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COUPLED FIXED POINT THEOREM FOR MULTI-VALUED MAPPING VIA GENERALIZED CONTRACTION IN PARTIALLY ORDERED METRIC SPACES WITH APPLICATIONS

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Abstract. In this manuscript, using generalized type contraction, multi-valued coupled fixed point results in the set up of partially ordered metric space are studied. For the validity of the derived result appropriate example and application are also provided.

1. Introduction and Preliminaries

All through the article $\mathbb{R}^+$, $\mathbb{N}$ and $\mathbb{N}_0$ will denote the sets of non-negative real numbers, positive integer and non-negative whole numbers respectively. Assume (Λ, ϱ) be a metric space, also consider $CB(\Lambda)$ be the class of all non-empty bounded and closed subsets of Λ. For $\vartheta \in \Lambda$ and $\Omega_1 \subset \Lambda$, denote

$$d(\vartheta, \Omega_1) = \inf\{\varrho(\vartheta, \zeta) : \zeta \in \Omega_1\},$$

and for $\Omega_1, \Omega_2 \subset \Lambda$ define

$$\delta(\Omega_1, \Omega_2) = \sup\{\varrho(\varsigma, \zeta) : \varsigma \in \Omega_1, \zeta \in \Omega_2\}.$$

Fixed point theory is an outstanding source which gives authentic techniques for the presence of fixed point, common fixed point, coincidence point and coupled fixed point for self maps under different conditions. Its main application comes forward by showing the uniqueness and existences of solution of a mathematical model like, functional equations, matrix equations, differential equations, integral equations, linear inequalities([3], [18], [21]). The pioneering and fruitful result regarding the said area were brought by Banach [5] entitled Banach contraction principle(BCP). Different researchers generalized BCP in numerous spaces via different contractive conditions([11], [14]) etc. Mainly, in 1969, Nadler [19] further modified the Banach contraction principle to multi-valued mapping with the Hausdorff metric.

In partially ordered metric space, metric fixed point theory has been flourished in the near past. Ran and Reurings [21] elaborated the Banach contraction principle(BCP) in partially ordered sets and also discussed its applications to non-linear and linear matrix equations. Nieto and Lopez [20] modified the consequence of Ran

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and Reurings and used their well-formed results to find a solution for first order ODEs. Jaggi [14] introduce the concept of rational type contraction in complete metric space and established unique fixed point theorems. The established result of Jaggi [14] extended by Harjani et. al [11] for the study fixed points in complete partial ordered metric space. For further details (see[2],[4],[6],[11],[15],[16]).

The notion of mixed monotone property of mappings established by Bhaskar and Lakshmikantham [7] for the existences of coupled fixed point results. Additionally, they applied their results on a first order ODEs (with along periodic boundary conditions) [12]. Lakshmikanthem and Ćirić [18], generalized the idea of mixed monotone mapping and established a coupled fixed point theorem on behalf of non-linear contractions in partially ordered metric spaces.

Abbas *et. al* [1] further developed coupled coincidence point and common fixed point theorem in complete metric space for multi-valued and single valued mappings. Huange and Fang [13] studied applications to integral inclusion with the help of fixed point results for multi-valued mixed increasing operator in ordered Banach spaces. In partially ordered metric spaces, Choudhery and Metya [9] established some result for multi-valued and single mapping. Yin and Gueo [22] defined the new notion of g-monotone mapping and established results for single and multi-valued mapping. After this the main results of [22] improved and extended by Choudhury and Metiya [8, 9]. Klanarong and Suantai [16] introduced a new type of mixed monotone multi-valued mapping in partially metric space. Furthermore, Kim and Sumit Chandok [17] proved coupled common fixed point theorems for generalized non-linear contraction in partially ordered metric spaces.

In the concerned work, using the idea of generalized rational type contraction condition, coupled fixed point theorem for multi-valued hybrid maps in the structure of complete partially ordered metric spaces have been investigated.

Definition 1.1. *Assume Ψ signify the set of functions $\kappa : \mathbb{R}+ \to (0, 1]$ with*
(i) $\mathbb{R}^+ = \{\theta \in \mathbb{R}/\theta > 0\}$,
(ii)$\kappa(\theta_n) \to 1 \Rightarrow \theta_n \to 0$.

Example 1.2. [16] *Let $\varphi : \mathbb{R}^+ \to [0, 1)$ define by,*

$$\varphi(\omega) = \begin{cases} 1 - \frac{\omega^3}{2} & \text{if } \omega \leq 1 \\ \beta < 1 & \text{if } \omega > 1. \end{cases}$$

The following definition can be found in [16] .

Definition 1.3. *Let $(\Lambda, \preceq)$ be a partially ordered set. Let Δ_1 and Δ_2 be two non-empty subsets of Λ. Then denote*
(i) $\Delta_1 <^{(I)} \Delta_2$ if for all $p \in \Delta_1$ there exist $q \in \Delta_2$ then $p < q$,
(ii)$\Delta_1 <^{(S)} \Delta_2$ if for all $q \in \Delta_2$ there exist $p \in \Delta_1$ then $p < q$.

Definition 1.4. *Presume $(\Lambda, \preceq)$ is a partially ordered set. Also let $\xi : \Lambda \times \Lambda \to 2^\Lambda$ and $\varsigma : \Lambda \to \Lambda$ be two maps ξ is mixed ς-monotone of type (A) if for any $r, s \in \Lambda$*

$$r_1, r_2 \in \Lambda,\ \varsigma(r_1) < \varsigma(r_2) \Longrightarrow \xi(r_1, s) <^{(I)} \xi(r_2, s),$$
$$s_1, s_2 \in \Lambda,\ \varsigma(s_1) > \varsigma(s_2) \Longrightarrow \xi(r, s_1) <^{(I)} \xi(r, s_2).$$

Definition 1.5. *Assume $(\Lambda, \preceq)$ is a partially ordered set. Let $\xi : \Lambda \times \Lambda \to 2^\Lambda$ and $\varsigma : \Lambda \to \Lambda$ be two maps ξ is mixed ς-monotone of type (B) if for any $r, s \in \Lambda$*

$$r_1, r_2 \in \Lambda,\ \varsigma(r_1) < \varsigma(r_2),\ \varsigma(s_1) > \varsigma(s_2) \Longrightarrow \xi(r_1, s_1) <^{(I)} \xi(r_1, s_2).$$

Proposition 1.6. [16] *Suppose $(\Lambda, \preceq)$ is a partially ordered set and let $\xi : \Lambda \times \Lambda \to 2^{\Lambda}$ and $\varsigma : \Lambda \to \Lambda$ be two maps. If it satisfy monotone property of type (A), then it is type (B), but converse not true.*

Proof. If for any $r_1, r_2, s_1, s_2 \in \Lambda$

$$r_1, r_2 \in \Lambda,\ \varsigma(r_1) < \varsigma(r_2) \Longrightarrow \xi(r_1, s_1) <^{(I)} \xi(r_2, s_1),$$

$$s_1, s_2 \in \Lambda,\ \varsigma(s_1) > \varsigma(s_2) \Longrightarrow \xi(r_2, s_1) <^{(I)} \xi(r_2, s_2).$$

Which implies that

$$\xi(r_1, s_1) <^{(I)} \xi(r_2, s_2).$$

□

But converse is not true it is verified from the next example.

Example 1.7. [16] *Let $\Lambda = [0, 1]$, taking the natural ordered relation in Λ. Let $\xi : \Lambda \times \Lambda \to CB(\Lambda)$, $\varsigma : \Lambda \to \Lambda$ define by,*

$$\xi(\theta, \varphi) = \begin{cases} [1, 4] & \text{if } (\varphi, \theta) = (1, 1) \\ \{1 + \frac{2}{3}(\varphi + \frac{1}{\theta}\} & \text{if } (\varphi, \theta) \neq (1, 1). \end{cases}$$

$$\varsigma(\varphi) = \varphi^2.$$

It hold monotone property of type (A) but not monotone property of type (B). It is easy to verify by taking $\varphi_1 = 1$, $\varphi_2 = 2$ and $\theta = 1$.

2. Main Results

Theorem 2.1. *Let $(\Lambda, \preceq)$ be a partially ordered set. Assume that there is a metric ϱ on Λ such the (Λ, ϱ) is complete metric space. Suppose $\xi : \Lambda \times \Lambda \to CB(\Lambda)$ and $\varsigma : \Lambda \to \Lambda$ are such that ξ is mixed ς-monotone of type (B) and*

$$\delta(\xi(\mu, \nu), \xi(u, v)) \leq \varphi\Big(\frac{\varrho(\varsigma\mu, \varsigma u) + \varrho(\varsigma\nu, \varsigma v)}{2}\Big)\Big(\frac{\varrho(\varsigma\mu, \varsigma u) + \varrho(\varsigma\nu, \varsigma v)}{2}\Big), \tag{2.1}$$

$\forall\ \mu, \nu, u, v \in \Lambda$ $\varsigma\mu > \varsigma u$ and $\varsigma\nu < \varsigma v$ and $\varphi \in \Psi$. Suppose that $\xi(\Lambda \times \Lambda) \subset \varsigma(\Lambda)$, $\varsigma(\Lambda)$ is a complete subset of Λ and suppose that

(i) There exist $\mu_0, \nu_0 \in \Lambda$ such that

$$\{\varsigma\mu_0\} <^{(I)} \xi(\mu_0, \nu_0), \{\varsigma\nu_0\} <^{(S)} \xi(\nu_0, \mu_0); \tag{2.2}$$

(ii) Λ hold the assumption given as;

(a) If an increasing sequences $\mu_n \to \mu$, $\nu_n \to \nu$. Then $\mu_n < \mu$, $\nu_n < \nu$ for all n.

Then $\exists\ \mu, \nu \in \Lambda$ such that

$$\varsigma(\mu) \in \xi(\mu, \nu), \varsigma(\nu) \in \xi(\nu, \mu). \tag{2.3}$$

Proof. Let $\mu_0, \nu_0 \in \Lambda$ be such that $\{\varsigma\mu_0\} <^{(I)} \xi(\mu_0, \nu_0)$ and $\{\varsigma\nu_0\} <^{(S)} \xi(\nu_0, \mu_0)$. Since $\xi(\Lambda \times \Lambda) \subset \varsigma\Lambda$. It follows that there exist $\mu_1, \nu_1 \in \Lambda$ be such that $\varsigma\mu_1 \in \xi(\mu_0, \nu_0)$ and $\varsigma\nu_1 \in \xi(\nu_0, \mu_0)$ and $\varsigma(\mu_0) < \varsigma(\mu_1)$, $\varsigma(\nu_0) > \varsigma(\nu_1)$. Since ξ is mixed ς-monotone of type (B), we obtain

$$\xi(\mu_0, \nu_0) <^{(I)} \xi(\mu_1, \nu_1),\ \xi(\nu_1, \mu_1) <^{(I)} \xi(\nu_0, \mu_0). \tag{2.4}$$

Again since $\xi(\Lambda \times \Lambda) \subset \varsigma\Lambda$. We can choose $\mu_2, \nu_2 \in \Lambda$ be such that $\varsigma\mu_2 \in \xi(\mu_1, \nu_1)$ and $\varsigma\nu_2 \in \xi(\nu_1, \mu_1)$ and $\varsigma(\mu_1) < \varsigma(\mu_2)$, $\varsigma(\nu_1) > \varsigma(\nu_2)$. By continuing this procedure, we achieve a sequence $\{\varsigma\mu_n\}$ and $\{\varsigma\nu_n\}$ in Λ such that

$$\{\varsigma\mu_{n+1}\} \in \xi(\mu_n, \nu_n) \ and \ \{\varsigma\nu_{n+1}\} \in \xi(\nu_n, \mu_n), \tag{2.5}$$

which implies that

$$\varsigma(\mu_0) < \varsigma(\mu_1) < ..... < \varsigma(\mu_n), \tag{2.6}$$

$$\varsigma(\nu_0) > \varsigma(\nu_1) > ..... > \varsigma(\nu_n). \tag{2.7}$$

Put

$$\alpha_n = \varrho(\varsigma(\mu_n), \varsigma(\mu_{n+1})) + \varrho(\varsigma(\nu_n), \varsigma(\nu_{n+1}))\forall n \geq 0. \tag{2.8}$$

We show that

$$\alpha_n \leq \varphi(\frac{\alpha_{n+1}}{2})(\frac{\alpha_{n+1}}{2}). \tag{2.9}$$

Since

$$\varsigma(\mu_{n-1}) < \varsigma(\mu_n) \ and \ \varsigma(\nu_{n-1}) > \varsigma(\nu_n).$$

From equation (2.5) and equation (2.1) we have

$$\varrho(\varsigma(\mu_n), \varsigma(\mu_{n+1})) \leq \delta(\xi(\mu_{n-1}, \nu_{n-1}), \xi(\mu_n, \nu_n))$$
$$\leq \varphi\Big(\frac{\varrho(\varsigma(\mu_{n-1}), \varsigma(\mu_n)) + \varrho(\varsigma(\nu_{n-1}), \varsigma(\nu_n))}{2}\Big)\Big(\frac{\varrho(\varsigma(\mu_{n-1}), \varsigma(\mu_n)) + \varrho(\varsigma(\nu_{n-1}), \varsigma(\nu_n))}{2}\Big),$$

which implies that

$$\varrho(\varsigma(\mu_n), \varsigma(\mu_{n+1})) \leq \varphi\big(\frac{\alpha_{n-1}}{2}\big)\big(\frac{\alpha_{n-1}}{2}\big). \tag{2.10}$$

Similarly, we have

$$\varrho(\varsigma(\nu_n), \varsigma(\nu_{n+1})) \leq \delta(\xi(\nu_{n-1}, \mu_{n-1}), \xi(\nu_n, \mu_n))$$
$$\leq \varphi\Big(\frac{\varrho(\varsigma(\nu_{n-1}), \varsigma(\mu_n)) + \varrho(\varsigma(\mu_{n-1}), \varsigma(\mu_n))}{2}\Big)\Big(\frac{\varrho(\varsigma(\nu_{n-1}), \varsigma(\nu_n)) + \varrho(\varsigma(\mu_{n-1}), \varsigma(\mu_n))}{2}\Big),$$

which implies that

$$\varrho(\varsigma(\nu_n), \varsigma(\nu_{n+1})) \leq \varphi\big(\frac{\alpha_{n-1}}{2}\big)\big(\frac{\alpha_{n-1}}{2}\big). \tag{2.11}$$

Adding equations (2.10) and (2.11), we have

$$\alpha_n \leq 2\varphi\big(\frac{\alpha_{n-1}}{2}\big)\big(\frac{\alpha_{n-1}}{2}\big),$$

which implies $\alpha_n \leq \alpha_{n-1}$. It follow that α_n is a monotone decreasing sequence of non-negative real numbers. Therefor, there is some $\alpha \geq 0$ such that

$$\lim_{n\to\infty} \alpha_n = \alpha. \tag{2.12}$$

Further, we need to show that $\alpha = 0$. Suppose $\alpha > 0$, then from

$$\frac{\alpha_n}{\alpha_{n-1}} \leq \varphi\big(\frac{\alpha_{n-1}}{2}\big) < 1.$$

Taking limit we have

$$\lim_{n\to\infty} \varphi\big(\frac{\alpha_{n-1}}{2}\big) = 1,$$

by using φ function we have
$\varrho(\varsigma(\mu_{n-1}), \varsigma(\mu_n)) \to 0$, $\varrho(\varsigma(\nu_{n-1}), \varsigma(\nu_n)) \to 0$. Therefore, $\alpha = 0$ that is

$$\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} (\varrho(\varsigma(\mu_{n+1}), \varsigma(\mu_n)) + \varrho(\varsigma(\nu_{n+1}), \varsigma(\nu_n)) = 0. \tag{2.13}$$

Now, we prove that $\{\varsigma(\mu_n)\}$ and $\{\varsigma(\nu_n)\}$ are Cauchy sequences. Suppose to the contrary, that at least one from $\{\varsigma(\mu_n)\}$ or $\{\varsigma(\nu_n)\}$ is not a Cauchy sequence. Then there is $\epsilon > 0$ and two subsequence of integers $\{r(\vartheta)\}$, $\{s(\vartheta)\}$ such that $s(\vartheta) > r(\vartheta) \geq \vartheta$ with

$$\gamma_\vartheta = \varrho(\varsigma(\mu_{r(\vartheta)}), \varsigma(\mu_{s(\vartheta)})) + \varrho(\varsigma(\nu_{r(\vartheta)}), \varsigma(\nu_{s(\vartheta)})) \geq \epsilon \ for \ \vartheta \in \mathbb{N}. \tag{2.14}$$

We also assume that

$$\varrho(\varsigma(\mu_{r(\vartheta)}), \varsigma(\mu_{s(\vartheta)-1})) + \varrho(\varsigma(\nu_{r(\vartheta)}), \varsigma(\nu_{s(\vartheta)-1})) < \epsilon. \tag{2.15}$$

By choosing $s(\vartheta)$ to be the smallest number exceeding $r(\vartheta)$ for which (2.14) hold. From equations (2.14), (2.15) and triangle inequality we have

$$\begin{aligned} \epsilon \leq \gamma_\vartheta &= \varrho(\varsigma(\mu_{r(\vartheta)}), \varsigma(\mu_{s(\vartheta)})) + \varrho(\varsigma(\nu_{r(\vartheta)}), \varsigma(\nu_{s(\vartheta)})) \\ &\leq \varrho(\varsigma(\mu_{r(\vartheta)}), \varsigma(\mu_{s(\vartheta)-1})) + \varrho(\varsigma(\mu_{s(\vartheta)-1}), \varsigma(\mu_{s(\vartheta)})) \\ &\quad + \varrho(\varsigma(\nu_{r(\vartheta)}), \varsigma(\nu_{s(\vartheta)-1})) + \varrho(\varsigma(\nu_{s(\vartheta)-1}), \varsigma(\nu_{s(\vartheta)})) \\ &= \varrho(\varsigma(\mu_{r(\vartheta)}), \varsigma(\mu_{s(\vartheta)-1})) + \varrho(\varsigma(\nu_{r(\vartheta)}), \varsigma(\nu_{s(\vartheta)-1})) + \alpha_{s(\vartheta)-1} \\ &< \epsilon + \alpha_{s(\vartheta)-1}. \end{aligned} \tag{2.16}$$

By passing limit as $\vartheta \to \infty$ we get

$$\lim_{\vartheta \to \infty} \gamma_\vartheta = \epsilon. \tag{2.17}$$

By triangle inequality we have

$$\begin{aligned} \gamma_\vartheta = \varrho(\varsigma(\mu_{r(\vartheta)}), \varsigma(\mu_{s(\vartheta)})) &+ \varrho(\varsigma(\nu_{r(\vartheta)}), \varsigma(\nu_{s(\vartheta)})) \\ &\leq \varrho(\varsigma(\mu_{r(\vartheta)}), \varsigma(\mu_{r(\vartheta)+1})) \\ &+ \varrho(\varsigma(\mu_{r(\vartheta)+1}), \varsigma(\mu_{s(\vartheta)+1})) + \varrho(\varsigma(\mu_{s(\vartheta)+1}), \varsigma(\mu_{s(\vartheta)})) \\ \varrho(\varsigma(\nu_{r(\vartheta)}), \varsigma(\nu_{r(\vartheta)+1})) &+ \varrho(\varsigma(\nu_{r(\vartheta)+1}), \varsigma(\nu_{s(\vartheta)+1})) + \varrho(\varsigma(\nu_{s(\vartheta)+1}), \varsigma(\nu_{s(\vartheta)})) \\ &= \varrho(\varsigma(\mu_{r(\vartheta)}), \varsigma(\mu_{r(\vartheta)+1})) + \varrho(\varsigma(\nu_{r(\vartheta)}), \varsigma(\nu_{r(\vartheta)+1})) + \\ &\varrho(\varsigma(\mu_{s(\vartheta)}), \varsigma(\mu_{s(\vartheta)+1})) + \varrho(\varsigma(\nu_{s(\vartheta)}), \varsigma(\mu_{s(\vartheta)+1})) + \\ &\varrho(\varsigma(\mu_{r(\vartheta)+1}), \varsigma(\mu_{s(\vartheta)+1})) + \varrho(\varsigma(\nu_{r(\vartheta)+1}), \varsigma(\mu_{s(\vartheta)+1})). \end{aligned} \tag{2.18}$$

$$\gamma_{(\vartheta)} \leq \alpha_{r(\vartheta)} + \alpha_{s(\vartheta)} + \varrho(\varsigma(\mu_{r(\vartheta)+1}), \varsigma(\mu_{s(\vartheta)+1})) + \varrho(\varsigma(\mu_{r(\vartheta)+1}), \varsigma(\mu_{s(\vartheta)+1})). \tag{2.19}$$

From equations (2.6) and (2.7), we have $\varsigma(\mu_{r(\vartheta)}) < \varsigma(\mu_{s(\vartheta)})$ and $\varsigma(\mu_{r(\vartheta)}) > \varsigma(\mu_{s(\vartheta)})$ by equation (2.5) and equation (2.1) we have

$$\begin{aligned} &\varrho(\varsigma(\mu_{r(\vartheta)+1}), \varsigma(\mu_{s(\vartheta)+1})) \leq \delta(\xi(\mu_{r(\vartheta)}, \nu_{r(\vartheta)}), \xi(\mu_{s(\vartheta)}, \nu_{s(\vartheta)})) \\ &\leq \varphi\Big(\frac{\varrho(\varsigma(\mu_{r(\vartheta)}), \varsigma(\mu_{s(\vartheta)})) + \varrho(\varsigma(\nu_{r(\vartheta)}), \varsigma(\nu_{s(\vartheta)}))}{2}\Big)\Big(\frac{\varrho(\varsigma(\mu_{r(\vartheta)}), \varsigma(\mu_{s(\vartheta)})) + \varrho(\varsigma(\nu_{r(\vartheta)}), \varsigma(\nu_{s(\vartheta)}))}{2}\Big). \end{aligned}$$

Which implies that

$$\varrho(\varsigma(\mu_{r(\vartheta)+1}), \varsigma(\mu_{s(\vartheta)+1})) \leq \varphi\big(\frac{\gamma_\vartheta}{2}\big)\big(\frac{\gamma_\vartheta}{2}\big). \tag{2.20}$$

Similarly we have

$$\begin{aligned} &\varrho(\varsigma(\nu_{r(\vartheta)+1}), \varsigma(\nu_{s(\vartheta)+1})) \leq \delta(\xi(\nu_{r(\vartheta)}, \mu_{r(\vartheta)}), \xi(\nu_{s(\vartheta)}, \mu_{s(\vartheta)})) \\ &\leq \varphi\Big(\frac{\varrho(\varsigma(\nu_{r(\vartheta)}), \varsigma(\nu_{s(\vartheta)})) + \varrho(\varsigma(\mu_{r(\vartheta)}), \varsigma(\mu_{s(\vartheta)}))}{2}\Big)\Big(\frac{\varrho(\varsigma(\nu_{r(\vartheta)}), \varsigma(\nu_{s(\vartheta)})) + \varrho(\varsigma(\mu_{r(\vartheta)}), \varsigma(\mu_{s(\vartheta)}))}{2}\Big). \end{aligned}$$

$$\varrho(\varsigma(\nu_{r(\vartheta)+1}), \varsigma(\nu_{s(\vartheta)+1})) \leq \varphi\big(\frac{\gamma_\vartheta}{2}\big)\big(\frac{\gamma_\vartheta}{2}\big). \tag{2.21}$$

Adding (2.20) and (2.21) and using (2.19) we have

$$\gamma_\vartheta \leq \alpha_{r(\vartheta)} + \alpha_{s(\vartheta)} + 2\varphi\big(\frac{\gamma_\vartheta}{2}\big)\big(\frac{\gamma_\vartheta}{2}\big). \tag{2.22}$$

By using (2.13) and (2.17) together with property of φ, we get contradiction. Thus $\{\varsigma(\mu_n)\}$ and $\{\varsigma(\nu_n)\}$ are Cauchy sequence. Since $\varsigma(\Lambda)$ is complete, there exist $\mu, \nu \in \Lambda$ such that

$$\lim_{n\to\infty} \varsigma(\mu_n) = \varsigma(\mu), \lim_{n\to\infty} \varsigma(\nu_n) = \varsigma(\nu). \tag{2.23}$$

Finally we claim $\varsigma(\mu) \in \xi(\mu, \nu)$ and $\varsigma(\nu) \in \xi(\nu, \mu)$. From equation (2.23) and (ii), we have $\varsigma(\mu_n) < \varsigma(\mu)$, $\varsigma(\nu_n) > \varsigma(\nu)$ $\forall$ $n \geq 0$ by (2.1) we have

$$\begin{aligned}\varrho(\xi(\mu,\nu), \varsigma(\mu_{n+1})) &\leq \delta(\xi(\mu,\nu), \xi(\mu_n, \nu_n)) \\ &\leq \varphi\Big(\frac{\varrho(\varsigma(\mu_n), \varsigma(\mu)) + \varrho(\varsigma(\nu_n), \varsigma(\nu))}{2}\Big)\Big(\frac{\varrho(\varsigma(\mu_n), \varsigma(\mu)) + \varrho(\varsigma(\nu_n), \varsigma(\nu))}{2}\Big).\end{aligned}$$

This implies that $\varrho(\xi(\mu,\nu), \varsigma\mu) = 0$. Hence $\varsigma\mu \in \xi(\mu,\nu)$. Similarly, we can show that $\varsigma(\nu) \in \xi(\nu,\mu)$. Thus (μ,ν) is a coupled coincidence point of ξ and ς. □

Corollary 2.2. *Let $(\Lambda, \preceq)$ is a partially ordered set. Further, suppose that there is a metric ϱ on Λ such the (Λ, ϱ) is complete metric space. Suppose $\xi : \Lambda \times \Lambda \to CB(\Lambda)$ and $\varsigma : \Lambda \to \Lambda$ are such that ξ is mixed ς- monotone of type (A) and*

$$\delta(\xi(\mu,\nu), \xi(u,v)) \leq \varphi\Big(\frac{\varrho(\varsigma\mu, \varsigma u) + \varrho(\varsigma\nu, \varsigma v)}{2}\Big)\Big(\frac{\varrho(\varsigma\mu, \varsigma u) + \varrho(\varsigma\nu, \varsigma v)}{2}\Big), \tag{2.24}$$

$\forall$ $\mu, \nu, u, v \in \Lambda$ $\varsigma\mu > \varsigma u$ and $\varsigma\nu < \varsigma v$ and $\varphi \in \Psi$. Suppose that $\xi(\Lambda \times \Lambda) \subset \varsigma(\Lambda)$ is a complete subset of Λ and suppose that

- *(i) There exist $\forall$ $\mu_0, \nu_0 \in \Lambda$ such that*

$$\{\varsigma\mu_0\} < \xi(\mu_0, \nu_0), \{\varsigma\nu_0\} < \xi(\nu_0, \mu_0); \tag{2.25}$$

- *(ii) Λ hold the assumption given as;*
- *(a) If an increasing sequences $\mu_n \to \mu$, $\nu_n \to \nu$. Then $\mu_n < \mu$, $\nu_n < \nu$ for all n.*

Then $\exists$ $\mu, \nu \in \Lambda$ such that

$$\varsigma(\mu) \in \xi(\mu,\nu), \varsigma(\nu) \in \xi(\nu,\mu). \tag{2.26}$$

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Example 2.3. *Let $\Lambda = [0,1]$ and assume the natural ordered relation in Λ. Let $\xi : \Lambda \times \Lambda \to CB(\Lambda)$, $\varsigma : \Lambda \to \Lambda$ and $\varphi : \mathbb{R}^+ \to [0,1)$ define by,*

$$\xi(\mu,\nu) = \begin{cases} [\frac{1}{2}, 1] & \text{if } (\mu,\nu) = (1,1) \\ \{\frac{\mu}{25} + \frac{1}{25(1+\nu)}\} & \text{if } (\mu,\nu) \neq (1,1). \end{cases}$$

$$\varsigma(\mu) = \mu^2.$$

$$\varphi(t) = \begin{cases} 1 - \frac{t^3}{2} & \text{if } t \leq 1 \\ \beta < 1 & \text{if } t > 1. \end{cases}$$

We discuss the below cases

Case 1.1 $(\mu,\nu)=(0,0),(u,v)=(0,1)$ *or* $(\mu,\nu)=(1,1),(u,v)=(0,1)$ *it is clear that* $(\varsigma\mu,\varsigma\nu)\preceq(\varsigma u,\varsigma v)$ *or* $(\varsigma\mu,\varsigma\nu)\succeq(\varsigma u,\varsigma v)$ *and*

we have $\delta\left(\xi(0,0),\xi(0,1)\right)=\delta\left(\{\frac{1}{25}\},\{\frac{1}{50}\}\right)=\frac{1}{50}$ *and*

$$\varphi\left(\frac{\varrho(\varsigma\mu,\varsigma u)+\varrho(\varsigma\nu,\varsigma v)}{2}\right)\left(\frac{\varrho(\varsigma\mu,\varsigma u)+\varrho(\varsigma\nu,\varsigma v)}{2}\right).$$

$$=\varphi\left(\frac{\varrho(0,0)+\varrho(0,1)}{2}\right)\left(\frac{\varrho(0,0)+\varrho(0,1)}{2}\right)=\frac{15}{32}.$$

The second choice is clearly hold.

Case 1.2 $(\mu,\nu)=(1,0),(u,v)=(0,0)$ *it is clear that* $(\varsigma\mu,\varsigma\nu)\preceq(\varsigma u,\varsigma v)$*or* $(\varsigma\mu,\varsigma\nu)\succeq(\varsigma u,\varsigma v)$ *and*

we have $\delta\left(\xi(1,0),\xi(0,0)\right)=\delta\left(\{\frac{2}{25}\},\{\frac{1}{25}\}\right)=\frac{1}{25}$ *and*

$$\varphi\left(\frac{\varrho(\varsigma\mu,\varsigma u)+\varrho(\varsigma\nu,\varsigma v)}{2}\right)\left(\frac{\varrho(\varsigma\mu,\varsigma u)+\varrho(\varsigma\nu,\varsigma v)}{2}\right).$$

$$\varphi\left(\frac{\varrho(1,0)+\varrho(0,0)}{2}\right)\left(\frac{\varrho(\varsigma\mu,\varsigma u)+\varrho(\varsigma\nu,\varsigma v)}{2}\right)=\frac{15}{32}.$$

Case 1.3 $(\mu,\nu)=(1,0),(u,v)=(0,1)$ *it is clear that* $(\varsigma\mu,\varsigma\nu)\preceq(\varsigma u,\varsigma v)$ *or* $(\varsigma\mu,\varsigma\nu)\succeq(\varsigma u,\varsigma v)$ *and*

we have $\delta\left(\xi(1,0),\xi(0,1)\right)=\delta\left(\{\frac{2}{25}\},\{\frac{1}{50}\}\right)=\frac{3}{50}$ *and*

$$\varphi\left(\frac{\varrho(\varsigma\mu,\varsigma u)+\varrho(\varsigma\nu,\varsigma v)}{2}\right)\left(\frac{\varrho(\varsigma\mu,\varsigma u)+\varrho(\varsigma\nu,\varsigma v)}{2}\right).$$

$$\varphi\left(\frac{\varrho(1,0)+\varrho(0,1)}{2}\right)\left(\frac{\varrho(1,0)+\varrho(0,1)}{2}\right)=\frac{1}{2}.$$

Case 1.4 $(\mu,\nu)=(1,0),(u,v)=(1,1)$ *again it is clear that* $(\varsigma\mu,\varsigma\nu)\preceq(\varsigma u,\varsigma v)$ *or* $(\varsigma\mu,\varsigma\nu)\succeq(\varsigma u,\varsigma v)$*and*

we have $\delta\left(\xi(1,0),\xi(1,1)\right)=\delta\left(\{\frac{2}{25}\},\{\frac{3}{50}\}\right)=\frac{1}{50}$ *and*

$$\varphi\left(\frac{\varrho(\varsigma\mu,\varsigma u)+\varrho(\varsigma\nu,\varsigma v)}{2}\right)\left(\frac{\varrho(\varsigma\mu,\varsigma u)+\varrho(\varsigma\nu,\varsigma v)}{2}\right).$$

$$\varphi\left(\frac{\varrho(1,0)+\varrho(1,1)}{2}\right)\left(\frac{\varrho(1,0)+\varrho(1,1)}{2}\right)=\frac{15}{32}.$$

Clearly for $(\varsigma\mu,\varsigma\nu)\preceq(\varsigma u,\varsigma v)$ *or* $(\varsigma\mu,\varsigma\nu)\succeq(\varsigma u,\varsigma v)$ *all the conditions of Theorem 2.1 holds. So* $(1,1)$ *is the coupled fixed point of* ξ *and* ς.

Theorem 2.4. *Suppose that the hypothesis of Theorem 2.1 are useable. In addition assume there be* $(\varsigma w,\varsigma z)\in\Lambda\times\Lambda$ *for each* $(\varsigma\mu,\varsigma\nu),(\varsigma u,\varsigma v)\in\Lambda\times\Lambda$ *which is comparable to* $(\varsigma\mu,\varsigma\nu)$ *and* $(\varsigma u,\varsigma v)$ *also* ς *is w-commutative with* ξ*. Then the coupled fixed point of* ξ *and* ς *has unique, that is there exist a unique* $(\mu,\nu)\in\Lambda\times\Lambda$ *such that* $\mu=\varsigma\mu\in\xi(\mu,\nu)$ *and* $\nu=\varsigma\nu\in\xi(\nu,\mu)$.

Proof. By Theorem 2.1 the set of coupled coincidence point is non-empty. Let (μ,ν) and (u,v) are two coupled coincidence points, that is,

$$\varsigma(\mu)\in\xi(\mu,\nu),\varsigma(\nu)\in\xi(\nu,\mu),$$
$$\varsigma(u)\in\xi(u,v),\varsigma(v)\in\xi(v,u).$$

By given hypotheses, there exists a $(\varsigma(s),\varsigma(t)) \in \Lambda \times \Lambda$ such that $(\varsigma(s),\varsigma(t))$ is comparable to both $(\varsigma(\mu),\varsigma(\nu))$ and $(\varsigma(u),\varsigma(v))$. Take $s_0 = s$ and $t_0 = t$ and define sequences s_n and t_n as follows

$$\varsigma(s_{n+1}) \in \xi(s_n,t_n) \ \textit{and}\ \varsigma(t_{n+1}) \in \xi(t_n,s_n)\ \forall\ n \geq 0.$$

Since $(\varsigma(s),\varsigma(t))$ is comparable to $(\varsigma(\mu),\varsigma(\nu))$, we may assume that $(\varsigma(s_0),\varsigma(t_0)) = (\varsigma(s),\varsigma(t)) \preceq (\varsigma(\mu),\varsigma(\nu))$ (the other case is similar). Now, by using the mixed ς-monotone property of ξ and induction we obtain $(\varsigma(s_n),\varsigma(t_n)) \preceq (\varsigma(\mu),\varsigma(\nu))$ for all n.

From equation (2.1) we have

$$\begin{aligned}\varrho(\varsigma(\mu),\varsigma(s_{n+1})) &\leq \delta(\xi(\mu,\nu),\xi(s_n,t_n))\\ &\leq \varphi\Big(\frac{\varrho(\varsigma(\mu),\varsigma(s_n)) + \varrho(\varsigma(\nu),\varsigma(t_n))}{2}\Big)\Big(\frac{\varrho(\varsigma(\mu),\varsigma(s_n)) + \varrho(\varsigma(\nu),\varsigma(t_n))}{2}\Big),\end{aligned}$$

hence we have

$$\varrho(\varsigma(\mu),\varsigma(s_{n+1})) \leq \varphi\big(\frac{\alpha_{n-1}}{2}\big)\big(\frac{\alpha_{n-1}}{2}\big). \tag{2.27}$$

Similarly we have

$$\begin{aligned}\varrho(\varsigma(\nu),\varsigma(t_{n+1})) &\leq \delta(\xi(\nu,\mu),\xi(t_n,\mu_n))\\ &\leq \varphi\Big(\frac{\varrho(\varsigma(\nu),\varsigma(t_n)) + \varrho(\varsigma(\mu),\varsigma(s_n))}{2}\Big)\Big(\frac{\varrho(\varsigma(\nu),\varsigma(t_n)) + \varrho(\varsigma(\mu),\varsigma(s_n))}{2}\Big),\end{aligned}$$

which implies that

$$\varrho(\varsigma(t_n),\varsigma(t_{n+1})) \leq \varphi\big(\frac{\alpha_{n-1}}{2}\big)\big(\frac{\alpha_{n-1}}{2}\big). \tag{2.28}$$

Adding equations (2.27) and (2.28) we have

$$\alpha_n \leq 2\varphi\big(\frac{\alpha_{n-1}}{2}\big)\big(\frac{\alpha_{n-1}}{2}\big),$$

which implies $\alpha_n \leq \alpha_{n-1}$. It follow that α_n is a monotone decreasing sequence of non-negative real numbers. Therefor, there is some $\alpha \geq 0$ such that

$$\lim_{n\to\infty} \alpha_n = \alpha. \tag{2.29}$$

Further, we show that $\alpha = 0$. Suppose $\alpha > 0$, then from

$$\frac{\alpha_n}{\alpha_{n-1}} \leq \varphi\big(\frac{\alpha_{n-1}}{2}\big) < 1,$$

by taking limit we have

$$\lim_{n\to\infty} \varphi\big(\frac{\alpha_{n-1}}{2}\big) = 1.$$

By using φ function we have

$$\varrho(\varsigma(\mu),\varsigma(s_n)) \to 0, \varrho(\varsigma(\nu),\varsigma(t_n)) \to 0. \tag{2.30}$$

Similarly we can show that

$$\varrho(\varsigma(u),\varsigma(s_n)) \to 0, \varrho(\varsigma(v),\varsigma(t_n)) \to 0. \tag{2.31}$$

Together with equations (2.30) and (2.31) we get

$$\varsigma(\mu) = \varsigma(u), \varsigma(\nu) = \varsigma(v). \tag{2.32}$$

Since $\varsigma(\mu) \in \xi(\mu,\nu)$ and $\varsigma(\nu) \in \xi(\nu,\mu)$, by w-commutativity of ξ and ς we have

$$\varsigma(\varsigma(\mu)) \in \varsigma(\xi(\mu,\nu)) \subseteq \xi(\varsigma(\mu),\varsigma(\nu)), \varsigma(\varsigma(\nu)) \in \varsigma(\xi(\nu,\mu)) \subseteq \xi(\varsigma(\nu),\varsigma(\mu)).$$

By replacing $\varsigma(\mu)$ and $\varsigma(\nu)$ by γ_1 and γ_2, respectively, we obtain

$$\varsigma(\gamma_1) \in \xi(\gamma_1, \gamma_2) \text{ and } \varsigma(\gamma_2) \in \xi(\gamma_2, \gamma_1).$$

Hence (γ_1, γ_2) is a coupled coincidence point. So u and v can be replaced by γ_1 and γ_2 in (2.32), respectively, which implies that $\varsigma(\gamma_1) = \varsigma(\mu)$ and $\varsigma(\gamma_2) = \varsigma(\nu)$, that is,

$$\gamma_1 = \varsigma(\gamma_1) \in \xi(\gamma_1, \gamma_2) \text{ and } \gamma_2 = \varsigma(\gamma_2) \in \xi(\gamma_2, \gamma_1).$$

Thus (γ_1, γ_2) is a coupled fixed point of ξ and ς.

To prove the uniqueness, let (z, w) be another coupled fixed point. Then (2.32) implies that $\gamma_1 = \varsigma(\gamma_1) = \varsigma(z) = z$ and $\gamma_2 = \varsigma(\gamma_2) = \varsigma(w) = w$.

□

3. Applications to Integral Equations

In this section, we give an existence theorem for the solution of the integral equations given as:

$$\left.\begin{aligned} \alpha(r) &= \int_{\zeta_1}^{\zeta_2} (\xi_1(r,\omega) + \xi_2(r,\omega))[f(r,\alpha(\omega),\beta(\omega)) + g(r,\alpha(\omega),\beta(\omega))]d\omega + h(r), r \in [\zeta_1,\zeta_2]; \\ \beta(r) &= \int_{\zeta_1}^{\zeta_2} (\xi_1(r,\omega) + \xi_2(r,\omega))[f(r,\beta(\omega),\alpha(\omega)) + g(r,\beta(\omega),\alpha(\omega))]d\omega + h(r), r \in [\zeta_1,\zeta_2]. \end{aligned}\right\} \tag{3.1}$$

The class of all continuous function denoted by $\Lambda := C(I, \mathbb{R})$ defined on $[\zeta_1, \zeta_2]$. Λ is the partially ordered set. The order relation in Λ is defined as:

$$\alpha, \beta \in C(I, \mathbb{R}), \alpha \leq \beta \Longleftrightarrow \alpha(r) \leq \beta(r) \ \forall r \in I.$$

Define $\varrho : \Lambda \times \Lambda \to \mathbb{R}^+$, by

$$\varrho(\alpha, \beta) = \sup_{t \in [\zeta_1, \zeta_2]} |\alpha(r) - \beta(r)| \forall \ \alpha, \ \beta \in \Lambda.$$

Then (Λ, ϱ) is a complete metric space on Λ. Now defined the following partial order on $\Lambda \times \Lambda$. For $(\alpha, \beta), (\gamma, v) \in \Lambda \times \Lambda$

$$(\alpha, \beta) \leq (\gamma, v) \Longleftrightarrow \alpha(r) \geq \gamma(r) \text{ and } \beta(r) \leq v(r) \ \forall r \in I.$$

For any $(\alpha, \beta) \in \Lambda \times \Lambda$, upper and lower bound of α, β are $(\max\{\alpha, \beta\}), \min\{\alpha, \beta\})$. Therefore for $(\alpha, \beta), (\gamma, v) \in \Lambda \times \Lambda$ there exist $(\max\{\alpha, \gamma\}), \max\{\beta, v\})$ which is comparable to (γ, v) and (α, β).

Theorem 3.1. *Suppose the following hypothesis hold:*

(1) $\xi_i : [\zeta_1, \zeta_2] \times [\zeta_1, \zeta_2] \to \mathbb{R}, i = 1, 2$ *and* $f, g : [\zeta_1, \zeta_2] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *and* $h : [\zeta_1, \zeta_2] \to \mathbb{R}$ *are continuous;*
(2) $\xi_1(r, \omega) \geq 0$, $\xi_2(r, \omega) \geq 0$*;*
(3) *There exist* $\lambda_1 > 0, \lambda_2 > 0$ *and* $\varphi \in \Psi$ *such that for all* $\alpha, \gamma, \beta, \gamma \in \mathbb{R}$ *with* $\alpha \geq \gamma$ *and* $\beta \leq \gamma$*, the following hypothesis holds:*

$$\left.\begin{aligned} 0 \leq f(\omega,\alpha,\beta) - f(\omega,\gamma,\gamma) \leq \lambda_1 \varphi\Big(\frac{\varrho(\alpha,\gamma) + \varrho(\beta,\gamma)}{2}\Big)\Big(\frac{\varrho(\alpha,\gamma) + \varrho(\beta,\gamma)}{2}\Big), \\ -\lambda_2 \varphi\Big(\frac{\varrho(\alpha,\gamma) + \varrho(\beta,\gamma)}{2}\Big)\Big(\frac{\varrho(\alpha,\gamma) + \varrho(\beta,\gamma)}{2}\Big) \leq g(\omega,\gamma,\gamma) - g(\omega,\alpha,\beta) \leq 0. \end{aligned}\right\} \tag{3.2}$$

(4) $\sup_{t\in[\zeta_1,\zeta_2]}\left|\int_{\zeta_1}^{\zeta_2}(\xi_1(r,\omega)+\xi_2(r,\omega))d\omega\right|\leq\frac{1}{2}$;
(5) $\max\{\lambda_1,\lambda_2\}\leq 1$;
(6) *There exist* $(\eta_1,\eta_2)\in\Lambda\times\Lambda$ *coupled upper and lower solution of the integral equation if* $\eta_1(r)\leq\eta_2(r)$ *and*

$$\eta_1(r)\leq\int_{\zeta_1}^{\zeta_2}(\xi_1(r,\omega)+\xi_2(r,\omega))\Big[f(r,\eta_1(\omega),\eta_2(\omega))+g(r,\eta_1(\omega),\eta_2(\omega))\Big]d\omega+h(r),r\in[\zeta_1,\zeta_2];$$

and

$$\eta_2(r)\geq\int_{\zeta_1}^{\zeta_2}(\xi_1(r,\omega)+\xi_2(r,\omega))\Big[f(r,\eta_2(\omega),\eta_1(\omega))+g(r,\eta_2(\omega),\eta_1(\omega))\Big]d\omega+h(r),r\in[\zeta_1,\zeta_2].$$

Then system of integral equations 3.1 has a solution in $C([\zeta_1,\zeta_2],\mathbb{R})$.

Proof. Define $\xi:C([\zeta_1,\zeta_2],\mathbb{R})\times C([\zeta_1,\zeta_2],\mathbb{R})\to C([\zeta_1,\zeta_2],\mathbb{R})$ by,

$$\left.\begin{aligned}\xi(\alpha,\beta)(r)=\int_{\zeta_1}^{\zeta_2}\xi_1(r,\omega)\Big[f(\omega,\alpha(\omega),\beta(\omega))+g(\omega,\alpha(\omega),\beta(\omega))\Big]d\omega\\+\int_{\zeta_1}^{\zeta_2}\xi_2(r,\omega)\Big[f(\omega,\alpha(\omega))+g(r,\alpha(\omega),\beta(\omega))\Big]d\omega+h(r),r\in[\zeta_1,\zeta_2].\end{aligned}\right\}\tag{3.3}$$

Now to show that ξ has mixed monotone property. For this if $\alpha_1\leq\alpha_2$ we have

$$\begin{aligned}\xi(\alpha_1,\beta)-\xi(\alpha_2,\beta)&=\int_{\zeta_1}^{\zeta_2}\xi_1(r,\omega)\Big[f(\omega,\alpha_1(\omega),\beta(\omega))+g(\omega,\alpha_1(\omega),\beta(\omega))\Big]d\omega\\&+\int_{\zeta_1}^{\zeta_2}\xi_2(r,\omega)\Big[f(\omega,\alpha_1(\omega),\beta(\omega))+g(\omega,\alpha_1(\omega),\beta(\omega))\Big]d\omega+h(r)\\&-\int_{\zeta_1}^{\zeta_2}\xi_1(r,\omega)\Big[f(\omega,\alpha_2(\omega),\beta(\omega))+g(\omega,\alpha_2(\omega),\beta(\omega))\Big]d\omega\\&-\int_{\zeta_1}^{\zeta_2}\xi_2(r,\omega)\Big[f(\omega,\alpha_2(\omega),\beta(\omega))+g(\omega,\alpha_2(\omega),\beta(\omega))\Big]d\omega-h(r)\\&=\Big[\int_{\zeta_1}^{\zeta_2}(\xi_1(r,\omega)\Big[\big(f(\omega,\alpha_1(\omega),\beta(\omega))-f(\omega,\alpha_2(\omega),\beta(\omega))\big)+\big(g(\omega,\alpha_1(\omega),\beta(\omega))-g(\omega,\alpha_2(\omega),\beta(\omega))\big)\Big]d\omega\\&+\int_{\zeta_1}^{\zeta_2}\xi_2(r,\omega)\Big[\big(f(\omega,\alpha_1(\omega),\beta(\omega))-f(\omega,\alpha_2(\omega),\beta(\omega))\big)+\big(g(\omega,\alpha_1(\omega),\beta(\omega))-g(\omega,\alpha_2(\omega),\beta(\omega))\big)\Big]d\omega\Big]\leq 0.\end{aligned}$$

By using given assumptions we get

$$\xi(\alpha_1,\beta(t))\leq\xi(\alpha_2,\beta(t)),\forall s\in I.$$

Similarly, if $\beta_1 \geq \beta_2$

$$\xi(\alpha,\beta_1)-\xi(\alpha,\beta_2)=\int_{\zeta_1}^{\zeta_2}\xi_1(r,\omega)\Big[f(\omega,\alpha(\omega),\beta_1(\omega))+g(\omega,\alpha(\omega),\beta_1(\omega))\Big]d\omega$$
$$+\int_{\zeta_1}^{\zeta_2}\xi_2(r,\omega)\Big[f(\omega,\alpha(\omega),\beta_1(\omega))+g(\omega,\alpha(\omega),\beta_1(\omega))\Big]d\omega+h(r)$$
$$-\int_{\zeta_1}^{\zeta_2}\xi_1(r,\omega)\Big[f(\omega,\alpha(\omega),\beta_2(\omega))+g(\omega,\alpha(\omega),\beta_2(\omega))\Big]d\omega$$
$$-\int_{\zeta_1}^{\zeta_2}\xi_2(r,\omega)\Big[f(\omega,\alpha(\omega),\beta_2(\omega))+g(\omega,\alpha(\omega),\beta_2(\omega))\Big]d\omega-h(r).$$
$$=\int_{\zeta_1}^{\zeta_2}(\xi_1(r,\omega)\Big[\big(f(\omega,\alpha(\omega),\beta_1(\omega))-f(\omega,\alpha(\omega),\beta_2(\omega))\big)+\big(g(\omega,\alpha(\omega),\beta_1(\omega))-g(\omega,\alpha(\omega),\beta_2(\omega))\big)\Big]d\omega$$
$$+\int_{\zeta_1}^{\zeta_2}\xi_2(r,\omega)\Big[\big(f(\omega,\alpha(\omega),\beta_1(\omega))-f(\omega,\alpha(\omega),\beta_2(\omega))\big)+\big(g(\omega,\alpha(\omega),\beta_1(\omega))-g(\omega,\alpha(\omega),\beta_2(\omega))\big)\Big]d\omega\leq 0.$$

By using given assumptions we obtained

$$\xi(\alpha,\beta_1(t))\leq\xi(\alpha,\beta_2(t)),\forall\omega\in I.$$

Now using assumption of equation (3.2) we have

$$\varrho(\xi(\alpha,\beta)-\xi(\gamma,\gamma))=\sup_{t\in[\zeta_1,\zeta_2]}\Bigg|\int_{\zeta_1}^{\zeta_2}\xi_1(r,\omega)\Big[f(\omega,\alpha(\omega),\beta(\omega))+g(\omega,\alpha(\omega),\beta(\omega))\Big]d\omega$$
$$+\int_{\zeta_1}^{\zeta_2}\xi_2(r,\omega)\Big[f(\omega,\alpha(\omega),\beta(\omega))+g(\omega,\alpha(\omega)),\beta(\omega)\Big]d\omega+h(r)$$
$$-\int_{\zeta_1}^{\zeta_2}\xi_1(r,\omega)\Big[f(\omega,\gamma(\omega),\gamma(\omega))+g(\omega,\alpha(\omega)),\beta(\omega)\Big]d\omega$$
$$-\int_{\zeta_1}^{\zeta_2}\xi_2(r,\omega)\Big[f(\omega,\gamma(\omega),\gamma(\omega))+g(\omega,\gamma(\omega),\gamma(\omega))\Big]d\omega-h(r)\Bigg|.$$

$$=\sup_{t\in[\zeta_1,\zeta_2]}\Bigg|\int_{\zeta_1}^{\zeta_2}\xi_1(r,\omega)\Big[\big(f(\omega,\alpha(\omega),\beta(\omega))-f(\omega,\gamma(\omega),\gamma(\omega))\big)+\big(g(\omega,\alpha(\omega),\beta(\omega))-g(\omega,\gamma(\omega),\gamma(\omega))\big)\Big]d\omega$$
$$+\int_{\zeta_1}^{\zeta_2}\xi_2(r,\omega)\Big[\big(f(\omega,\alpha(\omega),\beta(\omega))-f(\omega,\gamma(\omega),\gamma(\omega))\big)+\big(g(\omega,\alpha(\omega),\beta(\omega))-g(\omega,\gamma(\omega),\gamma(\omega))\big)\Big]d\omega\Bigg|.$$

$$\leq\sup_{t\in[\zeta_1,\zeta_2]}\Bigg|\int_{\zeta_1}^{\zeta_2}\xi_1(r,\omega)\Big[\lambda_1\varphi\Big(\frac{\varrho(\alpha,\gamma)+\varrho(\beta,\gamma)}{2}\Big)\Big(\frac{\varrho(\alpha,\gamma)+\varrho(\beta,\gamma)}{2}\Big)$$
$$+\lambda_2\varphi\Big(\frac{\varrho(\alpha,\gamma)+\varrho(\beta,\gamma)}{2}\Big)\Big(\frac{\varrho(\alpha,\gamma)+\varrho(\beta,\gamma)}{2}\Big)\Big]d\omega$$
$$+\int_{\zeta_1}^{\zeta_2}\xi_2(r,\omega)\Big[\lambda_1\varphi\Big(\frac{\varrho(\alpha,\gamma)+\varrho(\beta,\gamma)}{2}\Big)\Big(\frac{\varrho(\alpha,\gamma)+\varrho(\beta,\gamma)}{2}\Big)$$
$$+\lambda_2\varphi\Big(\frac{\varrho(\alpha,\gamma)+\varrho(\beta,\gamma)}{2}\Big)\Big(\frac{\varrho(\alpha,\gamma)+\varrho(\beta,\gamma)}{2}\Big)\Big]d\omega\Bigg|.$$

$$\leq 2\max\{\lambda_1,\lambda_2\}\sup_{t\in[\zeta_1,\zeta_2]}\left|\int_{\zeta_1}^{\zeta_2}((\xi_1(r,\omega)+\xi_2(r,\omega))\varphi\Big(\frac{\varrho(\alpha,\gamma)+\varrho(\beta,\gamma)}{2}\Big)\Big(\frac{\varrho(\alpha,\gamma)+\varrho(\beta,\gamma)}{2}\Big)d\omega\right|$$

$$=2\max\{\lambda_1,\lambda_2\}\sup_{t\in[\zeta_1,\zeta_2]}\left|\int_{\zeta_1}^{\zeta_2}(\xi_1(r,\omega)+\xi_2(r,\omega))d\omega\right|\varphi\Big(\frac{\varrho(\alpha,\gamma)+\varrho(\beta,\gamma)}{2}\Big)\Big(\frac{\varrho(\alpha,\gamma)+\varrho(\beta,\gamma)}{2}\Big).$$

By using assumptions (4) and (5), which implies that

$$\varrho(\xi(\alpha,\beta)-\xi(\gamma,\gamma))\leq\varphi\Big(\frac{\varrho(\alpha,\gamma)+\varrho(\beta,\gamma)}{2}\Big)\Big(\frac{\varrho(\alpha,\gamma)+\varrho(\beta,\gamma)}{2}\Big).$$

It is easy to verify condition $[(ii)]$ of Theorem 2.1. Let (η_1,η_2) be coupled upper and lower solution of the integral equation then, by assumption (6) we have $\eta_1(r)\leq\xi(\eta_1,\eta_2)$ and $\xi(\eta_2,\eta_1)\leq\eta_2(r)$. Since $\eta_1\leq\eta_2$. By defining $\varsigma:\Lambda\to\Lambda$ by $\varsigma(x)=x$. By Theorem 2.4 and proposition 1.6 there exist solutions of the system 3.1 in $C([\zeta_1,\zeta_2],\mathbb{R})$. □

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