Journal of Mathematical Analysis
ISSN: 2217-3412, URL: www.ilirias.com/jma
Volume 8 Issue 5 (2017), Pages 1-11.

TIMELIKE DIRECTIONAL TUBULAR SURFACES

CUMALI EKICI, HATICE TOZAK, MUSTAFA DEDE

Abstract. This study aimed to introduce a new version of timelike tubular surfaces in E_1^3. Firstly, we define a new adapted frame along a spacelike space curve, and denote this the q-frame. We then derive a relationship between the Frenet frame and the q-frame. Finally, we obtain a parametric representation of a timelike directional tubular surface.

1. Introduction

The offset at distance r to a curve $m(t)$ in 3-space can be defined as the envelope of the set of spheres with radius r which are centered at $m(t)$. Such a surface is called a pipe surface or tubular surface with spine curve $m(t)$ [16, 14]. Blaga (2005) describes a method of parameterization of the tubular surface which takes the parameter along the generating curve to be one of the parameters and denote by ψ the position vector of a point on the surface. This point lies on a circle of radius r, situated in the normal plane to the generating curve at a point s, with the center at the point $\alpha(s)$ from the curve. Denote by $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ the moving Frenet frame along the unit speed curve $\alpha(s)$. Assume that ρ denotes by the vector connecting the point from the curve with the point from the surface. Then, we have

$$\psi = \alpha(s) + \rho. \tag{1.1}$$

Let us denote by θ the angle between the vector $\mathbf{n}(s)$ and the vector ρ that lies in the normal plane. Then, as one can see immediately, we have

$$\rho = r(\mathbf{n}(s)\cos\theta + \mathbf{b}(s)\sin\theta). \tag{1.2}$$

Combining (1.1) and (1.2), we see that we obtained a parameterization of the tubular surface,

$$\psi(s,\theta) = r(s) + r\mathbf{n}(s)\cos\theta + r\mathbf{b}(s)\sin\theta.$$

A tubular surface often can be parameterized using the Frenet frame of a space curve [6]. However, various alternative methods have been proposed for computing the tubular surfaces such as by using the Darboux frame [9]. Tubular surfaces have also been studied in other ambient spaces [1, 7, 11, 12]. In addition, singularities of parallel surface and directional tubes are investigated in [2, 10].

2000 *Mathematics Subject Classification.* 53A04, 53A05.
Key words and phrases. Frenet frame, timelike pipe surface, Minkowski space, adapted frame.

Submitted April 10,2017. Published August 28, 2017.
Communicated by Siraj Uddin.

Bishop showed that we can define more frames along a space curve [3]. After that, Yilmaz and Turgut introduced the second type of Bishop frame [17]. Recently, Dede et al. [8] introduced the directional q-frame of a regular curve $\alpha(t)$ as follows

$$\mathbf{t} = \frac{\alpha'}{\|\alpha'\|}, \mathbf{n}_q = \frac{\mathbf{t} \wedge \mathbf{k}}{\|\mathbf{t} \wedge \mathbf{k}\|}, \mathbf{b}_q = \mathbf{t} \wedge \mathbf{n}_q \tag{1.3}$$

where $\mathbf{k}$ is the projection vector.

The variation equations of the directional q-frame are given by

$$\begin{bmatrix} \mathbf{t}' \\ \mathbf{n}_q' \\ \mathbf{b}_q' \end{bmatrix} = \|\alpha'\| \begin{bmatrix} 0 & k_1 & k_2 \\ -k_1 & 0 & k_3 \\ -k_2 & -k_3 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n}_q \\ \mathbf{b}_q \end{bmatrix} \tag{1.4}$$

where the q-curvatures are expressed as follows

$$k_1 = \frac{\langle \mathbf{t}', \mathbf{n}_q \rangle}{\|\alpha'\|}, k_2 = \frac{\langle \mathbf{t}', \mathbf{b}_q \rangle}{\|\alpha'\|}, k_3 = -\frac{\langle \mathbf{n}_q, \mathbf{b}_q' \rangle}{\|\alpha'\|} \tag{1.5}$$

In the three dimensional Minkowski space $\mathbb{R}_1^3$, the inner product of two vectors $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3) \in \mathbb{R}_1^3$ is defined as

$$< \mathbf{u}, \mathbf{v} >= u_1 v_1 + u_2 v_2 - u_3 v_3 \tag{1.6}$$

The norm of the vector $\mathbf{u}$ is given by

$$\|\mathbf{u}\| = \sqrt{|\langle u, u \rangle|} \tag{1.7}$$

We say that a Lorentzian vector $\mathbf{u}$ is spacelike, lightlike or timelike if $\langle \mathbf{u}, \mathbf{u} \rangle > 0$ and $\mathbf{u} = 0$, $\langle \mathbf{u}, \mathbf{u} \rangle = 0$, $\langle \mathbf{u}, \mathbf{u} \rangle < 0$, respectively [5].

Then Frenet formulas of spacelike curve may be written as

$$\frac{d}{dt} \begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix} = \begin{bmatrix} 0 & \kappa v & 0 \\ -\epsilon \kappa v & 0 & \tau v \\ 0 & \tau v & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix} \tag{1.8}$$

where $< \mathbf{n}, \mathbf{n} >= \epsilon$, $\epsilon = \pm 1$. The Minkowski curvature and torsion of spacelike curve $\alpha(t)$ are obtained by, respectively

$$\kappa = \|\mathbf{t}'\|, \tau = - < \mathbf{n}', \mathbf{b} > \tag{1.9}$$

Furthermore, for a timelike curve $\alpha(t)$, the following Frenet formulas are given:

$$\frac{d}{dt} \begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix} = \begin{bmatrix} 0 & \kappa v & 0 \\ \kappa v & 0 & \tau v \\ 0 & -\tau v & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix} \tag{1.10}$$

The Minkowski curvature and torsion of timelike curve are calculated by, respectively

$$\kappa = \|\mathbf{t}'\|, \tau =< \mathbf{n}', \mathbf{b} > \tag{1.11}$$

The Gauss and mean curvatures of a timelike surface are given by

$$K = \frac{LN - M^2}{EG - F^2}, 2H = \frac{LG - 2MF + NE}{EG - F^2}. \tag{1.12}$$

where E, F, G and L, M, N are the coefficients of first and second fundamental forms of a surface, respectively [13, 15].

2. Timelike D-Tubular Surfaces

In this section, we introduce a new form of timelike tubular surface, and call this surface a timelike directional tubular surface, or timelike D-tubular surface for short.

Let the center spacelike curve $\alpha(s)$ with a spacelike quasi-binormal be on the timelike D-tubular surface ψ and denote by v the angle between the vector $\mathbf{n}_q(s)$ (timelike) and the vector ρ (timelike) that lies in the normal plane. Then, we have

$$\rho = r(\sinh v\mathbf{n}_q + \cosh v\mathbf{b}_q). \tag{2.1}$$

Combining (1.1) and (2.1), we see that we obtained a parameterization of timelike D-tubular surface at a distance r with q-frame as

$$\psi^r(s,v) = \alpha(s) + r(\sinh v\mathbf{n}_q + \cosh v\mathbf{b}_q). \tag{2.2}$$

Let the center spacelike curve $\alpha(s)$ with a timelike quasi-binormal be on the timelike D-tubular surface ψ and denote by v the angle between the vector $\mathbf{b}_q(s)$ (timelike) and the vector ρ (timelike) that lies in the normal plane. Then, we have

$$\rho = r(\cosh v\mathbf{n}_q + \sinh v\mathbf{b}_q). \tag{2.3}$$

Combining (1.1) and (2.3), we see that we obtained a parameterization of timelike D-tubular surface at a distance r with q-frame as

$$\psi^r(s,v) = \alpha(s) + r(\cosh v\mathbf{n}_q - \sinh v\mathbf{b}_q). \tag{2.4}$$

Let the center timelike curve $\alpha(s)$ on the timelike D-tubular surface ψ and denote by v the angle between the vector $\mathbf{n}_q(s)$ (spacelike) and the vector ρ (spacelike) that lies in the normal plane. Then, we have

$$\rho = r(\cos v\mathbf{n}_q + \sin v\mathbf{b}_q). \tag{2.5}$$

Combining (1.1) and (2.5), we see that we obtained a parameterization of timelike D-tubular surface at a distance r with q-frame as

$$\psi^r(s,v) = \alpha(s) + r(\cos v\mathbf{n}_q + \sin v\mathbf{b}_q). \tag{2.6}$$

2.1. Timelike D-tubular Surface Around A Spacelike Curve.

In this section, we define the q-frame along a spacelike curve with projection vector $\mathbf{k}$.

Case i) Let $\alpha(s)$ be a spacelike curve that is parameterized by arc length s with timelike normal vector $\mathbf{n}$ and spacelike projection vector $\mathbf{k} = (0,1,0)$. Then the q-frame $\{\mathbf{t}, \mathbf{n}_q, \mathbf{b}_q, \mathbf{k}\}$ is given by

$$\mathbf{t} = \frac{\alpha'}{\|\alpha'\|}, \mathbf{n}_q = \frac{\mathbf{t}\wedge\mathbf{k}}{\|\mathbf{t}\wedge\mathbf{k}\|}, \mathbf{b}_q = \mathbf{t}\wedge\mathbf{n}_q \tag{2.7}$$

Since the curve is a spacelike curve, we have the unit tangent vector $\mathbf{t}$ (spacelike). Moreover, the cross product of $\mathbf{t}$ and $\mathbf{k}$ (spacelike) is a timelike vector, therefore the quasi-normal $\mathbf{n}_q$ is a timelike and the quasi-binormal vector $\mathbf{b}_q$ is a spacelike vector.

It follows that we can define the pseudo-Euclidean angle θ between vectors: the principal timelike normal $\mathbf{n}$ and timelike quasi-normal $\mathbf{n}_q$. Then, as one can see immediately, the relation matrix may be expressed as

$$\begin{bmatrix} \mathbf{t} \\ \mathbf{n}_q \\ \mathbf{b}_q \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh\theta & \sinh\theta \\ 0 & \sinh\theta & \cosh\theta \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix}. \tag{2.8}$$

$$\begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh\theta & -\sinh\theta \\ 0 & -\sinh\theta & \cosh\theta \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n}_q \\ \mathbf{b}_q \end{bmatrix}. \tag{2.9}$$

Differentiating (2.8) with respect to s, then substituting (2.9) and (1.8) into the results gives the variation equations of the q-frame in the following form

$$\begin{bmatrix} \mathbf{t}' \\ \mathbf{n}_q' \\ \mathbf{b}_q' \end{bmatrix} = \begin{bmatrix} 0 & -k_1 & k_2 \\ -k_1 & 0 & k_3 \\ -k_2 & k_3 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n}_q \\ \mathbf{b}_q \end{bmatrix} \tag{2.10}$$

where the q-curvatures are

$$\begin{aligned} k_1 &= \kappa\cosh\theta \\ k_2 &= -\kappa\sinh\theta \\ k_3 &= d\theta + \tau. \end{aligned} \tag{2.11}$$

Timelike directional tubular surface of type I:

It follows that the timelike D-tubular surface, at a distance r from the spine curve $\alpha(s)$, may be represented as

$$\psi^r(s,v) = \alpha(s) + r(\sinh v\mathbf{n}_q + \cosh v\mathbf{b}_q)$$

From (2.2), the partial derivatives of timelike directional tubular surface $\psi^r(s,v)$, with respect to s and v, are determined by

$$\psi^r_s = (1 - r(k_1\sinh v + k_2\cosh v))\mathbf{t} + rk_3\cosh v\mathbf{n}_q + rk_3\sinh v\mathbf{b}_q \tag{2.12}$$

and

$$\psi^r_v = r(\cosh v\mathbf{n_q} + \sinh v\mathbf{b}_q). \tag{2.13}$$

It follows that the unit normal vector of the timelike D-tubular surface of type I is

$$U = \pm(\sinh v\mathbf{n}_q - \cosh v\mathbf{b}_q). \tag{2.14}$$

Note that $\langle U, U\rangle = 1$ means that the D-tubular surface of type I is a timelike surface with a spacelike normal vector.

Corollary 2.1. It is well know that the points where $\psi^r_s \wedge \psi^r_v = 0$ are singular if and only if the singular points of the timelike D-tubular surface of type I can be obtained by

$$(1 - r(k_1\sinh v + k_2\cosh v)) = 0. \tag{2.15}$$

In this situation, if ψ^r_s vector is on the normal plane which spaned by n_q and b_q, then all points on the surface are singular. The surface $\psi^r(s,v)$ is a characterization for the singular points which are on the D-tubular surface $\psi^r(s,v)$.

Theorem 2.1. The Gaussian and mean curvatures of the timelike D-tubular surface of type I are given by

$$K = \frac{(k_1\sinh v + k_2\cosh v)}{r(1 - r(k_1\sinh v + k_2\cosh v))} \tag{2.16}$$

and

$$2H = \pm\frac{2r(k_1 \sinh v + k_2 \cosh v) + 1}{r(1 - r(k_1 \sinh v + k_2 \cosh v))}. \tag{2.17}$$

Proof: From (2.12) and (2.13), the coefficients $E = \langle \psi_s^r, \psi_s^r \rangle$, $F = \langle \psi_s^r, \psi_v^r \rangle$ and $G = \langle \psi_v^r, \psi_v^r \rangle$ of the first fundamental form are calculated by

$$E = (1 - r(k_1 \sinh v + k_2 \cosh v))^2 - r^2 k_3^2 \tag{2.18}$$

and

$$F = -r^2 k_3, G = -r^2. \tag{2.19}$$

By using $L = \langle \psi_{ss}^r, U \rangle$, $M = \langle \psi_{sv}^r, U \rangle$ and $N = \langle \psi_{vv}^r, U \rangle$, the coefficients of second fundamental form are obtained as

$$L = \pm \left[(k_1 \sinh v + k_2 \cosh v)(1 - r(k_1 \sinh v + k_2 \cosh v) + r k_3^2\right] \tag{2.20}$$

and

$$M = \pm r k_3, N = \pm r. \tag{2.21}$$

In E_1^3, the Gauss and mean curvatures of a surface are given by

$$K = \frac{LN - M^2}{EG - F^2}, 2H = \frac{LG - 2MF + NE}{EG - F^2}. \tag{2.22}$$

Combining (2.18)-(2.22), we obtain the Gauss and mean curvatures of the timelike D-tubular of type I as follows

$$K = \frac{(k_1 \sinh v + k_2 \cosh v)}{r(1 + r(k_1 \sinh v + k_2 \cosh v))} \tag{2.23}$$

and

$$2H = \mp\frac{2r(k_1 \sinh v + k_2 \cosh v) - 1}{r(1 + r(k_1 \sinh v + k_2 \cosh v))},$$

respectively.

Theorem 2.2. A Timelike D-tubular surface of type I is flat if and only if the q-curvatures of spacelike curve satisfies the following equation

$$\frac{k_1}{k_2} = -\coth v. \tag{2.24}$$

Proof: It is well known that a surface is flat if and only if the Gauss curvature vanishes. Therefore from (2.23) we have $k_1 \sinh v + k_2 \cosh v = 0$. It follows that

$$\frac{k_1}{k_2} = -\coth v.$$

Theorem 2.3. In E_1^3, timelike D-tubular surface of type I is a Weingarten surface.

Proof: A surface is a Weingarten surface if and only if its curvatures satisfies the following Jacobi equation

$$\Phi(H, K) = K_v H_s - H_v K_s = 0. \tag{2.25}$$

A straightforward computation shows that differentiating (2.16) and (2.17) with respect to s and v then, substituting results into (2.25) gives that timelike D-tubular surface is a Weingarten surface.

Case ii) Let $\alpha(s)$ be a spacelike curve that is parameterized by arc length s with spacelike normal vector $\mathbf{n}$ and spacelike projection vector $\mathbf{k} = (0, 1, 0)$.

Then we have the tangent $\mathbf{t}$ (spacelike), the quasi-normal $\mathbf{n}_q$ (spacelike) and the quasi-binormal vector $\mathbf{b}_q$ (timelike).

We can define the pseudo-Euclidean angle θ between the principal normal $\mathbf{b}$ (timelike) and quasi-normal $\mathbf{b}_q$ (timelike) vectors.

Differentiating the relation matrix (2.8) with respect to s, then substituting (2.9) and (1.8) into the results gives the variation equations of the q-frame in the following form

$$\begin{bmatrix} \mathbf{t}' \\ \mathbf{n}_q' \\ \mathbf{b}_q' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & -k_2 \\ -k_1 & 0 & -k_3 \\ -k_2 & -k_3 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n}_q \\ \mathbf{b}_q \end{bmatrix} \tag{2.26}$$

where the q-curvatures are

$$\begin{aligned} k_1 &= \kappa \cosh \theta \\ k_2 &= -\kappa \sinh \theta \\ k_3 &= -d\theta - \tau. \end{aligned} \tag{2.27}$$

Timelike directional tubular surface of type II:

It follows that the timelike D-tubular surface, at a distance r from the spine curve $\alpha(s)$, may be represented as

$$\psi^r(s,v) = \alpha(s) + r(\cosh v \mathbf{n}_q - \sinh v \mathbf{b}_q).$$

From (2.4), the partial derivatives of $\psi^r(s,v)$, with respect to s and v, are determined by

$$\psi_s^r = (1 - r(k_1 \cosh v - k_2 \sinh v))\mathbf{t} + r k_3 \cosh v \mathbf{n}_q - r k_3 \sinh v \mathbf{b}_q \tag{2.28}$$

and

$$\psi_v^r = r(\sinh v \mathbf{n}_q - \cosh v \mathbf{b}_q). \tag{2.29}$$

It follows that we obtain the unit normal vector of the timelike D-tubular surface of type II as

$$U = \pm(-\cosh v \mathbf{n_q} + \sinh v \mathbf{b}_q). \tag{2.30}$$

Observe that $\langle U, U \rangle = 1$ means that the D-tubular surface of type II is a timelike surface with a spacelike normal vector.

Corollary 2.2. It is well know that the points where $\psi_s^r \wedge \psi_v^r = 0$ are singular if and only if the singular points of the timelike D-tubular surface of type II can be obtained by

$$(1 - r(k_1 \cosh v - k_2 \sinh v)) = 0. \tag{2.31}$$

In this situation, if ψ_s^r vector is on the normal plane which spaned by n_q and b_q, then all points on the surface are singular. The surface $\psi^r(s,v)$ is a characterization for the singular points which are on the D-tubular surface $\psi^r(s,v)$.

Similar to the type I, from (2.28), (2.29) and (2.22), we can easily obtain the Gaussian and mean curvatures of the timelike D-tubular surface of type II as follows

$$K = \frac{-(k_1 \cosh v - k_2 \sinh v)}{r(1 - r(k_1 \cosh v - k_2 \sinh v))} \tag{2.32}$$

and

$$2H = \mp \frac{2r(k_1 \cosh v - k_2 \sinh v) - 1}{r(1 - r(k_1 \cosh v - k_2 \sinh v))}, \tag{2.33}$$

respectively.

Theorem 2.4. A timelike D-tubular surface of type II is flat if and only if its Gaussian curvature is zero. Then, we have

$$\frac{k_1}{k_2} = \tanh v. \tag{2.34}$$

Proof: Similar to proof of Theorem 2.2.

Theorem 2.5. Every timelike D-tubular surface of type II is a Weingarten surface.

Proof: It is well known that Weingarten surface satisfies identically the Jacobi equation (2.25). Thus, differentiating (2.32) and (2.33) with respect to s and v then, substituting the results into Jacobi equation leads that timelike D-tubular surface is a Weingarten surface.

2.2. **Timelike D-tubular Surface Around A Timelike Curve.** In this section, we define the q-frame along a timelike curve with projection vector $\mathbf{k}$.

Let $\alpha(s)$ be a timelike curve that is parameterized by arc length s with spacelike normal vector $\mathbf{n}$ and spacelike projection vector $\mathbf{k} = (0, 1, 0)$. Since the curve is a timelike curve, we have the unit tangent vector $\mathbf{t}$ (timelike). Moreover, the cross product of $\mathbf{t}$ and $\mathbf{k}$ (spacelike) is a spacelike vector, therefore the quasi-normal $\mathbf{n}_q$ is a spacelike and the quasi-binormal vector $\mathbf{b}_q$ is a spacelike vector.

We can define the pseudo-Euclidean angle θ between the principal binormal $\mathbf{n}$ (spacelike) and quasi-binormal $\mathbf{n}_q$ vectors. Then, as one can see immediately, the relation matrix may be expressed as

$$\begin{bmatrix} \mathbf{t} \\ \mathbf{n}_q \\ \mathbf{b}_q \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix}. \tag{2.35}$$

Thus,

$$\begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n}_q \\ \mathbf{b}_q \end{bmatrix}. \tag{2.36}$$

Differentiating (2.35) with respect to s, then substituting (2.36) and (1.8) into the results gives the variation equations of the q-frame in the following form

$$\begin{bmatrix} \mathbf{t}' \\ \mathbf{n}_q' \\ \mathbf{b}_q' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 \\ k_1 & 0 & k_3 \\ k_2 & -k_3 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n}_q \\ \mathbf{b}_q \end{bmatrix}, \tag{2.37}$$

where the q-curvatures are

$$\begin{aligned} k_1 &= \kappa\cos\theta \\ k_2 &= -\kappa\sin\theta \\ k_3 &= d\theta + \tau. \end{aligned} \tag{2.38}$$

It follows that the timelike D-tubular surface, at a distance r from the spine curve $\alpha(s)$, may be represented as

$$\psi^r(s, v) = \alpha(s) + r(\cos v\mathbf{n}_q + \sin v\mathbf{b}_q).$$

The partial derivatives of $\psi^r(s, v)$, with respect to s and v, are determined by

$$\psi_s^r = (1 + r(k_1\cos v + k_2\sin v))\mathbf{t} - rk_3\,(\sin v\mathbf{n}_q + \cos v\mathbf{b}_q) \tag{2.39}$$

and

$$\psi_v^r = r(-\sin v\mathbf{n_q} + \cos v\mathbf{b}_q). \tag{2.40}$$

It follows that the unit normal vector of the timelike D-tubular surface of type III is

$$U = \pm(\cos v\mathbf{n}_q + \sin v\mathbf{b}_q). \tag{2.41}$$

Note that $\langle U, U\rangle = 1$ means that the D-tubular surface of type III is a timelike surface with a spacelike normal vector.

Corollary 2.3. It is well know that the points where $\psi_s^r \wedge \psi_v^r = 0$ are singular if and only if the singular points of the timelike D-tubular surface of type III can be obtained by

$$(1 + r(k_1 \cos v + k_2 \sin v)) = 0. \tag{2.42}$$

We can then state the following theorem:

Theorem 2.6. The Gaussian and mean curvatures of the timelike D-tubular surface of type III are given by

$$K = \frac{k_1 \cos v + k_2 \sin v}{r(1 + r(k_1 \cos v + k_2 \sin v))} \tag{2.43}$$

and

$$2H = \mp\frac{2r(k_1 \cos v + k_2 \sin v) + 1}{r(1 + r(k_1 \cos v + k_2 \sin v)))}, \tag{2.44}$$

respectively.

Proof: From (2.39) and (2.40), the components $E = \langle \psi_s^r, \psi_s^r\rangle$, $F = \langle \psi_s^r, \psi_v^r\rangle$ and $G = \langle \psi_v^r, \psi_v^r\rangle$ of the first fundamental form are obtained by

$$E = -(1 - r(k_1 \cos v + k_2 \sin v)))^2 + r^2k_3^2 \tag{2.45}$$

and

$$F = r^2k_3, G = r^2. \tag{2.46}$$

Similarly we can derive the components $L = \langle \psi_{ss}^r, U\rangle$, $M = \langle \psi_{sv}^r, U\rangle$ and $N = \langle \psi_{vv}^r, U\rangle$ of the second fundamental form as

$$L = \pm\left[(k_1 \cos v + k_2 \sin v))(1 + r(k_1 \cos v + k_2 \sin v) - rk_3^2\right] \tag{2.47}$$

and

$$M = \mp rk_3, N = \mp r. \tag{2.48}$$

By substituting (2.45)-(2.48) into (2.22), the Gaussian and mean curvatures of the timelike D-tubular surface of type III surface are obtained by

$$K = \frac{k_1 \cos v + k_2 \sin v}{r(1 - r(k_1 \sinh v - k_2 \cosh v))}$$

and

$$2H = \mp\frac{2r(k_1 \cos v + k_2 \sin v) + 1}{r(1 - r(k_1 \cos v + k_2 \sin v))},$$

respectively.

Corollary 2.4. A timelike D-tubular surface of type III is flat if and only if its Gaussian curvature is zero. Then, we have

$$\frac{k_1}{k_2} = -\tan v.$$

Theorem 2.7. Every timelike D-tubular surface of type III is a Weingarten surface.

Proof: It is well known that Weingarten surface satisfies identically the Jacobi equation $\Phi(H, K) = K_v H_s - H_v K_s = 0$.

Thus, differentiating (2.43) and (2.44) with respect to s and v then, substituting results into Jacobi equation leads that timelike D-tubular surface is a Weingarten surface.

3. Conclusion

We obtained all possible parametrizations for timelike D-tubular surfaces. Therefore we investigated the there types of timelike D-tubular surfaces in this paper. We also investigated some special cases. We showed that all type of timelike D-tubular surfaces are Weingarten surface. As a future work, we will investigate the spacelike D-tubular surfaces.

4. Examples

Example 3.1. Let us consider a spacelike curve (line) parameterized by

$$\alpha(t) = (t, t, 0). \tag{4.1}$$

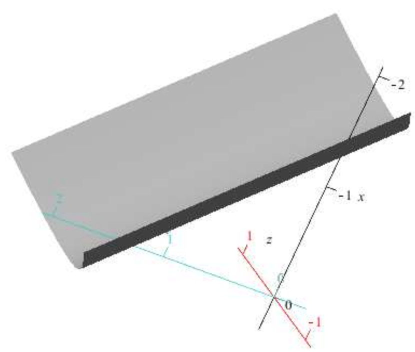

Figure 1.

The timelike D-tubular surface of type I is shown in Figure 1.

It is easy to see that,

$$\mathbf{t} = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0). \tag{4.2}$$

By using $\mathbf{k} = (0, 1, 0)$ and (2.7), the quasi-normal and the quasi-binormal vectors are obtained by

$$\mathbf{n}_q = (0, 0, 1) \tag{4.3}$$

and

$$\mathbf{b}_q = (-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0), \tag{4.4}$$

respectively.

For $r=1$, the timelike D-tubular surface of type I is parameterized by

$$\psi^r(t,v) = (t - \frac{\sqrt{2}}{2}\cosh v, t + \frac{\sqrt{2}}{2}\cosh v, -\sinh v). \tag{4.5}$$

Example 3.2. In E_1^3, assume that a spacelike curve is given by

$$\alpha(t) = (\sinh(t), t, \cosh(t))$$

The q-frame can be calculated by

$$\begin{aligned} \mathbf{t} &= (\frac{\sqrt{2}\cosh(t)}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\sinh(t)), \\ \mathbf{n}_q &= (-\sinh(t), 0, -\cosh(t)), \end{aligned}$$

and

$$\mathbf{b}_q = (-\frac{\sqrt{2}\cosh(t)}{2}, \frac{\sqrt{2}}{2}, \frac{-\sqrt{2}\sinh(t)}{2})$$

For $r=1$, the timelike D-tubular surface of type I is shown in Figure 2.

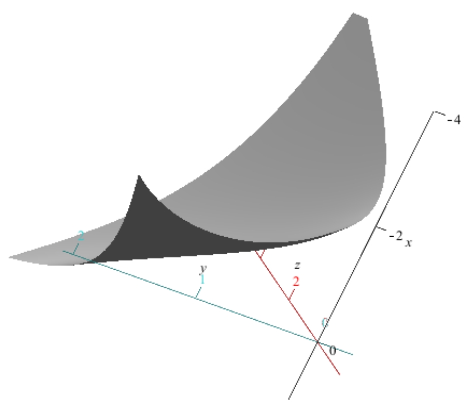

FIGURE 2.

REFERENCES

[1] H. S. Abdel-Aziz, M. K. Saad, *Weingarten timelike Tube surfaces around a spacelike curve*, Int. J. Math. Anal. **5** (2011) 1225-1236.

[2] A. Alghanemi, *On the Singularities of the D-Tubular Surfaces*, J. Math. Anal. **7 6** (2016) 97-104.

[3] R. L. Bishop, *There is more than one way to frame a curve*, Amer. Math. Monthly **82** (1975) 246–251.

[4] P. A. Blaga, *On Tubular surfaces in Computer Graphics*, Studia univ. Babes-Bolyai, Informatica **2** (2005) 81-90.

[5] B. Bukcu, M. K. Karacan, *Bishop frame of the spacelike curve with a spacelike binormal in Minkowski 3-space*, Selçuk J. Applied Math. **11 1** (2010) 15–25.

[6] M. Dede, C. Ekici, H. Tozak, *Directional Tubular Surfaces*, Int. J. Algebra **9** (2015) 527–535.

[7] M. Dede, *Tubular surfaces in Galilean space*, Math. Commun. **18** (2013) 209-217.

[8] M. Dede, C. Ekici, A. Görgülü, *Directional q-frame along a space curve*, IJARCSSE **5 12** (2015) 775-780.

[9] F. Dogan, Y. Yaylı, *Tubes with Darboux Frame*, Int. J. Contemp. Math. Sciences **7** (2012) 751-758.
[10] M. K. Karacan, H. Es, Y. Yaylı, *Singular Points of Tubular Surface in Minkowski 3-Space*, Sarajevo J. Math. **2 14** (2006) 73-82.
[11] S. Kızıltuğ, A. Çakmak, S. Kaya, *Timelike tubes around a spacelike curve with Darboux Frame of Weingarten Type*, Int. J. Pyhsics and Math. Sciences **4** (2013) 9-17.
[12] S. Kızıltuğ, A. Çakmak, *Developable ruled surface with Darboux Frame in Minkowski 3-space*, Life Science J. **10 4** (2013) 1906-1914.
[13] R. Lopez, *Differential geometry of curves and surfaces in Lorentz-Minkowski space*, Mini-Course taught at IME-USP, Brasil, 2008.
[14] T. Maekawa, N. M. Patrikalakis, T. Sakkalis, G. Yu, *Analysis and applications of pipe surfaces*, Comput. Aided Geom. Design **15 5** (1998) 437-458.
[15] B. O'Neill, *Semi-Riemannian Geometry with Applications to Relativity*, Academic Press, 1983.
[16] Z. Xu, R. Feng, J. Sun, *Analytic and Algebraic Properties of Canal Surfaces*, J. Computational and Applied Math. **195** (2006) 220-228.
[17] S. Yılmaz, M. Turgut, *A new version of Bishop frame and an application to spherical images*, J. Math. Anal. Appl. **371** (2010) 764-776.

Mustafa Dede, Department of Mathematics, Faculty of Arts and Sciences Kilis 7 Aralik University, 79000 Kilis, Turkey
E-mail address: `mustafadede@kilis.edu.tr`

Hatice Tozak, Department of Mathematics-Computer, Faculty of Arts and Sciences, Eskişehir Osmangazi University, 26480, Eskişehir-Turkey
E-mail address: `hatice.tozak@gmail.com`

Cumali Ekici, Department of Mathematics-Computer, Faculty of Arts and Sciences, Eskişehir Osmangazi University, 26480, Eskişehir-Turkey
E-mail address: `cekici@ogu.edu.tr`