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NEW FIXED POINT RESULTS ON BRANCIARI METRIC SPACES

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Abstract. Paper aims apply new contractive maps in terms of Branciari metric spaces in order to generalization of recently proved fixed point theorems. The findings of the present research contribute and enrich previous results reviewed in the literature.

1. Introduction

There are many generalizations of metric spaces and enormous fixed point results have been achieved. Nevertheless, all these generalizations usually will not lead to novel results. Most often, topologically such Branciari metric space is found to be equal to a metric space or a known Branciari metric space among others.This phrase does not imply that the obtained fixed point theorems on these spaces are redundant, since most of the given contractive conditions can not easily transformed to the new spaces. Notwithstanding, often this is case and subsequently results will be redundant.

Among aforementioned Branciari metric spaces, the Branciari metric spaces in terms of Branciari metric might not calculated in direct manner from, fixed point theorems upon specific metric spaces. In this paper we prove new fixed point results in Branciari metric spaces in order to generalization of recently proved fixed point theorems.

In 2000, a new Branciari metric space was introduced by Branciari [11].

Definition 1.1. *Let X be a non-empty set and $d : X \times X \to [0, \infty)$ be a mapping such that for all $x, y \in X$ and for all distinct points $u, v \in X$, each of them different from x and y, one has*

(i) $d(x, y) = 0 \Rightarrow x = y$;
(ii) $d(x, y) = d(y, x)$;
(iii) $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$.

Then (X, d) is called an Branciari metric space (or for short b.m.s.).

Definition 1.2. *Let (X, d) be a b.m.s. and $\{x_n\}$ be a sequence in X and $x \in X$.*

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(a) $\{x_n\}$ *is convergent to* x *if and only if* $d(x_n, x) \to 0$ *as* $n \to \infty$. *We denote this by* $x_n \to x$;
(b) $\{x_n\}$ *is Cauchy if and only if* $d(x_n, x_m) \to 0$ *as* $n, m \to \infty$;
(c) (X, d) *is complete if and only if every Cauchy sequence in* X *converges to some element in* X.

The following lemma was proved in [17].

Lemma 1.1. *Let* (X, d) *be a b.m.s.,* $\{x_n\}$ *be a Cauchy sequence in* (X, d), *and* $x, y \in X$. *Suppose that there exists a positive integer* N *such that*

(i) $x_n \neq x_m$, *for all* $n, m > N$;
(ii) x_n *and* x *are distinct points in* X, *for all* $n > N$;
(iii) x_n *and* y *are distinct points in* X, *for all* $n > N$;
(iv) $\lim_{n\to\infty} d(x_n, x) = \lim_{n\to\infty} d(x_n, y)$.

Then we have $x = y$.

It can be easily seen that in case one of the conditions (ii) or (iii) is not satisfied, the result of this lemma will be remain valid.
Among most mathematical problems, the existence of a solution equals to existence of fixed point of certain nonlinear operators. Banach fixed point theorem is one of the most suitable tool within classical functional analysis, which have been extensive by many authors (See [1–10, 12–16, 18–25]). Khan et al. in 1982 [22], Branciari in 2000 [12] and M. Jleli and B. Samet in 2014 [18] proved important generalization of fixed point theorems deals with in the present research. Motivated by the research going in this direction, we investigate some fixed point theorems for certain type of self-maps in the context of metric space. In fact, the fixed point theorem presented here can be considered as a continuation, in part, of the work of M. Jleli and B. Samet that is the given theorem investigate conditions are results of our work.

2. Main result

In this section, we represent and prove our main result.

Theorem 2.1. *Let* (X, d) *be a complete generalized metric space in the sense of Branciari* $(g.m.s)$ *and* $T : X \to X$ *be a given map. Suppose that there exist* $k \in (0, 1)$ *and a function* $\varphi : [0, \infty) \to [0, \infty)$, *satisfying the following conditions:*

(i) *for every* $\{t_n\} \subset (0, \infty)$ *and non-constant;*
$$\lim_{n\to\infty} \varphi(t_n) = 0 \iff \lim_{n\to\infty} t_n = 0.$$
(ii) *for every* $\{t_n\} \subset (0, \infty)$, *that* $t_n \to 0^+$, $\limsup_{n\to\infty} \sqrt[n]{\varphi(t_n)} < 1 \Rightarrow \sum_1^\infty t_n < \infty$.

such that
$$\varphi(d(Tx, Ty)) \leq k.\varphi(d(x, y)), \tag{2.1}$$
then T *has a unique fixed point.*

Proof. Let $x_0 \in X$ be an arbitrary point. We define the sequence $\{x_n\}$ by $x_n = T^n x = T x_{n-1}$.
If there exists $n_0 \in X$ such that $x_{n_0} = x_{n_0-1}$, then x_{n_0} is a fixed point of T and we have nothing to prove, thus we assume that $x_n \neq x_{n+1}$ for every $n \in \mathbb{N} \cup \{0\}$

and $d(x_n, x_{n+1}) > 0$ for all n. First we show that $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. Since T satisfies (2.1), for all $n \in \mathbb{N}$ we have

$$\varphi(d(x_n, x_{n+1})) \leq k\varphi(d(x_{n-1}, x_n)). \tag{2.2}$$

Since $k \in (0, 1)$, we have

$$\varphi(d(x_n, x_{n+1})) \leq k.\varphi(d(x_{n-1}, x_n)) < \varphi(d(x_{n-1}, x_n)) \quad \forall n \in \mathbb{N}. \tag{2.3}$$

Thus $\{\varphi(d(x_{n+1}, x_n))\}$ is a decreasing sequence, hence it is convergent and

$$\lim_{n\to\infty} \varphi(d(x_{n+1}, x_n)) = r \geq 0.$$

Now we show that $r = 0$. From (2.3), we have

$$0 < \varphi(d(x_{n+1}, x_n)) \leq k.\varphi(d(x_n, x_{n-1})) \leq ... \leq k^n\varphi(d(x_1, x_0)), \tag{2.4}$$

since $0 < k < 1$, therefore $\lim_{n\to\infty} \varphi(d(x_{n+1}, x_n)) = 0$. So $\lim_{n\to\infty} d(x_{n+1}, x_n) = 0$ by (i).
On the other hand from (2.4) we have

$$\varphi(d(x_{n+1}, x_n)) \leq k^n\varphi(d(x_1, x_0)) \quad \forall n \in \mathbb{N}.$$

Then

$$\sqrt[n]{\varphi(d(x_{n+1}, x_n))} \leq k\sqrt[n]{\varphi(d(x_1, x_0))} \quad \forall n \in \mathbb{N}$$

Thus

$$\limsup_{n\to\infty} \sqrt[n]{\varphi(d(x_{n+1}, x_n))} \leq k < 1 \tag{2.5}$$

Put $t_n = d(x_{n+1}, x_n)$, using (2.5) and condition (ii) of φ, we get

$$\sum_1^\infty t_n < \infty \quad \text{and also} \quad t_n \to 0. \tag{2.6}$$

Now we show that $d(x_n, x_{n+2}) \to 0$ as $n \to \infty$.

$$0 < \varphi(d(x_{n+2}, x_n)) \leq k.\varphi(d(x_{n+1}, x_{n-1})) \leq ... \leq k^n\varphi(d(x_2, x_0)), \tag{2.7}$$

therefore $d(x_{n+2}, x_n) \to 0$ as $n \to \infty$.

Now for proving the Cauchy of $\{x_n\}$, we consider two case
1. If $m = 2p + 1, p \geq 1$, then

$$\begin{aligned} d(x_n, x_{n+m}) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + ... + d(x_{n+2p}, x_{n+2p+1}) \\ &= \sum_n^{n+2p+1} t_n < \sum_n^\infty t_n. \end{aligned}$$

2. If $m = 2p, p \geq 2$, then

$$\begin{aligned} d(x_n, x_{n+m}) &\leq d(x_n, x_{n+2}) + d(x_{n+2}, x_{n+3}) + ... + d(x_{n+2p-1}, x_{n+2p}) \\ &\leq \sum_n^{n+2p+1} t_n < \sum_n^\infty t_n. \end{aligned}$$

Thus, combining these two cases and using (2.6),when $n \to \infty$ we have

$$d(x_n, x_{n+m}) \leq \sum_n^\infty t_n \to 0 \quad \text{as} \quad n \to \infty.$$

Thus, we deduce that$\{T^n x\}$ is a Cauchy sequence.

Completeness of (X, d) ensures $\lim_{m\to\infty} x_n = z$ for some $z \in X$.

Now we want to show that z is a fixed point of T. From (2.1) we have

$$\varphi(d(Tx_n, Tz)) \leq k\varphi(d(x_n, z)).$$

Hence $d(x_n, z) \to 0$, and $\varphi(d(x_n, z)) \to 0$, and therefore $\lim_{n\to\infty} \varphi(d(x_{n+1}, Tz)) = 0$ as $n \to \infty$. Again

$$\lim_{n\to\infty} d(x_{n+1}, Tz) = 0,$$

by (i).

$$d(z, Tz) \leq d(z, x_n) + d(x_n, x_{n+1}) + d(x_{n+1}, Tz).$$

Thus $z = Tz$ and hence z is a fixed point of T.

Finally we have to show that z is a unique fixed point of T. Suppose that there is $u \neq z$ such that $T(u) = u$. clearly $d(z, u) = d(Tz, Tu) > 0$.
So we can apply condition (2.1) for the pair (z, u), then we have

$$\varphi(d(Tz, Tu)) \leq k\varphi(d(z, u)),$$

hence

$$\varphi(d(z, u)) \leq k.\varphi(d(z, u)) < \varphi(d(z, u)),$$

which is a contradiction; means z is unique fixed point of T. □

Remark. *Note that $\varphi(0)$ is not necessary zero.*

From Theorem 2.1, we can immediately conclude the following result, because a metric space is a Branciari metric space.

Corollary 2.2. *Let (X, d) be a complete metric space and $T : X \to X$ be a given map. Suppose that there exist $k \in (0, 1)$ and a function $\varphi : [0, \infty) \to [0, \infty)$, satisfying the following conditions:*

(i) *for every $\{t_n\} \subset (0, \infty)$ and non constant;*

$$\lim_{n\to\infty} \varphi(t_n) = 0 \iff \lim_{n\to\infty} t_n = 0.$$

(ii) *for every $\{t_n\} \subset (0, \infty)$, that $t_n \to 0^+$, $\limsup_{n\to\infty} \sqrt[n]{\varphi(t_n)} < 1 \Rightarrow \sum_1^\infty t_n < \infty$.*

such that

$$\varphi(d(Tx, Ty)) \leq k\varphi(d(x, y)), \tag{2.8}$$

then T has a fixed point.

We see that Banach contraction immediately concludes from 2.2.
The following result (due to Jleli and Samet [18]) is another consequence of our main result.

Corollary 2.3. *([18, M. Jleli and B. Samet 2014]) Let (X, d) be a complete Branciari metric space and $f : X \to X$ be a given mapping. Suppose that there exists $k \in (0, 1)$ and $\theta : (0, \infty) \to (1, \infty)$, that is non-decreasing, for each sequence $t_n \subset (0, \infty)$,*

$$\lim_{n\to\infty} \theta(t_n) = 1 \iff \lim_{n\to\infty} t_n = 0,$$

and there exists $r \in (0, 1)$ and $l \in (0, \infty]$ such that $\lim_{n\to 0^+} \frac{\theta(t)-1}{t^r} = l$, such that

$$d(fx, fy) \neq 0 \Rightarrow \theta(d(fx, fy)) \leq [\theta(d(x, y))]^k, \quad \forall x, y \in X.$$

Then f has a unique fixed point.

Proof. Put $\varphi(t) = \ln(\theta(t))$ for each $t \in [0, \infty)$, we show that $\varphi \in \phi$ and then apply Theorem 2.1.
It's easy to show that φ satisfies condition (i) of ϕ.

$$\lim_{n\to\infty} \varphi(t_n) = \lim_{n\to\infty} \ln(\theta(t_n)) = \ln(1) = 0 \iff \lim_{n\to\infty} t_n = 0.$$

Now we want to show that φ satisfies condition (ii) of ϕ. From condition of θ, there exists $r \in (0, 1)$ and $l \in (0, \infty]$ such that

$$\lim_{t\to 0^+} \frac{\theta(t) - 1}{t^r} = l$$

suppose that $l < \infty$. In this case, let $B = l/2 > 0$. From the definition of limit, there exists $\delta > 0$ such that

$$|\frac{\theta(t) - 1}{t^r} - l| \leq B, \ for\ all\ t,\ 0 < t < \delta.$$

This implies that

$$\frac{\theta(t) - 1}{t^r} \geq l - B = B, \ for\ all\ 0 < t < \delta,$$

then we have

$$Bt^r + 1 \leq \theta(t)\ for\ all\ 0 < t < \delta,$$

therefore

$$Bt^r < \theta(t), \ for\ all\ 0 < t < \delta.$$

In case $l = \infty$, Let $B > 0$ be an arbitrary positive number. From definition of limit, there exists $\delta > 0$ such that

$$\frac{\theta(t) - 1}{t^r} \geq B, \ for\ all\ 0 < t < \delta.$$

Again we obtain

$$Bt^r < \theta(t), \ for\ all\ 0 < t < \delta.$$

Now we suppose that $\limsup_{n\to\infty}(\ln\theta(t_n))^{1/n} < 1$, Then we have

$$\inf_{k\geq 1} \sup_{n\geq k}(\ln\theta(t_n))^{1/n} < 1 \Rightarrow \beta = \sup_{n\geq k}(\ln\theta(t_n))^{1/n} < 1 \Rightarrow (\ln\theta(t_n))^{1/n} \leq \beta < 1 \Rightarrow$$

$$\ln\theta(t_n) \leq \beta^n < 1 \Rightarrow \theta(t_n) \leq e^{\beta^n} < 1 \Rightarrow \limsup_{n\to\infty}(\theta(t_n))^{1/n} \leq \lim_{n\to\infty} e^{(1/n.\beta^n)} < 1.$$

On the other hand we obtained that $Bt^r < \theta(t)$, then we have

$$\limsup_{n\to\infty}(Bt^r)^{1/n} < \limsup_{n\to\infty}(\theta(t_n))^{1/n} \leq \lim_{n\to\infty} e^{(1/n.\beta^n)} < 1.$$

Hence $\limsup_{n\to\infty}(t_n)^{1/n} < 1$, thus φ satisfies in condition (ii) of ϕ. There fore using Theorem 2.1 this corollary is proved. □

Denote by Λ the set of all functions $\lambda : [0, +\infty) \to [0, +\infty)$ satisfying the following hypotheses:

(i) λ is a lebegue integrable mapping on each compact subset of $[0, +\infty)$,
(ii) for every $\epsilon > 0$, we have $\int_0^\epsilon \lambda(s)ds > 0$,
(iii) $||\lambda|| \leq 1$, where $||\lambda|| = \sup\{\lambda(t) : t \in [0, \infty)\}$.

Now, we have the following result.

Corollary 2.4. *Let (X, d) be a complete Branciari metric space and $T : X \to X$ be a mapping such that:*

$$\int_0^{d(Tx,Ty)} \lambda(s)ds \leq k \int_0^{d(x,y)} \lambda(s)ds, \tag{2.9}$$

where $\lambda \in \Lambda$ and $k \in (0, 1)$. Then T has a unique fixed point.

Proof. Define $\varphi : [0, +\infty) \to [0, +\infty)$ by $\varphi(t) = \int_0^t \lambda(s)ds$. Condition (i) is clear, so it is sufficient that we prove (ii). If

$$\limsup_{n\to\infty} \sqrt[n]{\varphi(t_n)} < 1,$$

so by Root Test we have, $\sum_{n=1}^{\infty} \varphi(t_n) < \infty$. On the other hand

$$\varphi(t_n) = |\varphi(t_n)| = \left| \int_0^{t_n} \lambda(s)ds \right| \leq \int_0^{t_n} |\lambda(s)|ds \leq ||\lambda||t_n < t_n,$$

for any n, and

$$\lim_{n\to\infty} \frac{\varphi(t_n)}{t_n} \leq \lim_{n\to\infty} \frac{t_n}{t_n} = 1.$$

By limit comparison test and convergence of $\sum_{n=1}^{\infty} \varphi(t_n)$ we get $\sum_{n=1}^{\infty} t_n < \infty$. Therefore φ satisfies all the conditions of φ in Theorem 2.1. So T has a unique fixed point.

□

Example 2.1. *Let $X = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{2, 3\}$ and $d : X \times X \to [0, \infty)$ defined as follow*

$$d(x, y) = \begin{cases} 0, & x{=}y; \\ 2, & x, y \in \{\frac{1}{n} : n \in \mathbb{N}\}; \\ \frac{1}{2n}, & x = \frac{1}{n}, y \in \{2, 3\}; \\ 2, & x = 2, y = 3 \text{ or } x = 3, y = 2. \end{cases}$$

We observe that

$$d(\frac{1}{2}, \frac{1}{3}) > d(\frac{1}{2}, 2) + d(2, \frac{1}{3}),$$

Hence $d(x, y)$ is not a metric. Now we show that $d(x, y)$ is a Branciari metric. If $x = y$ that's visible. For case $x = \frac{1}{n}$ and $y = \frac{1}{m}$ and $(u, v \neq x, y)$ $u = 2$ and $v = 3$ we have

$$d(\frac{1}{n}, \frac{1}{m}) \leq d(\frac{1}{n}, 2) + d(2, 3) + d(3, \frac{1}{m});$$

$$2 \leq \frac{1}{2n} + 2 + \frac{1}{2n}.$$

If $x = 2$ and $y = \frac{1}{n}$ $(u, v \neq 2, \frac{1}{n})$. Then we have

$$d(2, \frac{1}{n}) \leq d(2, 3) + d(3, \frac{1}{m}) + d(\frac{1}{m}, \frac{1}{n});$$

$$\frac{1}{2n} \leq 2 + \frac{1}{2m} + 2.$$

If $x = 2$ and $y = 3$ $(u, v \neq 2, 3)$. Then we have

$$d(2, 3) \leq d(2, \frac{1}{n}) + d(\frac{1}{n}, \frac{1}{m}) + d(\frac{1}{m}, 3);$$

$$2 \leq \frac{1}{2n} + 2 + \frac{1}{2m}.$$

So we conclude that $d(x,y)$ is a Branciari. On the other hand

$$\lim_{n\to\infty} d(\frac{1}{n}, 2) = \lim_{n\to\infty} d(\frac{1}{n}, 3) = \lim_{n\to\infty} \frac{1}{2n} = 0.$$

Thus limit is not unique. $\{\frac{1}{n}\}$ is not Cauchy although it is convergent, since

$$\lim_{n\to\infty} d(\frac{1}{n}, \frac{1}{n+k}) = 2 \neq 0, \quad \forall k.$$

Therefore we conclude that this space is not Hausdorff.

Let $T : X \to X$ be a map defined by

$$Tx = \begin{cases} \frac{1}{n+1}, & x = \frac{1}{n}; \\ 2, & x = 2, 3; \end{cases}$$

for each $n \in \mathbb{N}$. Clearly the Banach contraction is not satisfied. In fact

$$2 = d(T\frac{1}{2}, T\frac{1}{3}) = d(\frac{1}{3}, \frac{1}{4}) \geq kd(\frac{1}{2}, \frac{1}{3}) = 2k, \quad \forall k \in [0, 1).$$

Now, define $\varphi : [0, \infty) \to [0, \infty)$ by

$$\varphi(t) = \begin{cases} 0, & t \in \{2, 3\}; \\ t, & t \in c_0; \\ \sqrt{te^t}, & \text{otherwise}, \end{cases}$$

where c_0 is the set of sequences which converge to zero. Easily we can show that φ satisfies conditions (i) and (ii) of Theorem 2.1; T satisfies (2.1). To verifying, it is enough that we consider three cases:

(1) *$x = \frac{1}{n}$ and $y = \frac{1}{m}$;*
(2) *$x = \frac{1}{n}$ and $y = 2$ (or $y = 3$);*
(3) *$x = 2$ and $y = 3$.*

$x^* = 2$ is fixed point of T.

Example 2.2. *Let $X = [-2, -1] \cup \{0\} \cup [1, 2]$ and $d : X \times X \to [0, \infty)$ defined as follow $d(x, x) = 0$, for all $x \in X$*

$$d(1, 2) = d(2, 1) = 3, \ d(1, -1) = d(-1, 1) = d(-1, 2) = d(2, -1) = 1,$$

$d(x, y) = |x - y|$, otherwise.

We observe that

$$d(1, 2) > d(1, -1) + d(-1, 2).$$

Hence $d(x, y)$ is not a metric. It is obvious that $d(x, y)$ is a complete Branciari metric space.

Let $T : X \to X$ be a map defined by

$$Tx = \begin{cases} \frac{3}{4}x, & x \in [-2, -\frac{3}{2}) \cup (\frac{3}{2}, 2]; \\ 0, & \text{other where}; \end{cases}$$

Now, we define $\varphi : [0, \infty) \to [0, \infty)$ by $\varphi(t) = \sqrt{t}$.

Easily we can show that φ satisfies conditions (i) and (ii) of Theorem 2.1, T satisfies (2.1) and $x^* = 0$ is fixed point of T.

Theorem 2.5. *Let (X, d) be a complete generalized metric space in the sense of Branciari (g.m.s) and $T : X \to X$ be a given map. Suppose that there exist $k \in (0, 1)$ and a function $\varphi : [0, \infty) \to [0, \infty)$, satisfying the following conditions:*

(i) *for every $\{t_n\} \subset (0, \infty)$ and non-constant;*

$$\lim_{n\to\infty} \varphi(t_n) = 0 \iff \lim_{n\to\infty} t_n = 0.$$

(ii) *φ is continuous.*

such that

$$\varphi(d(Tx, Ty)) \leq k\varphi(d(x, y)), \tag{2.10}$$

then T has a unique fixed point.

Proof. Using the proof of Theorem 2.1, it's sufficient we show that sequence $\{x_n\}$ is a Cauchy sequence.
Suppose that $\{x_n\}$ don't be a Cauchy sequence. Then there exists $\epsilon > 0$, for which we can find subsequences $\{x_{m_k}\}$ and $\{x_{n_k}\}$ of $\{x_n\}$ with $n_k > m_k > k$, such that

$$d(x_{m_k}, x_{n_k}) \geq \epsilon.$$

Further, corresponding to m_k, we can choose n_k in such a way that, it is smallest integer with $n_k > m_k$ and satisfying $d(x_{m_k}, x_{n_k}) \geq \epsilon$ then $d(x_{m_k}, x_{n_k-1}) < \epsilon$.
Then from the rectangular inequality, we have

$$\begin{aligned} \epsilon &\leq d(x_{n_k}, x_{m_k}) \\ &\leq d(x_{n_k}, x_{n_k-2}) + d(x_{n_k-2}, x_{n_k-1}) + d(x_{n_k-1}, x_{m_k}) \\ &\leq d(x_{n_k}, x_{n_k-2}) + d(x_{n_k-2}, x_{n_k-1}) + \epsilon, \end{aligned}$$

thus $\lim_{k\to\infty} d(x_{n_k}, x_{m_k}) = \epsilon$. Again we have

$$d(x_{n_k}, x_{m_k}) \leq d(x_{n_k}, x_{n_k-1}) + d(x_{n_k-1}, x_{m_k-1}) + d(x_{m_k-1}, x_{m_k}),$$

and

$$d(x_{n_k-1}, x_{m_k-1}) \leq d(x_{n_k}, x_{n_k-1}) + d(x_{m_k}, x_{n_k}) + d(x_{m_k-1}, x_{m_k}).$$

Letting $k \to \infty$, $\lim_{k\to\infty} d(x_{n_k-1}, x_{m_k-1}) = \epsilon$.
Using inequality (2.10), we have

$$\varphi(d(x_{m_k}, x_{n_k})) \leq k\varphi(d(x_{m_k-1}, x_{n_k-1})).$$

Letting $k \to \infty$, since φ is continuous,

$$\varphi(\epsilon) \leq k\varphi(\epsilon) < \varphi(\epsilon),$$

which is a contradiction.Thus $\{x_n\}$ is a Cauchy sequence. □

Now we get the following fixed point theorem of integral type due to Branciari [12].

Corollary 2.6. *([12, Branciari 2002]) Let (X, d) be a complete metric space, $c \in (0, 1)$ and let $T : X \to X$ be a mapping such that for each $x, y \in X$,*

$$\int_0^{d(Tx,Ty)} \alpha(s)ds \leq c \int_0^{d(x,y)} \alpha(s)ds,$$

where $\alpha : [0, +\infty) \to [0, +\infty)$ is a Lebesgue-integrable mapping which is summable (i.e., with finite integral) on each compact subset of $[0, +\infty)$, nonnegative, and such

that for each $\epsilon > 0$, $\int_0^\epsilon \alpha(s)ds > 0$*; Then* T *has a unique fixed point* $a \in X$ *such that for each* $x \in X, \lim_{n\to\infty} T^n x = a$.

Proof. Put $\varphi(t) = \int_0^t \alpha(s)ds$ for each $t \in [0, \infty)$ and apply Theorem 2.5. □

Conclusions

Among mentioned Branciari metric spaces in this work, by the new condition that presented in Theorem 2.1 which is weaker than of conditions in the work of M. Jleli and B. Samet in 2014 [18], since here $\varphi(0)$ is not necessary equal to zero by the Remark 2. Hence, two important facts must be considered in this procedure:

(1) Fixed point theorems on Branciari metric spaces in the sense of Branciari may not be directly obtained from the fixed point theorems on certain metric spaces.
(2) Moreover, the mapping $\varphi : [0, \infty) \to [0, \infty)$ which is defined by a nondecreasing function, continuous and $\sum_{n=0}^{\infty} \varphi^n(t) < \infty$ for all $t > 0$; is stronger that of our conditions included in Theorem 2.1.

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