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COMMON FIXED POINT THEOREMS FOR GENERALIZED CONTRACTIONS

DURDANA LATEEF, JAMSHAID AHMAD, ABDULLAH EQAL AL-MAZROOEI

Abstract. The aim of this article is to introduce new family of functions $M(S,T)$ and $N(S,T)$ and obtain some new common fixed point theorems in the context of multiplicative metric spaces. The established results carry some well known results from the literature to multiplicative metric space.

1. Introduction and Preliminaries

Banach contraction principle has been a very advantageous and efficacious means in nonlinear analysis. Various authors have generalized Banach contraction principle in different spaces. In 1999, Dhage [12] introduced the notion of D-metric space and proved the above result for the D-metric space. Subsequently, Mustafa and Sims presented some remarks on D-metric space and gave the concept of generalized metric space. They also proved the Banach contraction principle for the following generalized space. In the last few years, many authors gave different generalized space such as quasi-metric spaces, fuzzy metric spaces, cone metric spaces, complex valued metric space, partial metric spaces and C^*-algebra valued metric space (see, for instance, and the references therein).

In 2008, Bashirov et al. [9] introduced the notion of multiplicative metric spaces, and studied the concept of multiplicative calculus and proved the fundamental theorem of multiplicative calculus. In 2012, Florack and Assen [13] displayed the use of the concept of multiplicative calculus in biomedical image analysis. In 2011, Bashirov et al. [10] exploit the efficiency of multiplicative calculus over the Newtonian calculus. They demonstrated that the multiplicative differential equations are more suitable than the ordinary differential equations in investigating some problems in various fields. Furthermore, Bashirov et al. [9] illustrated the usefulness of multiplicative calculus with some interesting applications. With the help of multiplicative absolute value function, they defined the multiplicative distance between two nonnegative real numbers as well as between two positive square matrices. This provides the basis for multiplicative metric spaces. In 2012, Özavşar and Çevikel [27] investigate multiplicative metric spaces by remarking its topological properties, and introduced concept of multiplicative contraction mapping and

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proved some fixed point theorems of multiplicative contraction mappings on multiplicative spaces. Yamaod and Sintunavarat [32] defined cyclic (α, β)-admissible contractions in the context of multiplicative metric spaces and obtained some fixed point results for these contractions. Mongkolkeha et al. [20] introduced the concept of multiplicative proximal contraction mapping which is a Banach's contraction for non-self mapping in the framework of multiplicative metric spaces and also proved best proximity point theorems for such mappings. They [21] also introduced the new concept of -multiplicative proximal contraction mappings in the setting of multiplicative metric spaces and we also solved optimal approximate solution for such mappings. Recently, Abbas et ai. [1] proved some common fixed point results of quasi-weak commutative mappings on a closed ball in the setting of multiplicative metric spaces. At the same time, they also studied the sufficient conditions for the existence of a common solution of multiplicative boundary value problem. Abbas et al. [2] also studied the sufficient conditions for the existence of common fixed points of pair of rational contractive types mappings involved in cocyclic representation of a nonempty subset of a multiplicative metric space and established some new comparable results in this way. For more details in the direction of multiplicative metric space, we refer the reader to [3, 5, 6, 7, 14, 15, 16, 17, 19, 22, 23, 24, 25, 28, 29, 30, 31].

Now, we present some necessary definitions and results in multiplicative metric spaces, which will be needed in the sequel.

Definition 1.1. [9]*Let X be a nonempty set. A multiplicative metric is a mapping $d : X \times X \longrightarrow \mathbb{R}^+$ satisfying the following assertions:*

- $(MMS1)$ $d(x, y) \geq 1$ for all $x, y \in X$ and $d(x, y) = 1 \Leftrightarrow x = y$;
- $(MMS2)$ $d(x, y) = d(y, x)$ for all $x, y \in X$;
- $(MMS3)$ $d(x, z) \leq d(x, y) \cdot d(y, z)$ for all $x, y, z \in X$.

The pair (X, d) is called then a multiplicative metric space.

Definition 1.2. [9]*Let (X, d) be a multiplicative metric space, $\{x_n\}$ be a sequence in X and $x \in X$. If for each $\epsilon > 1$ and every multiplicative open ball $B_\epsilon(x) = \{y : d(x, y) < \epsilon\}$, there exists a natural number $n_0 \in \mathbb{N}$ such that $n \geq n_0$, then $x_n \in B_\epsilon(x)$. Then the sequence $\{x_n\}$ is said to be multiplicative converging to x, denoted by $x_n \to x$ as $n \to \infty$.*

Proposition 1.3. [27]*Let (X, d) be a multiplicative metric space, $\{x_n\}$ be a sequence in X and $x \in X$. Then $x_n \to x$ as $n \to \infty$ if and only if $d(x_n, x) \to 1$ as $n \to \infty$.*

Definition 1.4. [27]*Let (X, d) be a multiplicative metric space and $\{x_n\}$ be a sequence in X. The sequence $\{x_n\}$ is called multiplicative Cauchy sequence if, for each $\epsilon > 1$, there exists a natural number $n_0 \in \mathbb{N}$ such that*

$$d(x_n, x_m) < \epsilon$$

for all $n, m \geq n_0$.

Proposition 1.5. [27]*Let (X, d) be a multiplicative metric space and $\{x_n\}$ be a sequence in X. Then the sequence $\{x_n\}$ is a multiplicative Cauchy sequence if and only if $d(x_n, x_m) \to 1$ as $n, m \to \infty$.*

Definition 1.6. [27]*A multiplicative metric space (X, d) is said to be multiplicative complete if every multiplicative Cauchy sequence in (X, d) is multiplicative convergent in X.*

Theorem 1.7. [27]*Let* (X, d) *be a complete multiplicative metric space and let* $S : X \to X$ *be self-mapping. If there exists a constant* $k \in [0, 1)$ *such that*

$$d(Sx, Sy) \leq (d(x, y))^k$$

for all $x, y \in X$*. Then* S *has a unique fixed point.*

In this article, we prove a common fixed point theorem for self mappings regarding some new families of control functions and generalized contractions.

2. Main Results

Very recently, Ahmad et al. [4] defined the family $M(S, T)$ of all functions $a : X \times X \to [0, 1)$ with following assertions

$$a(TSx, y) \leq a(x, y) \text{ and } a(x, STy) \leq a(x, y)$$

and the family $N(S, T)$ of all functions $\lambda : X \to [0, 1)$ such that for all $x, y \in X$ with

$$\lambda(TSx) \leq \lambda(x)$$

on a metric space (X, d) and for two self mappings $S, T : X \to X$.

The following proposition plays an important role in the proofs of our main theorems.

Proposition 2.1. [4]*Let* (X, d) *be a metric space and* $S, T : X \to X$ *be self-mappings. Let* $x_0 \in X$*, we define the sequence* $\{x_n\}$ *by* $x_{2n+1} = Sx_{2n}$*,* $x_{2n+2} = Tx_{2n+1}$ *for all integers* $n \geq 0$*.*

If $a \in M(S, T)$, then $a(x_{2n}, y) \leq a(x_0, y)$ and $a(x, x_{2n+1}) \leq a(x, x_1)$ for all $x, y \in X$ and integers $n \geq 0$.

Now we state our main theorem.

Theorem 2.2. *Let* (X, d) *be a complete multiplicative metric space and* $S, T : X \to X$ *be self-mappings. If there mappings* $\lambda, \mu, \gamma, \delta \in M(S, T)$ *such that for all* $x, y \in X$*:*

$$\lambda(x, y) + \mu(x, y) + \gamma(x, y) + \delta(x, y) < 1,$$

$$\begin{aligned} & d(Sx, Ty) \\ \leq \ & (d(x, y))^{\lambda(x,y)}.(d(x, Sx))^{\mu(x,y)}.(d(y, Ty))^{\gamma(x,y)}.(\frac{d(y, Sx)d(x, Ty)}{1 + d(x, y)})^{\delta(x,y)}. \end{aligned}$$

Then S *and* T *have a unique common fixed point.*

Proof. Let $x_0 \in X$, we define the sequence $\{x_n\}$ by

$$x_{2n+1} = Sx_{2n} \text{ and } x_{2n+2} = Tx_{2n+1}$$

for all $n = 0, 1, 2, ...$ From Proposition 2.1, for all $n = 0, 1, 2, ...$,we have

$$\begin{aligned}
d(x_{2n+1}, x_{2n+2}) &= d(Sx_{2n}, TSx_{2n}) \\
&\leq (d(x_{2n}, Sx_{2n}))^{\lambda(x_{2n}, Sx_{2n})}.(d(x_{2n}, Sx_{2n}))^{\mu(x_{2n}, Sx_{2n})} \\
&\quad .(d(Sx_{2n}, TSx_{2n}))^{\gamma(x_{2n}, Sx_{2n})} \\
&\quad .(\frac{d(Sx_{2n}, Sx_{2n})d(x_{2n}, TSx_{2n})}{1 + d(x_{2n}, Sx_{2n})})^{\delta(x_{2n}, Sx_{2n})} \\
&= (d(x_{2n}, x_{2n+1}))^{\lambda(x_{2n}, x_{2n+1})}.(d(x_{2n}, x_{2n+1}))^{\mu(x_{2n}, x_{2n+1})} \\
&\quad .(d(x_{2n+1}, x_{2n+2}))^{\gamma(x_{2n}, x_{2n+1})}.(\frac{d(x_{2n}, x_{2n+2})}{1 + d(x_{2n}, x_{2n+1})})^{\delta(x_{2n}, x_{2n+1})} \\
&\leq (d(x_{2n}, x_{2n+1}))^{\lambda(x_0, x_{2n+1})}.(d(x_{2n}, x_{2n+1}))^{\mu(x_0, x_{2n+1})} \\
&\quad .(d(x_{2n+1}, x_{2n+2}))^{\gamma(x_0, x_{2n+1})} \\
&\quad .(\frac{(d(x_{2n}, x_{2n+1}).d(x_{2n+1}, x_{2n+2})}{1 + d(x_{2n}, x_{2n+1})})^{\delta(x_0, x_{2n+1})}.
\end{aligned}$$

Again from the Proposition 2.1, we have

$$\begin{aligned}
d(x_{2n+1}, x_{2n+2}) &\leq (d(x_{2n}, x_{2n+1}))^{\lambda(x_0, x_1)}.(d(x_{2n}, x_{2n+1}))^{\mu(x_0, x_1)} \\
&\quad .(d(x_{2n+1}, x_{2n+2}))^{\gamma(x_0, x_1)}.(d(x_{2n+1}, x_{2n+2}))^{\delta(x_0, x_1)} \\
&= (d(x_{2n}, x_{2n+1}))^{(\lambda(x_0, x_1) + \mu(x_0, x_1))}..(d(x_{2n+1}, x_{2n+2}))^{(\gamma(x_0, x_1) + \delta(x_0, x_1))}.
\end{aligned}$$

Hence,

$$d(x_{2n+1}, x_{2n+2}) \leq (d(x_{2n}, x_{2n+1}))^{\frac{\lambda(x_0, x_1) + \mu(x_0, x_1)}{1 - \gamma(x_0, x_1) - \delta(x_0, x_1)}} = (d(x_{2n}, x_{2n+1}))^k. \qquad (2.1)$$

Similarly from the Proposition 2.1, we have

$$\begin{aligned}
d(x_{2n}, x_{2n+1}) &= d(Tx_{2n-1}, STx_{2n-1}) = d(STx_{2n-1}, Tx_{2n-1}) \\
&\leq (d(Tx_{2n-1}, x_{2n-1}))^{\lambda(Tx_{2n-1}, x_{2n-1})} \\
&\quad .(d(Tx_{2n-1}, STx_{2n-1}))^{\mu(Tx_{2n-1}, x_{2n-1})} \\
&\quad .(d(x_{2n-1}, Tx_{2n-1}))^{\gamma(Tx_{2n-1}, x_{2n-1})} \\
&\quad .(\frac{d(Tx_{2n-1}, Tx_{2n-1})d(x_{2n-1}, STx_{2n-1})}{1 + d(Tx_{2n-1}, x_{2n-1})})^{\delta(Tx_{2n-1}, x_{2n-1})} \\
&= (d(x_{2n}, x_{2n-1}))^{\lambda(x_{2n}, x_{2n-1})}.(d(x_{2n}, x_{2n+1}))^{\mu(x_{2n}, x_{2n-1})} \\
&\quad .(d(x_{2n-1}, x_{2n}))^{\gamma(x_{2n}, x_{2n-1})}.(\frac{d(x_{2n-1}, x_{2n+1})}{1 + d(x_{2n}, x_{2n-1})})^{\delta(x_{2n}, x_{2n-1})} \\
&\leq (d(x_{2n}, x_{2n-1}))^{\lambda(x_0, x_{2n-1})}.((d(x_{2n}, x_{2n+1}))^{\mu(x_0, x_{2n-1})} \\
&\quad .(d(x_{2n-1}, x_{2n}))^{\gamma(x_0, x_{2n-1})} \\
&\quad .(\frac{d(x_{2n-1}, x_{2n}).d(x_{2n}, x_{2n+1})}{1 + d(x_{2n}, x_{2n-1})})^{\delta(x_0, x_{2n-1})} \\
&\leq (d(x_{2n}, x_{2n-1}))^{\lambda(x_0, x_1)}.(d(x_{2n}, x_{2n+1}))^{\mu(x_0, x_1)} \\
&\quad .(d(x_{2n-1}, x_{2n}))^{\gamma(x_0, x_1)} \\
&\quad .(\frac{d(x_{2n-1}, x_{2n}).d(x_{2n}, x_{2n+1})}{1 + d(x_{2n}, x_{2n-1})})^{\delta(x_0, x_1)} \\
&\leq (d(x_{2n-1}, x_{2n}))^{(\lambda(x_0, x_1) + \gamma(x_0, x_1))} \\
&\quad .(d(x_{2n}, x_{2n+1}))^{(\mu(x_0, x_1) + \delta(x_0, x_1))}.
\end{aligned}$$

Hence

$$d(x_{2n}, x_{2n+1}) \leq (d(x_{2n-1}, x_{2n}))^{\frac{\lambda(x_0, x_1) + \gamma(x_0, x_1)}{1 - \mu(x_0, x_1) - \delta(x_0, x_1)}} = (d(x_{2n-1}, x_{2n}))^k. \tag{2.2}$$

Thus from (2.1) and (2.2), we conclude that

$$d(x_n, x_{n+1}) \leq (d(x_{n-1}, x_n))^k \leq (d(x_{n-2}, x_{n-1}))^{k^2} \leq ... \leq (d(x_0, x_1))^{k^n}. \tag{2.3}$$

Now we show that $\{x_n\}$ is Cauchy in (X, d). Therefore for each $m, n \in \mathbb{N}$ with $m > n$ such that

$$\begin{aligned}
d(x_n, x_m) &\leq d(x_n, x_{n+1}).d(x_{n+1}, x_{n+2})...d(x_{m-1}, x_m) \\
&\leq (d(x_0, x_1))^{k^n}.(d(x_0, x_1))^{k^{n+1}}...(d(x_0, x_1))^{k^{m-1}} \\
&\leq (d(x_0, x_1))^{(k^n + k^{n+1} + ... + k^{m-1})} \\
&\leq (d(x_0, x_1))^{(k^n + k^{n+1} + ...)} \\
&\leq (d(x_0, x_1))^{(\frac{k^n}{1-k})}.
\end{aligned}$$

Taking limit as $m, n \longrightarrow \infty$, we get $d(x_n, x_m) \longrightarrow 1$. Hence $\{x_n\}$ is a multiplicative Cauchy sequence. By the completeness of (X, d) it follows that $x_n \longrightarrow x^* \in X$.

From Proposition 2.1, for all $n = 0, 1, 2, ...$,we have

$$\begin{aligned} d(x_{2n+1}, Tx^*) &= d(Sx_{2n}, Tx^*) \\ &\leq (d(x_{2n}, x^*))^{\lambda(x_{2n},x^*)}.(d(x_{2n}, Sx_{2n}))^{\mu(x_{2n},x^*)} \\ &\quad .(d(x^*, Tx^*))^{\gamma(x_{2n},x^*)} \\ &\quad .(d(x^*, Sx_{2n}))^{L(x_{2n},x^*)}.(\frac{d(x_{2n}, Tx^*)}{1 + d(x_{2n}, x^*)})^{\delta(x_{2n},x^*)} \\ &= (d(x_{2n}, x^*))^{\lambda(x_{2n},x^*)}.(d(x_{2n}, x_{2n+1}))^{\mu(x_{2n},x^*)} \\ &\quad .(d(x^*, Tx^*))^{\gamma(x_{2n},x^*)} \\ &\quad .(\frac{d(x^*, x_{2n+1})d(x_{2n}, Tx^*)}{1 + d(x_{2n}, x^*)})^{\delta(x_{2n},x^*)} \\ &\leq (d(x_{2n}, x^*))^{\lambda(x_0,x^*)}.(d(x_{2n}, x_{2n+1}))^{\mu(x_0,x^*)} \\ &\quad .(d(x^*, Tx^*))^{\gamma(x_0,x^*)} \\ &\quad .(\frac{d(x^*, x_{2n+1})d(x_{2n}, Tx^*)}{1 + d(x_{2n}, x^*)})^{\delta(x_0,x^*)} \end{aligned}$$

which on taking limit as n tends to infinity gives

$$\begin{aligned} d(x^*, Tx^*) &\leq (d(x^*, Tx^*))^{\gamma(x_0,x^*)}.(d(x^*, Tx^*))^{\delta(x_0,x^*)} \\ &= (d(x^*, Tx^*))^{\gamma(x_0,x^*)+\delta(x_0,x^*)} \end{aligned}$$

this implies that

$$(d(x^*, Tx^*))^{1-\gamma(x_0,x^*)-\delta(x_0,x^*)} \leq 1$$

which further implies that

$$(d(x^*, Tx^*)) \leq (1)^{\frac{1}{1-\gamma(x_0,x^*)-\delta(x_0,x^*)}} \leq 1.$$

Thus x^* is a fixed point of mapping T. Similarly, from Proposition 2.1, for all $n = 0, 1, 2, ...$,we have

$$\begin{aligned} d(Sx^*, x_{2n+2}) &= d(Sx^*, Tx_{2n+1}) \\ &\leq (d(x^*, x_{2n+1}))^{\lambda(x^*,x_{2n+1})}.(d(x^*, Sx^*))^{\mu(x^*,x_{2n+1})} \\ &\quad .(d(x_{2n+1}, Tx_{2n+1}))^{\gamma(x^*,x_{2n+1})} \\ &\quad .(\frac{d(x_{2n+1}, Sx^*)d(x^*, Tx_{2n+1})}{1 + d(x^*, x_{2n+1})})^{\delta(x^*,x_{2n+1})} \\ &= (d(x^*, x_{2n+1}))^{\lambda(x^*,x_{2n+1})}.(d(x^*, Sx^*))^{\mu(x^*,x_{2n+1})} \\ &\quad .(d(x_{2n+1}, x_{2n+2}))^{\gamma(x^*,x_{2n+1})} \\ &\quad .(\frac{d(x_{2n+1}, Sx^*)d(x^*, x_{2n+2})}{1 + d(x^*, x_{2n+1})})^{\delta(x^*,x_{2n+1})} \\ &\leq (d(x^*, x_{2n+1}))^{\lambda(x^*,x_1)}.(d(x^*, Sx^*))^{\mu(x^*,x_1)} \\ &\quad .(d(x_{2n+1}, x_{2n+2}))^{\gamma(x^*,x_1)} \\ &\quad .(\frac{d(x_{2n+1}, Sx^*)d(x^*, x_{2n+2})}{1 + d(x^*, x_{2n+1})})^{\delta(x^*,x_1)}. \end{aligned}$$

Taking the limit as n approaches to ∞, we get

$$\begin{aligned} d(Sx^*, x^*) &\leq (d(x^*, Sx^*))^{\mu(x^*,x_1)}.(d(x^*, Sx^*))^{\delta(x^*,x_1)} \\ &= (d(x^*, Sx^*))^{\mu(x^*,x_1)+\delta(x^*,x_1)}. \end{aligned}$$

This implies that

$$(d(x^*, Sx^*))^{1-\mu(x_0,x^*)-\delta(x_0,x^*)} \leq 1$$

which further implies that

$$(d(x^*, Sx^*)) \leq (1)^{\frac{1}{1-\mu(x_0,x^*)-\delta(x_0,x^*)}} \leq 1.$$

Thus x^* is a fixed point of mapping S. Hence x^* is a common fixed point of mappings S and T.

Let z be another common fixed point of the mappings S and T other than x^*. Now consider

$$\begin{aligned} d(x^*, z) &= d(Sx^*, Tz) \\ &\leq (d(x^*, z))^{\lambda(x^*,z)}.(d(x^*, Sx^*))^{\mu(x^*,z)}.(d(z, Tz))^{\gamma(x^*,z)}.(\frac{d(z, Sx^*)d(x^*, Tz)}{1 + d(x^*, z)})^{\delta(x^*,z)} \\ &= (d(x^*, z))^{\lambda(x^*,z)}.(d(x^*, x^*))^{\mu(x^*,z)}.(d(z, z))^{\gamma(x^*,z)}.(\frac{d(z, x^*)d(x^*, z)}{1 + d(x^*, z)})^{\delta(x^*,z)} \\ &\leq (d(x^*, z))^{\lambda(x^*,z)}.(d(x^*, z))^{\delta(x^*,z)} = (d(x^*, z))^{\lambda(x^*,z)+\delta(x^*,z)}. \end{aligned}$$

This implies that

$$(d(x^*, z))^{1-\lambda(x^*,z)-\delta(x^*,z)} \leq 1$$

which further implies that

$$d(x^*, z) \leq (1)^{\frac{1}{1-\lambda(x^*,z)-\delta(x^*,z)}} \leq 1$$

which is a contradiction to the fact that $x^* \neq z$. Thus x^* is a unique common fixed point of the mappings S and T. □

Theorem 2.3. *Let (X, d) be a complete multiplicative metric space and $S : X \to X$ be self-mapping. If there mappings $\lambda, \mu, \gamma, \delta \in M(S, S)$ such that for all $x, y \in X$:*

$$\lambda(x, y) + \mu(x, y) + \gamma(x, y) + \delta(x, y) < 1,$$

$$\begin{aligned} & d(Sx, Sy) \\ \leq\ & (d(x, y))^{\lambda(x,y)}.(d(x, Sx))^{\mu(x,y)}.(d(y, Sy))^{\gamma(x,y)}.(\frac{d(y, Sx)d(x, Sy)}{1 + d(x, y)})^{\delta(x,y)}. \end{aligned}$$

Then S has a unique fixed point.

Theorem 2.4. *Let (X, d) be a complete multiplicative metric space and $S, T : X \to X$ be self-mappings. If there mappings $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in N(S, T)$ such that for all $x, y \in X$:*

$$\lambda_1(x) + \lambda_2(x) + \lambda_3(x) + \lambda_4(x) < 1,$$

$$\begin{aligned} & d(Sx, Ty) \\ \leq\ & (d(x, y))^{\lambda_1(x)}.(d(x, Sx))^{\lambda_2(x)}.(d(y, Ty))^{\lambda_3(x)}.(\frac{d(y, Sx)d(x, Ty)}{1 + d(x, y)})^{\lambda_4(x)}. \end{aligned}$$

Then S and T have a unique common fixed point.

Proof. Define $\lambda, \mu, \gamma, \delta, L : X \times X \to [0, 1)$ by $\lambda(x, y) = \lambda_1(x)$, $\mu(x, y) = \lambda_2(x)$, $\gamma(x, y) = \lambda_3(x)$ and $\delta(x, y) = \lambda_4(x)$ for all $x, y \in X$. Then for all $x, y \in X$;

$$\lambda(TSx, y) = \lambda_1(TSx) \leq \lambda_1(x) = \lambda(x, y) \text{ and } \lambda(x, STy) = \lambda_1(x) = \lambda(x, y)$$

$$\mu(TSx, y) = \lambda_2(TSx) \leq \lambda_2(x) = \mu(x, y) \text{ and } \mu(x, STy) = \lambda_2(x) = \mu(x, y),$$

$$\gamma(TSx, y) = \lambda_3(TSx) \leq \lambda_3(x) = \gamma(x, y) \text{ and } \gamma(x, STy) = \lambda_3(x) = \gamma(x, y);$$

$$\delta(TSx, y) = \lambda_4(TSx) \leq \lambda_4(x) = \delta(x, y) \text{ and } \delta(x, STy) = \lambda_4(x) = \delta(x, y);$$

and

$$\lambda(x, y) + \mu(x, y) + \gamma(x, y) + \delta(x, y) = \lambda_1(x) + \lambda_2(x) + \lambda_3(x) + \lambda_4(x) < 1,$$

$$\begin{aligned} & d(Sx, Ty) \\ \leq\ & (d(x, y))^{\lambda_1(x)}.(d(x, Sx))^{\lambda_2(x)}.(d(y, Ty))^{\lambda_3(x)}.(\frac{d(y, Sx)d(x, Ty)}{1 + d(x, y)})^{\lambda_4(x)} \\ =\ & (d(x, y))^{\lambda(x,y)}.(d(x, Sx))^{\mu(x,y)}.(d(y, Ty))^{\gamma(x,y)}.(\frac{d(y, Sx)d(x, Ty)}{1 + d(x, y)})^{\delta(x,y)}. \end{aligned}$$

By Theorem (2.2), S and T have unique common fixed point. □

Theorem 2.5. *Let (X, d) be a complete multiplicative metric space and $S : X \to X$ be self-mappings. If there mappings $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in N(S, S)$ with*

$$\lambda_1(x) + \lambda_2(x) + \lambda_3(x) + \lambda_4(x) < 1,$$

such that for all $x, y \in X$:

$$\begin{aligned} & d(Sx, Sy) \\ \leq\ & (d(x, y))^{\lambda_1(x)}.(d(x, Sx))^{\lambda_2(x)}.(d(y, Sy))^{\lambda_3(x)}.(\frac{d(y, Sx)d(x, Sy)}{1 + d(x, y)})^{\lambda_4(x)}. \end{aligned}$$

Then S has a unique fixed point.

Theorem 2.6. *Let (X, d) be a complete multiplicative metric space and $S, T : X \to X$ be self-mappings. If there exist non negative reals $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in [0, 1)$ with*

$$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 < 1,$$

such that for all $x, y \in X$:

$$\begin{aligned} & d(Sx, Ty) \\ \leq\ & (d(x, y))^{\lambda_1}.(d(x, Sx))^{\lambda_2}.(d(y, Ty))^{\lambda_3}.(\frac{d(y, Sx)d(x, Ty)}{1 + d(x, y)})^{\lambda_4}. \end{aligned}$$

Then S and T have a unique common fixed point.

Theorem 2.7. *Let (X, d) be a complete multiplicative metric space and $S : X \to X$ be self-mapping. If there exist non negative reals $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in [0, 1)$ with*

$$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 < 1,$$

such that for all $x, y \in X$:

$$\begin{aligned} & d(Sx, Sy) \\ \leq\ & (d(x, y))^{\lambda_1}.(d(x, Sx))^{\lambda_2}.(d(y, Sy))^{\lambda_3}.(\frac{d(y, Sx)d(x, Sy)}{1 + d(x, y)})^{\lambda_4}. \end{aligned}$$

Then S has a unique fixed point.

Conflict of Interests

The authors declare that they have no competing interests.

Authors' Contribution

All authors contributed equally and significantly in writing this paper. All authors read and approved the final paper.

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Durdana Lateef
Deanery of Academic Services, Taibah University, Al-Madinah Al-Munawwarah, 41411, Kingdom of Saudi Arabia
E-mail address: drdurdanamaths@gmail.com

Jamshaid Ahmad
Department of Mathematics, University of Jeddah, P.O.Box 80327, Jeddah 21589, Saudi Arabia
E-mail address: jamshaid_jasim@yahoo.com

Abdullah Eqal Al-Mazrooei
Department of Mathematics, University of Jeddah, P.O.Box 80327, Jeddah 21589, Saudi Arabia
E-mail address: aealmazrooei@uj.edu.sa