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ON THE LIMITS OF LOGARITHMIC SUMMABLE FUZZY-NUMBER-VALUED FUNCTIONS AT INFINITY

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Abstract. We introduce logarithmic summability of fuzzy-number-valued functions by using logarithmic averages and establish some Tauberian conditions which guarantee the existence of limits of logarithmic summable fuzzy-number-valued functions at infinity. We also obtained analogous results in case of improper fuzzy Riemann integrals.

1. Introduction

Let s be a $\mathbb{C}-$valued locally integrable function defined on $[1,\infty)$. Then the *logarithmic average* of s is defined by

$$\tau(x) := \frac{1}{\log x}\int_1^x \frac{s(u)}{u}du, \quad x \in (1,\infty).$$

Originated from the discrete version which has been successfully applied to problems occuring in probability theory and summability theory [1–6], *logarithmic averages* has played an important role in handling limit problems in continuous-time. In probability theory, authors have established integral type almost sure limit theorems using logarithmic averages and applied these theorems to different types of processes [7–12]. In case of summability theory, authors [13–15] have recently used the logarithmic averages as a convergence method for functions of real and complex numbers at infinity and introduced the logarithmic summability method of functions. In these studies, they have also investigated the Tauberian conditions under which convergence of improper integrals follows from logarithmic summability and compared the method with well-known Cesàro summability method [16–19].

In this paper, motivated by the studies above in classical analysis, we introduce the logarithmic summability of fuzzy-number-valued functions and gave one-sided Tauberian conditions for logarithmic summability method. We have also obtained slow decrease and Landau type Tauberian results. Moreover, obtained results concerning logarithmic summability of fuzzy-number-valued functions are extended to improper fuzzy Riemann integrals.

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2. Preliminaries and notations

A *fuzzy number* is a fuzzy set on the real axis, i.e. μ is normal, fuzzy convex, upper semi-continuous and $\sup \mu = \overline{\{t \in \mathbb{R} : \mu(t) > 0\}}$ is compact [20]. We denote the space of fuzzy numbers by E^1. *α-level set* $[\mu]_\alpha$ of $\mu \in E^1$ is defined by

$$[\mu]_\alpha := \begin{cases} \{t \in \mathbb{R} : \mu(t) \geq \alpha\} \quad , & if \quad 0 < \alpha \leq 1, \\ \overline{\{t \in \mathbb{R} : \mu(t) > \alpha\}} \quad , & if \quad \alpha = 0. \end{cases}$$

$r \in \mathbb{R}$ can be considered as a fuzzy number $\overline{r}$ defined by

$$\overline{r}(t) := \begin{cases} 1 \quad , & if \quad t = r, \\ 0 \quad , & if \quad t \neq r. \end{cases}$$

Let $\mu, \nu \in E^1$ and $k \in \mathbb{R}$. The addition and scalar multiplication are defined by

$$[\mu + \nu]_\alpha = [\mu]_\alpha + [\nu]_\alpha = [\mu_\alpha^- + \nu_\alpha^-, \mu_\alpha^+ + \nu_\alpha^+], [k\mu]_\alpha = k[\mu]_\alpha$$

where $[\mu]_\alpha = [\mu_\alpha^-, \mu_\alpha^+]$, for all $\alpha \in [0,1]$. Fuzzy number $\overline{0}$ is identity element in $(E^1, +)$ and none of $u \neq \overline{r}$ has inverse in $(E^1, +)$. For any $k_1, k_2 \in \mathbb{R}$ with $k_1 k_2 \geq 0$, distribution property $(k_1 + k_2)u = k_1 u + k_2 u$ holds but for general $k_1, k_2 \in \mathbb{R}$ it fails to hold. On the other hand properties $k(u + v) = ku + kv$ and $k_1(k_2 u) = (k_1 k_2)u$ holds for any $k, k_1, k_2 \in \mathbb{R}$ [21]. It should be noted that E^1 with addition and scalar multiplication defined above is not a linear space over $\mathbb{R}$. The metric D on E^1 is defined as

$$D(u, v) := \sup_{\alpha \in [0,1]} \max\{|u_\alpha^- - v_\alpha^-|, |u_\alpha^+ - v_\alpha^+|\}.$$

Partial ordering on E^1 is defined as follows:

$$\mu \preceq \nu \Longleftrightarrow [\mu]_\alpha \preceq [\nu]_\alpha \Longleftrightarrow \mu_\alpha^- \leq \nu_\alpha^- \quad \text{and} \quad \mu_\alpha^+ \leq \nu_\alpha^+ \quad \text{for all} \quad \alpha \in [0,1].$$

We say a fuzzy number μ is negative if and only if $\mu(t) = 0$ for all $t \geq 0$ (see [22]). In view of Lemma 6 in [23], Lemma 5 in [24], Lemma 3.4, Theorem 4.9 in [25] and Lemma 14 in [26] next Lemma follows.

Lemma 2.1. *Let $\mu, \nu, w, z \in E^1$ and $\varepsilon > 0$. Then following statements hold:*

(i) *$D(\mu, \nu) \leq \varepsilon$ if and only if $\mu - \overline{\varepsilon} \preceq \nu \preceq \mu + \overline{\varepsilon}$*
(ii) *If $\mu \preceq \nu + \overline{\varepsilon}$ for every $\varepsilon > 0$, then $\mu \preceq \nu$.*
(iii) *If $\mu \preceq w$ and $w \preceq \nu$, then $\mu \preceq \nu$.*
(iv) *If $\mu \preceq w$ and $\nu \preceq z$, then $\mu + \nu \preceq w + z$.*
(v) *If $\mu + w \preceq \nu + w$ then $\mu \preceq \nu$.*

Definition 2.2. *A fuzzy-number-valued function $f : [a, b] \to E^1$ is said to be continuous at $x_0 \in [a, b]$ if for each $\varepsilon > 0$ there is a $\delta > 0$ such that $D(f(x), f(x_0)) < \varepsilon$ whenever $x \in [a, b]$ with $|x - x_0| < \delta$. If f is continuous at each $x \in [a, b]$, then we say f is continuous on $[a, b]$.*

Definition 2.3. *A fuzzy-number-valued function $f : [a, b] \to E^1$ is called Riemann integrable on $[a, b]$, if there exists $I \in E^1$ with the property : $\forall \varepsilon > 0$, $\exists \delta > 0$ such that for any division of $[a, b]$ $d : a = x_0 < x_1 < \cdots < x_n = b$ of norm $v(d) < \delta$, and for any points $\xi_i \in [x_i, x_{i+1}]$ $i = \overline{0, n-1}$, we have*

$$D\left(\sum_{i=0}^{n-1} f(\xi_i)(x_{i+1} - x_i), I\right) < \varepsilon.$$

Then $I = \int_a^b f(x)dx$.

Theorem 2.4. *[27] If fuzzy-number-valued function* $f : [a, b] \to E^1$ *is continuous (with respect to the metric D) and for each* $x \in [a, b]$, f *has the parametric representation*

$$[f(x)]_\alpha = [f_\alpha^-(x), f_\alpha^+(x)],$$

then $\int_a^b f(x)dx$ *exists, belongs to* E^1 *and is parametrized by*

$$\left[\int_a^b f(x)dx\right]_\alpha = \left[\int_a^b f_\alpha^-(x)dx, \int_a^b f_\alpha^+(x)dx\right].$$

Using the results of Anastassiou [28] we have

Theorem 2.5. *If* $f : [a, b] \to E^1$ *and* $g : [a, b] \to E^1$ *are continuous then*

(i) $\int_a^b (\alpha f(x) + \beta g(x))dx = \alpha \int_a^b f(x)dx + \beta \int_a^b g(x)dx$ *where* α *and* β *are real numbers.*

(ii) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ *where* $a < c < b$.

(iii) *The function* $F : [a, b] \to \mathbb{R}_+$ *defined by* $F(x) = D(f(x), g(x))$ *is continuous on* $[a, b]$ *and*

$$D\left(\int_a^b f(x)dx, \int_a^b g(x)dx\right) \leq \int_a^b F(x)dx.$$

(iv) $\int_a^x f(u)du$ *is a continuous function in* $x \in [a, b]$.

(v) $\int_a^b f(x)dx \preceq \int_a^b g(x)dx$ *whenever* $f(x) \preceq g(x)$ *for all* $x \in [a, b]$.

3. Logarithmic summability method for fuzzy-number-valued functions

Definition 3.1. *Let* $s : [1, \infty) \to E^1$ *be a continuous fuzzy-number-valued function. The logarithmic average of* s *is defined by*

$$\tau(x) = \frac{1}{\log x} \int_1^x \frac{s(u)}{u} du, \quad x \in (1, \infty). \tag{3.1}$$

Fuzzy-number-valued function s *is said to be logarithmic summable to a fuzzy number* L *if* $\lim_{x\to\infty} \tau(x) = L$.

Theorem 3.2. *If continuous fuzzy-number-valued function* s *converges to a fuzzy number* L *as* $x \to \infty$, *then it is logarithmic summable to* L.

Proof. Let s be a continuous fuzzy-number-valued function and $s(x) \to L$ as $x \to \infty$. Then given any $\varepsilon > 0$ there exists $x_0 > 1$ such that $D(s(x), L) < \frac{\varepsilon}{2}$ whenever

$x \geq x_0$ and $M > 0$ such that $D(s(x), L) < M$ whenever $x < x_0$. So we have

$$\begin{aligned}
D(\tau(x), L) &= D\left(\frac{1}{\log x}\int_1^x \frac{s(u)}{u}du, L\right) \\
&= D\left(\frac{1}{\log x}\int_1^x \frac{s(u)}{u}du, \frac{1}{\log x}\int_1^x \frac{L}{u}du\right) \\
&= \frac{1}{\log x} D\left(\int_1^x \frac{s(u)}{u}du, \int_1^x \frac{L}{u}du\right) \\
&\leq \frac{1}{\log x}\int_1^x \frac{D(s(u), L)}{u}du \\
&= \frac{1}{\log x}\int_1^{x_0} \frac{D(s(u), L)}{u}du + \frac{1}{\log x}\int_{x_0}^y \frac{D(s(u), L)}{u}du \\
&\leq M \cdot \frac{\log x_0}{\log x} + \frac{\varepsilon}{2}\frac{(\log x - \log x_0)}{\log x} < M \cdot \frac{\log x_0}{\log x} + \frac{\varepsilon}{2}
\end{aligned}$$

Taking limit as $x \to \infty$, we get $D(\tau(x), L) \to 0$ and this completes the proof. □

By following example, we see that a logarithmic summable fuzzy-number-valued function need not to converge at infinity.

Example 3.3. *Take fuzzy-number-valued function $s : [1, \infty) \to E^1$ such that*

$$(s(x))(t) = \begin{cases} t - \sin x, & if \quad \sin x \leq t \leq \sin x + 1, \\ -t + \sin x + 2, & if \quad \sin x + 1 \leq t \leq \sin x + 2, \\ 0, & otherwise. \end{cases}$$

Considering (3.1) and Theorem 2.4, we have

$$\begin{aligned}
&s_\alpha^-(x) = \sin x + \alpha \quad , \quad s_\alpha^+(x) = \sin x + 2 - \alpha, \\
&\lim_{x\to\infty} \tau_\alpha^-(x) = \lim_{x\to\infty} \frac{1}{\log x}\int_1^x \frac{s_\alpha^-(u)}{u}du = \alpha, \\
&\lim_{x\to\infty} \tau_\alpha^+(x) = \lim_{x\to\infty} \frac{1}{\log x}\int_1^x \frac{s_\alpha^+(u)}{u}du = 2 - \alpha.
\end{aligned}$$

Since $\lim_{x\to\infty} D(\tau(x), L) = 0$ where $[L]_\alpha = [\alpha, 2-\alpha]$, fuzzy-number-valued function s is logarithmic summable to fuzzy number

$$L(t) = \begin{cases} t & if \quad 0 \leq t \leq 1, \\ 2 - t & if \quad 1 \leq t \leq 2, \\ 0 & otherwise. \end{cases}$$

But $\lim_{x\to\infty} s(x)$ does not exist.

The next lemma is necessary for our main results.

Lemma 3.4. *Let s be a continuous fuzzy-number-valued function. If s is logarithmic summable to a fuzzy number L, then for every $\lambda > 1$*

$$\lim_{x\to\infty} \frac{1}{(\lambda - 1)\log x}\int_x^{x^\lambda} \frac{s(u)}{u}du = L \tag{3.2}$$

and for every $0 < \ell < 1$

$$\lim_{x\to\infty} \frac{1}{(1 - \ell)\log x}\int_{x^\ell}^x \frac{s(u)}{u}du = L. \tag{3.3}$$

Proof. Let consider the facts that

$$\frac{1}{(\lambda-1)\log x}\int_x^{x^\lambda}\frac{s(u)}{u}du+\frac{1}{\lambda-1}\tau(x)=\tau(x^\lambda)+\frac{1}{\lambda-1}\tau(x^\lambda)\quad\text{for}\quad\lambda>1 \tag{3.4}$$

and

$$\frac{1}{(1-\ell)\log x}\int_{x^\ell}^{x}\frac{s(u)}{u}du+\frac{\ell}{1-\ell}\tau(x^\ell)=\tau(x)+\frac{\ell}{1-\ell}\tau(x)\quad\text{for}\quad 0<\ell<1. \tag{3.5}$$

Taking the limit of both sides of (3.4) and (3.5) as $x\to\infty$, we get (3.2) and (3.3) respectively. □

In the following theorem we give necessary and sufficient conditions for a logarithmic summable fuzzy-number-valued function to have a limit at infinity.

Theorem 3.5. *Let s be a continuous fuzzy-number-valued function. If s is logarithmic summable to a fuzzy number L, then $\lim_{x\to\infty}s(x)=L$ if and only if for every $\varepsilon>0$ there exist $x_0>1$ and $\lambda>1$ such that for $x\geq x_0$*

$$\frac{1}{(\lambda-1)\log x}\int_x^{x^\lambda}\frac{s(u)}{u}du\succeq s(x)-\overline{\varepsilon} \tag{3.6}$$

and another $0<\ell<1$ such that

$$\frac{1}{(1-\ell)\log x}\int_{x^\ell}^{x}\frac{s(u)}{u}du\preceq s(x)+\overline{\varepsilon}. \tag{3.7}$$

Proof. Necessity. Let $\lim_{x\to\infty}s(x)=L$. If we consider the inequalities

$$D\left(\frac{1}{(\lambda-1)\log x}\int_x^{x^\lambda}\frac{s(u)}{u}du,s(x)\right)\leq D\left(\frac{1}{(\lambda-1)\log x}\int_x^{x^\lambda}\frac{s(u)}{u}du,L\right)+D(L,s(x))$$

$$D\left(\frac{1}{(1-\ell)\log x}\int_{x^\ell}^{x}\frac{s(u)}{u}du,s(x)\right)\leq D\left(\frac{1}{(1-\ell)\log x}\int_{x^\ell}^{x}\frac{s(u)}{u}du,L\right)+D(L,s(x))$$

and Lemma 3.4 and take limit as $x\to\infty$ then validity of (3.6) and (3.7) is obtained.

Sufficiency. Let fuzzy-number-valued functions s be logarithmic summable to L and (3.6), (3.7) be satisfied.

From (3.6), given $\varepsilon>0$ there exist $x_1>1$ and $\lambda>1$ such that

$$\frac{1}{(\lambda-1)\log x}\int_x^{x^\lambda}\frac{s(u)}{u}du\succeq s(x)-\frac{\overline{\varepsilon}}{3}$$

whenever $x\geq x_1$. Besides, since

$$\lim_{x\to\infty}D\left(\frac{1}{\lambda-1}\tau(x),\frac{1}{\lambda-1}\tau(x^\lambda)\right)=0,$$

there exists $x_2>1$ such that for $x\geq x_2$ we have

$$D\left(\frac{1}{\lambda-1}\tau(x),\frac{1}{\lambda-1}\tau(x^\lambda)\right)\leq\frac{\varepsilon}{3}\cdot$$

So by (i) of Lemma 2.1 we get that

$$\frac{1}{\lambda-1}\tau(x)-\frac{\overline{\varepsilon}}{3}\preceq\frac{1}{\lambda-1}\tau(x^\lambda)\preceq\frac{1}{\lambda-1}\tau(x)+\frac{\overline{\varepsilon}}{3}\cdot$$

Also, since $\lim_{tx\to\infty} \tau(x^\lambda) = L$, there exists $x_3 > 1$ such that $D(\tau(x^\lambda), L) \leq \frac{\varepsilon}{3}$ for $x \geq x_3$, meaning

$$L - \frac{\overline{\varepsilon}}{3} \preceq \tau(x^\lambda) \preceq L + \frac{\overline{\varepsilon}}{3}.$$

Then in view of the equality (3.4) we obtain

$$s(x) - \frac{\overline{\varepsilon}}{3} + \frac{1}{\lambda - 1}\tau(x) \preceq L + \frac{\overline{\varepsilon}}{3} + \frac{1}{\lambda - 1}\tau(x) + \frac{\overline{\varepsilon}}{3}$$

whenever $x \geq x_4$ where $x_4 = \max\{x_1, x_2, x_3\}$. So by (v) of Lemma 2.1 we have

$$s(x) \preceq L + \overline{\varepsilon}. \tag{3.8}$$

Analogously, considering (3.7), (3.5) and Lemma 2.1, there exists $x_4^* > 1$ such that

$$s(x) \succeq L - \overline{\varepsilon} \tag{3.9}$$

for $x \geq x_4^*$. So by (3.8) and (3.9), we have

$$L - \overline{\varepsilon} \preceq s(x) \preceq L + \overline{\varepsilon}$$

whenever $x \geq \max\{x_4, x_4^*\}$, and this completes the proof. □

Definition 3.6. *A fuzzy-number-valued function s is said to be slowly decreasing with respect to logarithmic summability if for every $\varepsilon > 0$ there exist $x_0 > 1$ and $\lambda > 1$ such that*

$$s(u) \succeq s(x) - \overline{\varepsilon}$$

whenever $x_0 \leq x < u \leq x^\lambda$.

Lemma 3.7. *If fuzzy-number-valued function s is slowly decreasing with respect to logarithmic summability, then for every $\varepsilon > 0$ there exist $x_0 > 1$ and $0 < \lambda < 1$ such that for every $x \geq x_0$*

$$s(x) \succeq s(u) - \overline{\varepsilon} \qquad \textit{whenever} \qquad x^\lambda < u \leq x. \tag{3.10}$$

Proof. Assume that fuzzy-number-valued function s is slowly decreasing with respect to logarithmic summability and there exists $\varepsilon_0 > 0$ such that for all $0 < \lambda < 1$ and $x_0 > 1$ there exist real numbers u and $x \geq x_0$ for which

$$s(x) \not\succeq s(u) - \overline{\varepsilon}_0 \qquad \text{whenever} \qquad x^\lambda < u \leq x. \tag{3.11}$$

Therefore, there exists $\alpha_0 \in [0, 1]$ such that

$$s^-_{\alpha_0}(x) < s^-_{\alpha_0}(u) - \varepsilon_0 \quad \textit{or} \quad s^+_{\alpha_0}(x) < s^+_{\alpha_0}(u) - \varepsilon_0.$$

None of the real valued functions $s^-_{\alpha_0}(x)$ and $s^+_{\alpha_0}(x)$ satisfy the following condition given by Moricz [13] for a slowly decreasing real valued function h

$$\lim_{\lambda\to 1^-} \liminf_{x\to\infty} \min_{x^\lambda < u \leq x} [h(x) - h(u)] \geq 0.$$

This contradicts with that s is slowly decreasing and this completes the proof. □

Corollary 3.8. *Let s be a continuous fuzzy-number-valued function. If s is logarithmic summable to a fuzzy number L and slowly decreasing with respect to logarithmic summability, then $\lim_{x\to\infty} s(x) = L$.*

4. Logarithmic summability method in case of improper fuzzy Riemann integrals

Recently, Yavuz *et al.* [29] defined the Cesàro summability of integrals of fuzzy-number-valued functions and given Tauberian conditions under which convergence of improper fuzzy Riemann integrals of fuzzy-number-valued functions follows from Cesàro summability. We may also define logarithmic summability of fuzzy Riemann integrals and give corresponding Tauberian theorems in view of the results concerning logarithmic summability of fuzzy-number-valued functions in Section 3.

Definition 4.1. *Let $f : [1, \infty) \to E^1$ be a continuous fuzzy-number-valued function and $s(x) = \int_1^x f(u)du$. The logarithmic average of s is defined by*

$$\tau(x) = \frac{1}{\log x} \int_1^x \frac{s(u)}{u} du, \quad x \in (1, \infty).$$

The integral

$$\int_1^\infty f(u)du$$

is said to be logarithmic summable to a fuzzy number L if $\lim_{x\to\infty} \tau(x) = L$. The value of this limit is said to be the logarithmic sum of the integral.

Theorem 4.2. *If integral $\int_1^\infty f(u)du$ converges to a fuzzy number L, then it is logarithmic summable to L.*

Theorem 4.3. *Let $f : [1, \infty) \to E^1$ be a continuous fuzzy-number-valued function. If integral $\int_1^\infty f(u)du$ is logarithmic summable to a fuzzy number L, then it converges to L if and only if for every $\varepsilon > 0$ there exist $x_0 > 1$ and $\lambda > 1$ such that for $x \geq x_0$*

$$\frac{1}{(\lambda - 1)\log x} \int_x^{x^\lambda} \frac{s(u)}{u} du \succeq s(x) - \overline{\varepsilon}$$

and another $0 < \ell < 1$ such that

$$\frac{1}{(1 - \ell)\log x} \int_{x^\ell}^x \frac{s(u)}{u} dy \preceq s(x) + \overline{\varepsilon}$$

where $s(x) = \int_1^x f(u)du$.

Corollary 4.4. *Let f be a continuous fuzzy-number-valued function on $[1, \infty)$. If integral $\int_1^\infty f(u)du$ is logarithmic summable to a fuzzy number L and integral function s of function f is slowly decreasing, then integral $\int_1^\infty f(u)du$ converges to L.*

Theorem 4.5. *Let f be a continuous fuzzy-number-valued function on $[1, \infty)$. If there exist negative constant fuzzy number μ and a real number $x_0 \geq 1$ such that*

$$(x \log x) f(x) \succeq \mu \quad \text{for} \quad x > x_0, \tag{4.1}$$

then fuzzy-number-valued function $s(x) = \int_1^x f(u)du$ is slowly decreasing.

Proof. Assume that $(x\log x)f(x) \succeq \mu$ is satisfied. Then for $x > x_0$ we have

$$x\log x f_\alpha^-(x) \geq \mu_\alpha^- \geq \mu_0^- \qquad , \qquad x\log x f_\alpha^+(x) \geq \mu_\alpha^+ \geq \mu_1^+ \geq \mu_0^- .$$

For the sake of simplicity take $\mu_0^- = -H$ where $H > 0$. Then for $x > x_0$

$$x\log x f_\alpha^-(x) \geq -H \Rightarrow f_\alpha^-(x) \geq \frac{-H}{x\log x} \quad \textit{and} \quad x\log x f_\alpha^+(x) \geq -H \Rightarrow f_\alpha^+(x) \geq -\frac{H}{x\log x}$$

are satisfied. So, for $x_0 \leq x < u \leq x^\lambda$ where $\lambda > 1$ we get

$$s_\alpha^-(u) - s_\alpha^-(x) = \int_x^u f_\alpha^-(y)dy \geq -H\int_x^u \frac{dy}{y\log y} = -H\log\left(\frac{\log u}{\log x}\right) \geq -H\log\lambda$$

$$s_\alpha^+(u) - s_\alpha^+(x) = \int_x^u f_\alpha^+(y)dy \geq -H\int_x^u \frac{dy}{y\log y} = -H\log\left(\frac{\log u}{\log x}\right) \geq -H\log\lambda.$$

Then choosing $\lambda := e^{\varepsilon/H}$, the inequalities

$$s_\alpha^-(u) \geq s_\alpha^-(x) - \varepsilon \qquad , \qquad s_\alpha^+(u) \geq s_\alpha^+(x) - \varepsilon$$

follows immediately and $s(u) \succeq s(x) - \overline{\varepsilon}$ holds for $x_0 \leq x < u \leq x^\lambda$. □

Corollary 4.6. *Let f be a continuous fuzzy-number-valued function on $[1,\infty)$. If integral $\int\limits_1^\infty f(u)du$ is logarithmic summable to a fuzzy number L and condition (4.1) is satisfied, then integral $\int\limits_1^\infty f(u)du$ converges to L.*

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