized proximal contraction mappings. This family was introduced in [37].

Let $\psi\colon [0, \infty) \to [0, \infty)$ be a nondecreasing mapping such that$\sum_{n=1} \psi^n(t) < \infty$ for all $t \geq 0$ and $\psi(t) < t$ for all $t > 0$. By Φ_ψ we denote the family of functions $\phi : (\mathbb{R}^+)^4 \to \mathbb{R}^+$ satisfying the following conditions:

(i) ϕ is continuous and nondecreasing in each coordinate.

(ii) Let $u_1, u_2 \in \mathbb{R}^+$ such that if $u_1 < u_2$ and $u_1 \leq \phi(u_2, u_2, u_1, u_2)$, then $u_1 \leq \psi(u_2)$. If $u_1 \geq u_2$ and $u_1 \leq \phi(u_1, u_2, u_1, u_1)$, then $u_1 = 0$.

(iii) If $u \in \mathbb{R}^+$ such that $u \leq \phi(0, 0, u, 1/2u)$, then $u = 0$.

Following are some examples of $\phi \in \Phi_\psi$ which are mentioned in [26]:

(i) Let $\phi_1(u_1, u_2, u_3, u_4) = \alpha \max\{u_1, u_2, u_3, u_4\}$ with $\psi(t) = \alpha t$, where $\alpha \in [0, 1)$.

(ii) Let $\phi_2(u_1, u_2, u_3, u_4) = \alpha u_1$ with $\psi(t) = \alpha t$, where $\alpha \in [0, 1)$.

(iii) Let $\phi_3(u_1, u_2, u_3, u_4) = \alpha \max\{u_1, 1/2(u_2 + u_3), u_4\}$ with $\psi(t) = \alpha t$, where $\alpha \in [0, 1)$.

(iv) Let $\phi_4(u_1, u_2, u_3, u_4) = au_1 + b(u_2 + u_3) + cu_4$ with $\psi(t) = (a + 2b + c)t$, where a, b, c are non-negative real numbers such that $a + 2b + c \in [0, 1)$.

3. Main results

Throughout this section $(X, \mathfrak{T}(\mathfrak{F}))$ be a gauge space with respect to the family $\mathfrak{F} = \{d_v | v \in \mathfrak{A}\}$ of pseudo metrics on X. Further, the following notations have same meanings. Given that A and B are nonempty subsets of X. Then

$$d_v(A,B) = \inf\{d_v(a,b) : a \in A, b \in B\}$$
$$A_0 = \{a \in A : d_v(a,b) = d_v(A,B) \text{ for each } v \in A, \text{ for some } b \in B\}$$
$$B_0 = \{b \in B : d_v(a,b) = d_v(A,B) \text{ for each } v \in A, \text{ for some } a \in A\}.$$

The following definition is extended form of the definition given by Basha and Shahzad [28].

Definition 3.1. Let A and B are nonempty subsets of X. Then B is said to be approximatively compact with respect to A if each $\{v_n\}$ in B with $d_v(x, v_n) \to d_v(x, B)$ $\forall v \in \mathfrak{A}$ for some x in A, has a convergent subsequence.

Following we introduce the notion of implicit generalized proximal contraction mappings of first kind.

Definition 3.2. Let A and B be nonempty subsets of X. A mapping $T\colon A \to B$ is called implicit generalized proximal contraction of first kind if for each $x_1, x_2, u_1, u_2 \in A$, there exists $\phi \in \Phi_\psi$ such that $d_v(u_1, Tx_1) = d_v(A,B) = d_v(u_2, Tx_2)$ implies

$$d_v(u_1,u_2) \le \phi(d_v(x_1,x_2), d_v(x_1,u_1), d_v(x_2,u_2), 1/2(d_v(x_2,u_1) + d_v(x_1,u_2))) \quad (3.1)$$

for each $v \in \mathfrak{A}$.

Following we state and prove the best proximity point theorem for implicit generalized proximal contraction mappings of first kind.

Theorem 3.1. *Let $(X, \mathfrak{T}(\mathfrak{F}))$ be a complete gauge space induced by separating family of pseudo metrics $\mathfrak{F} = \{d_v | v \in \mathfrak{A}\}$. Let A and B be nonempty closed subsets of X such that B is approximately compact with respect to A and A_0 is nonempty. Let $T\colon A \to B$ be implicit generalized proximal contraction of first kind with $T(A_0) \subseteq B_0$. Then T has a best proximity point, that is, there exists an element x in A such that $d_v(x, Tx) = d_v(A,B)$ $\forall v \in \mathfrak{A}$.*

Proof. Let $x_0 \in A_0$. Since $T(A_0) \subseteq B_0$, so $Tx_0 \in B_0$, thus we have $x_1 \in A_0$ such that $d_v(x_1, Tx_0) = d_v(A,B)$, $\forall v \in \mathfrak{A}$. Similarly for $x_1 \in A_0$, we have $Tx_1 \in B_0$, thus we get $x_2 \in A_0$ such that $d_v(x_2, Tx_1) = d_v(A,B)$, $\forall v \in \mathfrak{A}$. By continuing this process, for each $n \in \mathbb{N}$ we have $x_n, x_{n+1} \in A_0$ such that

$$d_v(x_{n+1}, Tx_n) = d_v(A,B)\ \forall v \in \mathfrak{A}.$$

Assume that $x_n \neq x_{n+1}$, otherwise x_n is a best proximity point. Thus from (3.1), for each $n \in \mathbb{N}$ we have

$$\begin{aligned} d_v(x_n, x_{n+1}) &\le \phi\big(d_v(x_{n-1},x_n), d_v(x_{n-1},x_n), d_v(x_n,x_{n+1}),\\ &\quad 1/2(d_v(x_{n-1},x_{n+1}) + d_v(x_n,x_n))\big)\\ &= \phi\big(d_v(x_{n-1},x_n), d_v(x_{n-1},x_n), d_v(x_n,x_{n+1}),\\ &\quad 1/2(d_v(x_{n-1},x_{n+1}))\big)\\ &\le \phi\big(d_v(x_{n-1},x_n), d_v(x_{n-1},x_n), d_v(x_n,x_{n+1}),\\ &\quad 1/2(d_v(x_{n-1},x_n) + d_v(x_n,x_{n+1}))\big)\ \forall v \in \mathfrak{A}. \end{aligned} \quad (3.2)$$

We claim that $d_v(x_n, x_{n+1}) < d_v(x_{n-1}, x_n)$ $\forall v \in \mathfrak{A}$ for each $n \in \mathbb{N}$.

Suppose on contrary that $d_v(x_n, x_{n+1}) \geq d_v(x_{n-1}, x_n)$ $\forall v \in \mathfrak{A}$ for some n. Since ϕ is non-decreasing, by using this in (3.2), we have

$$d_v(x_n, x_{n+1}) \leq \phi(d_v(x_n, x_{n+1}), d_v(x_{n-1}, x_n), d_v(x_n, x_{n+1}), d_v(x_n, x_{n+1})), \quad (3.3)$$

for all $v \in \mathfrak{A}$. Using property (ii) of Φ_ψ in (3.3), we obtain

$$d_v(x_n, x_{n+1}) = 0 \ \forall v \in \mathfrak{A}$$

which is contradiction to our assumption, since $x_{n+1} \neq x_n$ for each $n \in \mathbb{N} \cup \{0\}$. Thus $d_v(x_n, x_{n+1}) < d_v(x_{n-1}, x_n)$ $\forall v \in \mathfrak{A}$ for each $n \in \mathbb{N}$. Therefore, for each $n \in \mathbb{N}$, (3.2) becomes

$$d_v(x_n, x_{n+1}) \leq \phi\big(d_v(x_{n-1}, x_n), d_v(x_{n-1}, x_n), d_v(x_n, x_{n+1}), (d_v(x_{n-1}, x_n))\big), \quad (3.4)$$

for all $v \in \mathfrak{A}$. By using (3.4) and property (ii) of Φ_ψ, for each $n \in \mathbb{N}$ we have

$$d_v(x_n, x_{n+1}) \leq \psi\big(d_v(x_{n-1}, x_n)\big) \ \forall v \in \mathfrak{A}.$$

Consequently, for each $n \in \mathbb{N}$ we get

$$d_v(x_n, x_{n+1}) \leq \psi^n\big(d_v(x_0, x_1)\big) \ \forall v \in \mathfrak{A}.$$

Let $n > m$, we have

$$\begin{aligned} d_v(x_m, x_n) &\leq d_v(x_m, x_{m+1}) + d_v(x_{m+1}, x_{m+2}) + \cdots + d_v(x_{n-1}, x_n) \\ &\leq \psi^m(d_v(x_0, x_1)) + \psi^{m+1}(d_v(x_0, x_1)) + \cdots + \psi^{n-1}(d_v(x_0, x_1)) \\ &= \sum_{i=m}^{n-1} \psi^i(d_v(x_0, x_1)) < \infty \ \forall v \in \mathfrak{A}. \end{aligned}$$

Hence $\{x_n\}$ is Cauchy sequence in $(X, \mathfrak{T}(\mathfrak{F}))$. Since A is closed subset of X and X is complete. Then there exists a point x^* in A such that $x_n \to x^*$. Furthermore,

$$\begin{aligned} d_v(x^*, B) &\leq d_v(x^*, Tx_n) \\ &\leq d_v(x^*, x_{n+1}) + d_v(x_{n+1}, Tx_n) \\ &= d_v(x, x_{n+1}) + d_v(A, B) \\ &\leq d_v(x^*, x_{n+1}) + d_v(x^*, B) \ \forall v \in \mathfrak{A}. \end{aligned}$$

Therefore, $d_v(x^*, Tx_n) \to d_v(x^*, B), \forall v \in \mathfrak{A}$ as $n \to \infty$. Since B is approximatively compact with respect to A, the sequence $\{Tx_n\}$ has a subsequence $\{Tx_{n_k}\}$ converging to some point y^* in B. It results that

$$d_v(x^*, y^*) = \lim_{n \to \infty} d_v(x_{n_{k+1}}, Tx_{n_k}) = d_v(A, B).$$

Since for $x^* \in A_0$, we have $Tx^* \in B_0$, thus we have $u \in A$ such that $d_v(u, Tx^*) = d_v(A, B)$ $\forall v \in \mathfrak{A}$. Also we have $d_v(x_{n+1}, Tx_n) = d_v(A, B)$ $\forall v \in \mathfrak{A}$. Thus, from (3.1), we have

$$d_v(x_{n+1}, u) \leq \phi\big(d_v(x_n, x^*), d_v(x_n, x_{n+1}), d_v(x^*, u), 1/2(d_v(x_n, u) + d_v(x^*, x_{n+1}))\big),$$

for all $v \in \mathfrak{A}$. Letting $n \to \infty$ in the above inequality, we get

$$d_v(x^*, u) \leq \phi(0, 0, d_v(x^*, u), 1/2 d_v(x^*, u)) \ \forall v \in \mathfrak{A}.$$

By the property (iii) of ϕ, we have $d_v(x^*, u) = 0\ \forall v \in \mathfrak{A}$. Since X is separating gauge space, thus we conclude that $x^* = u$. Therefore

$$d_v(x^*, Tx^*) = d_v(u, Tx^*) = d_v(A, B)\ \forall v \in \mathfrak{A},$$

and this completes the proof. □

Example 3.1. Let $X = C([0, 10], \mathbb{R}) \times C([0, 10], \mathbb{R})$ be the space of all pairs of continuous and bounded real functions defined on $[0, 10]$, endowed with pseudo metrics $d_n(x(t), y(t)) = \max_{t\in[0,n]}\{|x_1(t) - y_1(t)| + |x_2(t) - y_2(t)|\}\ \forall\ x(t) = (x_1(t), x_2(t))$, $y(t) = (y_1(t), y_2(t)) \in X$ and $n \in \{1, 2, 3, \ldots, 10\}$. Take $A = \{(0, x(t)) : t \in [0, 10]\}$ and $B = \{(10, x(t)) : t \in [0, 10]\}$. Define $T\colon A \to B$ by

$$T(0, x(t)) = \left(10, \frac{x(t)}{2}\right) \text{ for each } t \in [0, 10].$$

It is easy to see that all the conditions of Theorem 3.1 holds with $\phi(u_1, u_2, u_3, u_4) = \frac{u_1}{2}$. Thus T has a best proximity point.

Corollary 3.1. *Let $(X, \mathfrak{T}(\mathfrak{F}))$ be a complete gauge space induced by separating family of pseudo metrics $\mathfrak{F} = \{d_v | v \in \mathfrak{A}\}$. Let A and B be nonempty closed subsets of X such that B is approximatively compact with respect to A and A_0 is nonempty. Further assume that a mapping $T\colon A \to B$ satisfies the following conditions:*

(a) There exists a nonnegative real number $\alpha < 1$ such that for all u_1, u_2, x_1, x_2 in A, we have

$$d_v(u_1, Tx_1) = d_v(A, B) = d_v(u_2, Tx_2) \Rightarrow d_v(u_1, u_2) \le \alpha d_v(x_1, x_2)\ \forall v \in \mathfrak{A};$$

(b) $T(A_0) \subseteq B_0$.

Then T has a best proximity point, that is, there exists an element x in A such that $d_v(x, Tx) = d_v(A, B)\ \forall v \in \mathfrak{A}$.

Proof. Let $\phi(u_1, u_2, u_3, u_4) = \phi_2(u_1, u_2, u_3, u_4) = \alpha u_1$ with $\psi(t) = \alpha t$, where $\alpha \in [0, 1)$. From (3.1), we have $d_v(u_1, Tx_1) = d_v(A, B) = d_v(u_2, Tx_2) \Rightarrow d_v(u_1, u_2) \le \alpha d_v(x_1, x_2)\ \forall v \in \mathfrak{A}$, for $u_1, u_2, x_1, x_2 \in A$. Thus Theorem 3.1 implies that T has a best proximity point $x \in A$ such that $d_v(x, Tx) = d_v(A, B)\ \forall v \in \mathfrak{A}$. □

In the following definition we define second kind of implicit generalized proximal contraction.

Definition 3.3. Let A and B be nonempty subsets of X. A mapping $T\colon A \to B$ is called implicit generalized proximal contraction of second kind if for each $x_1, x_2, u_1, u_2 \in A$, there exists $\phi \in \Phi_\psi$ such that $d_v(u_1, Tx_1) = d_v(A, B) = d_v(u_2, Tx_2)$ implies

$$\begin{aligned} d_v(Tu_1, Tu_2) \quad \le \quad & \phi\big(d_v(Tx_1, Tx_2), d_v(Tx_1, Tu_1), d_v(Tx_2, Tu_2), \\ & 1/2(d_v(Tx_2, Tu_1) + d_v(Tx_1, Tu_2))\big) \end{aligned} \tag{3.5}$$

for each $v \in \mathfrak{A}$.

Following, we state and proof the best proximity point theorem for second kind of implicit generalized proximal contraction.

Theorem 3.2. *Let $(X, \mathfrak{T}(\mathfrak{F}))$ be a complete gauge space induced by separating family of pseudo metrics $\mathfrak{F} = \{d_v | v \in \mathfrak{A}\}$. Let A and B be nonempty closed subsets of X such that A is approximately compact with respect to B and A_0 is nonempty. Let $T\colon A \to B$ be a continuous implicit generalized proximal contraction of second* *kind such that $T(A_0) \subseteq B_0$. Then T has a best proximity point, that is, there exists an element x in A such that $d_v(x, Tx) = d_v(A, B)$ $\forall v \in \mathfrak{A}$.*

Proof. Let $x_0 \in A_0$. Then, following the same steps as in Theorem 3.1, we can find a sequence $\{x_n\}$ in A_0 such that for all $n \in \mathbb{N} \cup \{0\}$, we have

$$d_v(x_{n+1}, Tx_n) = d_v(A, B) \ \forall v \in \mathfrak{A}.$$

Assume that $Tx_n \neq Tx_{n+1}$ for each $n \in \mathbb{N} \cup \{0\}$, otherwise x_{n+1} is a fixed point. Thus from (3.5), we have

$$\begin{aligned} d_v(Tx_n, Tx_{n+1}) &\leq \phi\big(d_v(Tx_{n-1}, Tx_n), d_v(Tx_{n-1}, Tx_n), d_v(Tx_n, Tx_{n+1}), \\ &\quad 1/2(d_v(Tx_{n-1}, Tx_{n+1}) + d_v(Tx_n, Tx_n))\big) \\ &\leq \phi\big(d_v(Tx_{n-1}, Tx_n), d_v(Tx_{n-1}, Tx_n), d_v(Tx_n, Tx_{n+1}), \\ &\quad 1/2(d_v(Tx_{n-1}, Tx_{n+1}))\big) \\ &\leq \phi\big(d_v(Tx_{n-1}, Tx_n), d_v(Tx_{n-1}, Tx_n), d_v(Tx_n, Tx_{n+1}), \\ &\quad 1/2(d_v(Tx_{n-1}, Tx_n) + d_v(Tx_n, Tx_{n+1}))\big) \ \forall v \in \mathfrak{A}. \end{aligned} \tag{3.6}$$

We claim that $d_v(Tx_n, Tx_{n+1}) < d_v(Tx_{n-1}, Tx_n)$ $\forall v \in \mathfrak{A}$ for each $n \in \mathbb{N} \cup \{0\}$.

Suppose on contrary that $d_v(Tx_n, Tx_{n+1}) \geq d_v(Tx_{n-1}, Tx_n)$ for all $v \in \mathfrak{A}$, and some n. Since ϕ is nondecreasing, by using this in (3.6), we have

$$d_v(Tx_n, Tx_{n+1}) \leq \phi\left(\begin{array}{c} d_v(Tx_n, Tx_{n+1}), d_v(Tx_{n-1}, Tx_n), \\ d_v(Tx_n, Tx_{n+1}), d_v(Tx_n, Tx_{n+1}) \end{array} \right), \tag{3.7}$$

for all $v \in \mathfrak{A}$. By using property (ii) of Φ_ψ in (3.7), we have

$$d_v(Tx_n, Tx_{n+1}) = 0 \ \forall v \in \mathfrak{A}$$

which is contradiction to our assumption, i.e. $Tx_{n+1} \neq Tx_n$ for each $n \in \mathbb{N} \cup \{0\}$. Thus $d_v(Tx_n, Tx_{n+1}) < d_v(Tx_{n-1}, Tx_n)$ $\forall v \in \mathfrak{A}$ for all n. Therefore (3.6) becomes

$$\begin{aligned} d_v(Tx_n, Tx_{n+1}) &\leq \phi\big(d_v(Tx_{n-1}, Tx_n), d_v(Tx_{n-1}, Tx_n), d_v(Tx_n, Tx_{n+1}), \\ &\quad d_v(Tx_{n-1}, Tx_n)\big) \ \forall v \in \mathfrak{A}. \end{aligned} \tag{3.8}$$

By using (3.8) and property (ii) of Φ_ψ, we have

$$d_v(Tx_n, Tx_{n+1}) \leq \psi\big(d_v(Tx_{n-1}, Tx_n)\big) \ \forall v \in \mathfrak{A} \text{ for all } n \in \mathbb{N}.$$

Consequently, we get

$$d_v(Tx_n, Tx_{n+1}) \leq \psi^n\big(d_v(Tx_0, Tx_1)\big) \ \forall v \in \mathfrak{A} \text{ for each } n \in \mathbb{N} \cup \{0\}.$$

Let $n > m$, we have

$$\begin{aligned} d_v(Tx_m, Tx_n) &\leq d_v(Tx_m, Tx_{m+1}) + d_v(Tx_{m+1}, Tx_{m+2}) + ... + d_v(Tx_{n-1}, Tx_n) \\ &\leq \psi^m(d_v(Tx_0, Tx_1)) + \psi^{m+1}(d_v(Tx_0, Tx_1)) + ... + \psi^{n-1}(d_v(Tx_0, Tx_1)) \\ &= \sum_{i=m}^{n-1} \psi^i(d_v(Tx_0, Tx_1)) < \infty \ \forall v \in \mathfrak{A}. \end{aligned}$$

Hence $\{Tx_n\}$ is Cauchy sequence in B, which is closed subset of $(X, \mathfrak{T}(\mathfrak{F}))$. Since X is complete gauge space so the sequence converges to some point y^* in B. By using triangular inequality we get

$$\begin{aligned} d_v(y^*, A) &\leq d_v(y^*, x_n) \\ &\leq d_v(y^*, Tx_{n-1}) + d_v(Tx_{n-1}, x_n) \\ &= d_v(y^*, Tx_{n-1}) + d_v(A, B) \\ &\leq d_v(y^*, Tx_{n-1}) + d_v(y^*, A)\ \forall v \in \mathfrak{A}. \end{aligned}$$

Therefore, $d_v(y^*, x_n) \to d_v(y^*, A)\ \forall v \in \mathfrak{A}$ as $n \to \infty$. Since A is approximatively compact with respect to B, the sequence $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ convergent to some point x^* in A. Since T is continuous , thus we reach the following

$$d_v(x^*, Tx^*) = \lim_{n\to\infty} d_v(x_{n_{k+1}}, Tx_{n_k}) = d_v(A, B)\ \forall v \in \mathfrak{A},$$

and this completes the proof. □

Corollary 3.2. *Let $(X, \mathfrak{T}(\mathfrak{F}))$ be a complete gauge space induced by separating family of pseudo metrics $\mathfrak{F} = \{d_v | v \in \mathfrak{A}\}$. Let A and B be nonempty closed subsets of X such that A is approximately compact with respect to B and A_0 is nonempty. Further assume that the mapping $T\colon A \to B$ satisfies the following conditions:*

(a) There exists a nonnegative real number $\alpha < 1$ such that for all u_1, u_2, x_1, x_2 in A,

$$d_v(u_1, Tx_1) = d_v(A, B) = d_v(u_2, Tx_2) \Rightarrow d_v(Tu_1, Tu_2) \leq \alpha d_v(Tx_1, Tx_2),$$

for all $v \in \mathfrak{A}$;

(b) $T(A_0) \subseteq B_0$;

(c) T is continuous.

Then there exists an element x in A such that $d_v(x, Tx) = d_v(A, B)\ \forall v \in \mathfrak{A}$.

4. Consequences

Assume that (X, d) is a complete metric space. For the family $\mathfrak{F} = \{d_v = d | v \in \mathfrak{A}\}$, we get a gauge space which is complete as well as separating. Thus, we get the following results from Theorem 3.1 and Theorem 3.2, respectively.

Theorem 4.1. *Let A and B be nonempty closed subsets of complete metric space (X, d) such that B is approximately compact with respect to A and A_0 is nonempty. Let $T\colon A \to B$ be a mapping satisfying the following conditions:*

(i) for each $x_1, x_2, u_1, u_2 \in A$, there exists $\phi \in \Phi_\psi$ such that $d(u_1, Tx_1) = d(A, B) = d(u_2, Tx_2)$ implies

$$d(u_1, u_2) \leq \phi(d(x_1, x_2), d(x_1, u_1), d(x_2, u_2), 1/2(d(x_2, u_1) + d(x_1, u_2)));$$

(ii) $T(A_0) \subseteq B_0$.

Then T has a best proximity point.

Theorem 4.2. *Let A and B be nonempty closed subsets of complete metric space (X, d) such that A is approximately compact with respect to B and A_0 is nonempty. Let $T\colon A \to B$ be a mapping satisfying the following conditions:*

(i) for each $x_1, x_2, u_1, u_2 \in A$, there exists $\phi \in \Phi_\psi$ such that the conditions $d(u_1, Tx_1) = d(A, B) = d(u_2, Tx_2)$ imply

$$d(Tu_1, Tu_2) \leq \phi(d(Tx_1, Tx_2), d(Tx_1, Tu_1), d(Tx_2, Tu_2), 1/2(d(Tx_2, Tu_1) + d(Tx_1, Tu_2)));$$

(ii) $T(A_0) \subseteq B_0$;

(iii) T is continuous.

Then T has a best proximity point.

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