Journal of Mathematical Analysis
ISSN: 2217-3412, URL: www.ilirias.com/jma
Volume 8 Issue 2 (2017), Pages 80-88.

A COMMON FIXED POINT THEOREM FOR GENERALIZED (ψ, φ)-WEAK CONTRACTIONS OF SUZUKI TYPE

FEI HE, YU-QI SUN, XIAO-YUE ZHAO

Abstract. The purpose of this paper is to prove a conjecture proposed by Singh et al. in [Filomat 29:7 (2015), 1481-1490].

1. Introduction

The Banach contraction principle is a basic and remarkable result in fixed point theory. Over the years, it has been extended in many different directions and spaces, see [1-24] and the references therein. In 1974, Ćirić [8] established a fixed point theorem which is one of the most important results in various generalizations of this principle. Since then, many extensions and generalizations of Ćirić's result have appeared in the literature; see for instance [9, 10, 14, 15, 16, 22, 23].

Recently, Suzuki [20] obtained a powerful generalization of Banach contraction theorem. Using the idea of the Suzuki contraction, various fixed point results have been extended in many directions; see for instance [12, 17, 18, 19, 20, 21]. Particularly, Singh et al. [19] give a weakly contractive version of Suzuki type and Ćirić type contractions and generalize some results of Dorić [11], Zhang et al. [24] and others. A question and a new conjecture were proposed by Singh et al. in [19].

In this paper, we give a proof of the conjecture of Singh et al. [19] and answer their question.

2. Main results

Now we begin to state our result, which is the conjecture arisen in [19].

Theorem 2.1. *Let X be a complete metric space and $S, T : X \to X$ such that for every $x, y \in X$,*

$$\frac{1}{2}\min\{d(x, Sx), d(y, Ty)\} \leq d(x, y)$$

implies

$$\psi(d(Sx, Ty)) \leq \psi(m(x, y)) - \varphi(m(x, y)), \tag{2.1}$$

2000 *Mathematics Subject Classification.* 54H25; 47H10.

Key words and phrases. Common fixed point; generalized (ψ, φ)-weak contraction; Suzuki type contraction; metric space.

Submitted February 23, 2017. Published April 8, 2017.

Communicated by Hassen Aydi.

This research was supported by the National Natural Science Foundation of China (11561049, 11471236).

where

(i) $\psi : [0, \infty) \to [0, \infty)$ *is a continuous and monotone nondecreasing function with* $\psi(t) = 0$ *if and only if* $t = 0$*,*

(ii) $\varphi : [0, \infty) \to [0, \infty)$ *is a lower semi-continuous function with* $\varphi(t) = 0$ *if and only if* $t = 0$*, and*

$$m(x, y) = \max\{d(x, y), d(x, Sx), d(y, Ty), \frac{d(x, Ty) + d(y, Sx)}{2}\}.$$

Then S *and* T *have a unique common fixed point.*

Proof. Let $x_0 \in X$ be given. Construct a sequence $\{x_n\}$ in X such that $x_{2n-1} = Sx_{2n-2}$ and $x_{2n} = Tx_{2n-1}$, $n = 1, 2, \cdots$. The following fact will be used in the sequel.

$$\frac{1}{2} \min\{d(x, Sx), d(y, Ty)\} \leq d(x, y)$$

if and only if

$$\frac{1}{2} d(x, Sx) \leq d(x, y), \text{ or } \frac{1}{2} d(y, Ty) \leq d(x, y).$$

Next, we always assume that $x_n \neq x_{n-1}$ for all $n \in \mathbb{N}$. If not, the existence of the common fixed point can be proved.

Indeed, if $x_{2n} = x_{2n-1}$ for some $n \in \mathbb{N}$, we can prove x_{2n-1} is a common fixed point of T and S. In fact, using (2.1), we obtain that

$$\frac{1}{2} d(x_{2n-1}, Tx_{2n-1}) = \frac{1}{2} d(x_{2n-1}, x_{2n}) = 0 \leq d(x_{2n}, x_{2n-1})$$

implies

$$\psi(d(Sx_{2n}, Tx_{2n-1})) \leq \psi(m(x_{2n}, x_{2n-1})) - \varphi(m(x_{2n}, x_{2n-1})), \tag{2.2}$$

where

$$\begin{aligned} m(x_{2n}, x_{2n-1}) &= \max\{d(x_{2n}, x_{2n-1}), d(x_{2n}, Sx_{2n}), d(x_{2n-1}, Tx_{2n-1}), \\ &\qquad \frac{d(x_{2n}, Tx_{2n-1}) + d(x_{2n-1}, Sx_{2n})}{2}\} \\ &= \max\{0, d(x_{2n}, Sx_{2n}), 0, \frac{0 + d(x_{2n}, Sx_{2n})}{2}\} \\ &= d(x_{2n}, Sx_{2n}). \end{aligned}$$

Thus (2.2) becomes that

$$\psi(d(Sx_{2n}, x_{2n})) \leq \psi(d(Sx_{2n}, x_{2n})) - \varphi(d(Sx_{2n}, x_{2n})),$$

which implies $\varphi(d(Sx_{2n}, x_{2n})) \leq 0$. By the property of φ, we get that $d(Sx_{2n}, x_{2n}) = 0$ and $Sx_{2n} = x_{2n}$. From $Tx_{2n-1} = x_{2n} = x_{2n-1}$, it follows that $Sx_{2n-1} = Tx_{2n-1} = x_{2n-1}$, that is x_{2n-1} is a common fixed point of T and S. Similarly, if $x_{2n-1} = x_{2n-2}$ for some $n \in \mathbb{N}$, then x_{2n-2} is a common fixed point of T and S.

Now, we divide our proof in the following five steps.

Step 1. We prove

$$\lim_{n \to \infty} d(x_n, x_{n+1}) = 0 \tag{2.3}$$

and

$$\lim_{n \to \infty} d(x_n, x_{n+2}) = 0. \tag{2.4}$$

Notice that, for any $n \in \mathbb{N}$,

$$\frac{1}{2} d(x_{2n-1}, Tx_{2n-1}) = \frac{1}{2} d(x_{2n-1}, x_{2n}) \leq d(x_{2n}, x_{2n-1}).$$

Therefore by (2.1), we have

$$\psi(d(Sx_{2n}, Tx_{2n-1})) \leq \psi(m(x_{2n}, x_{2n-1})) - \varphi(m(x_{2n}, x_{2n-1})), \tag{2.5}$$

where

$$\begin{aligned} m(x_{2n}, x_{2n-1}) &= \max\{d(x_{2n}, x_{2n-1}), d(x_{2n}, Sx_{2n}), d(x_{2n-1}, Tx_{2n-1}), \\ &\qquad \frac{d(x_{2n}, Tx_{2n-1}) + d(x_{2n-1}, Sx_{2n})}{2}\} \\ &= \max\{d(x_{2n}, x_{2n-1}), d(x_{2n}, x_{2n+1}), d(x_{2n-1}, x_{2n}), \\ &\qquad \frac{d(x_{2n}, x_{2n}) + d(x_{2n-1}, x_{2n+1})}{2}\} \\ &= \max\{d(x_{2n}, x_{2n-1}), d(x_{2n}, x_{2n+1})\}. \end{aligned}$$

If $m(x_{2n}, x_{2n-1}) = d(x_{2n}, x_{2n+1})$, then (2.5) becomes

$$\psi(d(x_{2n}, x_{2n+1})) \leq \psi(d(x_{2n}, x_{2n+1})) - \varphi(d(x_{2n}, x_{2n+1})),$$

which implies $\varphi(d(x_{2n}, x_{2n+1})) \leq 0$ and $d(x_{2n}, x_{2n+1}) = 0$. This is a contradiction with $x_{2n} \neq x_{2n+1}$. Consequently, $m(x_{2n}, x_{2n-1}) = d(x_{2n}, x_{2n-1})$ and (2.5) becomes that

$$\psi(d(x_{2n}, x_{2n+1})) \leq \psi(d(x_{2n}, x_{2n-1})) - \varphi(d(x_{2n}, x_{2n-1})). \tag{2.6}$$

Similarly, we can deduce that

$$\psi(d(x_{2n+1}, x_{2n+2})) \leq \psi(d(x_{2n}, x_{2n+1})) - \varphi(d(x_{2n}, x_{2n+1})). \tag{2.7}$$

Combining (2.6) and (2.7), we see that

$$\psi(d(x_{n+1}, x_n)) \leq \psi(d(x_n, x_{n-1})) - \varphi(d(x_n, x_{n-1})), \tag{2.8}$$

for any $n \in \mathbb{N}$. Since $\varphi(d(x_n, x_{n-1})) > 0$, we have that

$$\psi(d(x_{n+1}, x_n)) < \psi(d(x_n, x_{n-1})).$$

Hence from the property of ψ we conclude that

$$d(x_{n+1}, x_n) < d(x_n, x_{n-1}).$$

Thus the sequence $\{d(x_{n+1}, x_n)\}$ is monotone nonincreasing and bounded below, which implies that $\lim_{n\to\infty} d(x_{n+1}, x_n) = r$ for some $r \geq 0$. We claim that $r = 0$. In fact, taking upper limits as $n \to \infty$ on each side of (2.8), we see that

$$\psi(r) \leq \psi(r) - \varphi(r).$$

Consequently, we get that $\varphi(r) \leq 0$ and $r = 0$. This means that $\lim_{n\to\infty} d(x_{n+1}, x_n) = 0$ and (2.3) holds. Using the triangular inequality, we see that

$$d(x_n, x_{n+2}) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}),$$

and by (2.3) we get that (2.4) holds.

Step 2. We prove the following claim.

Claim: for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that if $m > n > N$ with $m - n \equiv 1 \mod 2$ then $d(x_m, x_n) < \varepsilon$.

Suppose, to the contrary that there exists $\varepsilon_0 > 0$ such that for any $N \in \mathbb{N}$ we can find $m > n > N$ with $m - n \equiv 1 \mod 2$ satisfying $d(x_m, x_n) > \varepsilon_0$. Using (2.3) and (2.4), corresponding to this ε_0, we find N_0 such that $n > N_0$ implies

$$d(x_n, x_{n+1}) < \varepsilon_0 \text{ and } d(x_n, x_{n+2}) < \varepsilon_0. \tag{2.9}$$

Next, by induction we can get two subsequences $\{x_{m_k}\}$ and $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$d(x_{m_k}, x_{n_k}) \geq \varepsilon_0,\ d(x_{m_k-2}, x_{n_k}) < \varepsilon_0 \text{ and } m_k - n_k \equiv 1 \mod 2. \tag{2.10}$$

Indeed, if we take $N = N_0$, we can find that $l_1 > n_1 > N_0$ with $l_1 - n_1 \equiv 1 \mod 2$ such that $d(x_{l_1}, x_{n_1}) \geq \varepsilon_0$. Due to (2.9), we can choose a $m_1 \in \{n_1+3, n_1+5, \cdots, l_1\}$ in such a way that it is the smallest integer satisfying $d(x_{m_1}, x_{n_1}) \geq \varepsilon_0$. Then we obtain

$$d(x_{m_1}, x_{n_1}) \geq \varepsilon_0,\ d(x_{m_1-2}, x_{n_1}) < \varepsilon_0 \text{ and } m_1 - n_1 \equiv 1 \mod 2.$$

If we take $N = m_1$, we can find that $l_2 > n_2 > m_1$ with $l_2 - n_2 \equiv 1 \mod 2$ such that $d(x_{l_2}, x_{n_2}) \geq \varepsilon_0$. Similar to the choice of m_1, we can choose a $m_2 \in \{n_2 + 3, n_2 + 5, \cdots, l_2\}$ such that

$$d(x_{m_2}, x_{n_2}) \geq \varepsilon_0,\ d(x_{m_2-2}, x_{n_2}) < \varepsilon_0 \text{ and } m_2 - n_2 \equiv 1 \mod 2.$$

Continuing this process, we obtain two subsequences $\{x_{m_k}\}$ and $\{x_{n_k}\}$ of $\{x_n\}$ satisfying (2.10).

Observe that

$$\begin{aligned} \varepsilon_0 \leq d(x_{m_k}, x_{n_k}) &\leq d(x_{m_k}, x_{m_k-2}) + d(x_{m_k-2}, x_{n_k}) \\ &\leq d(x_{m_k}, x_{m_k-2}) + \varepsilon_0. \end{aligned}$$

Then by(2.4) we have

$$\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = \varepsilon_0.$$

From this and (2.3), it follows that

$$\lim_{k\to\infty} d(x_{m_k+1}, x_{n_k}) = \varepsilon_0.$$

By a similar way, we get

$$\lim_{k\to\infty} d(x_{n_k+1}, x_{m_k}) = \varepsilon_0$$

and

$$\lim_{k\to\infty} d(x_{m_k+1}, x_{n_k+1}) = \varepsilon_0.$$

Now, using $m_k - n_k \equiv 1 \mod 2$, we consider the following two cases.

Case i. If $m_k = 2p_k - 1$ and $n_k = 2q_k$ for some $p_k, q_k \in \mathbb{N}$. From (2.9) and (2.10), we see that

$$\frac{1}{2} d(x_{m_k}, Tx_{m_k}) = \frac{1}{2} d(x_{m_k}, x_{m_k+1}) < \frac{1}{2}\varepsilon_0 < \varepsilon_0 \leq d(x_{n_k}, x_{m_k}).$$

Therefore by (2.1) we have

$$\psi(d(x_{m_k+1}, x_{n_k+1})) = \psi(d(Sx_{n_k}, Tx_{m_k})) \leq \psi(m(x_{n_k}, x_{m_k})) - \varphi(m(x_{n_k}, x_{m_k})), \tag{2.11}$$

where

$$\begin{aligned} m(x_{n_k}, x_{m_k}) &= \max\{d(x_{n_k}, x_{m_k}), d(x_{m_k}, Tx_{m_k}), d(x_{n_k}, Sx_{n_k}), \\ &\qquad \frac{d(x_{n_k}, Tx_{m_k}) + d(x_{m_k}, Sx_{n_k})}{2}\} \\ &= \max\{d(x_{n_k}, x_{m_k}), d(x_{m_k}, x_{m_k+1}), d(x_{n_k}, x_{n_k+1}), \\ &\qquad \frac{d(x_{n_k}, x_{m_k+1}) + d(x_{m_k}, x_{n_k+1})}{2}\}. \end{aligned}$$

Then

$$\lim_{k\to\infty} m(x_{n_k}, x_{m_k}) = \max\{\varepsilon_0, 0, 0, \frac{\varepsilon_0+\varepsilon_0}{2}\} = \varepsilon_0.$$

Thus, taking the upper limits as $k \to \infty$ in (2.11), we obtain $\psi(\varepsilon_0) \le \psi(\varepsilon_0) - \varphi(\varepsilon_0)$, and therefore $\varphi(\varepsilon_0) = 0$, which is a contradiction with $\varepsilon_0 > 0$.

Case ii. If $m_k = 2p_k$ and $n_k = 2q_k - 1$ for some $p_k, q_k \in \mathbb{N}$. From (2.9) and (2.10), we see that

$$\frac{1}{2}d(x_{m_k}, Sx_{m_k}) = \frac{1}{2}d(x_{m_k}, x_{m_k+1}) < \frac{1}{2}\varepsilon_0 < \varepsilon_0 \le d(x_{m_k}, x_{n_k}).$$

Therefore by (2.1) we have

$$\psi(d(x_{m_k+1}, x_{n_k+1})) = \psi(d(Sx_{m_k}, Tx_{n_k})) \le \psi(m(x_{m_k}, x_{n_k})) - \varphi(m(x_{m_k}, x_{n_k})),$$

where

$$\begin{aligned} m(x_{m_k}, x_{n_k}) &= \max\{d(x_{m_k}, x_{n_k}), d(x_{m_k}, Sx_{m_k}), d(x_{n_k}, Tx_{n_k}), \\ &\qquad \frac{d(x_{n_k}, Sx_{m_k}) + d(x_{m_k}, Tx_{n_k})}{2}\} \\ &= \max\{d(x_{m_k}, x_{n_k}), d(x_{m_k}, x_{m_k+1}), d(x_{n_k}, x_{n_k+1}), \\ &\qquad \frac{d(x_{n_k}, x_{m_k+1}) + d(x_{m_k}, x_{n_k+1})}{2}\}. \end{aligned}$$

Similar to the proof of Case i, we see that this case is impossible.

Thus we proved the Claim.

Step 3. We prove that $\{x_n\}$ is a Cauchy sequence.

Let $\varepsilon > 0$ be given. Using the Claim, we find $N_1 \in \mathbb{N}$ such that if $m > n > N_1$ with $m - n \equiv 1 \mod 2$ then

$$d(x_n, x_m) < \frac{\varepsilon}{2}.$$

On the other hand, Using (2.3), we also find $N_2 \in \mathbb{N}$ such that for any $n > N_2$ we have

$$d(x_n, x_{n+1}) < \frac{\varepsilon}{2}.$$

Let $m, n > N = \max\{N_1, N_2\}$ with $m > n$. We get the following two cases.

(1) If $m - n \equiv 1 \mod 2$, then

$$d(x_m, x_n) < \frac{\varepsilon}{2} < \varepsilon.$$

(2) If $m - n \equiv 0 \mod 2$, then

$$\begin{aligned} d(x_m, x_n) &\le d(x_m, x_{n+1}) + d(x_{n+1}, x_n) \\ &< \frac{\varepsilon}{2} \\ &< \varepsilon. \end{aligned}$$

Thus we obtain $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists $z \in X$ such that

$$\lim_{n\to\infty} d(x_n, z) = 0.$$

Step 4. We prove that z is a common fixed point of T and S.

We claim that

$$\frac{1}{2}d(x_{2n}, Sx_{2n}) \le d(x_{2n}, z) \quad \text{or} \quad \frac{1}{2}d(x_{2n+1}, Tx_{2n+1}) \le d(x_{2n+1}, z).$$

Otherwise, we have

$$\begin{aligned} d(x_{2n}, x_{2n+1}) &\leq d(x_{2n}, z) + d(z, x_{2n+1}) \\ &< \frac{1}{2}d(x_{2n}, x_{2n+1}) + \frac{1}{2}d(x_{2n+1}, x_{2n+2}) \\ &< \frac{1}{2}d(x_{2n}, x_{2n+1}) + \frac{1}{2}d(x_{2n}, x_{2n+1}) \\ &= d(x_{2n}, x_{2n+1}). \end{aligned}$$

This is a contraction. Then there exists a subsequence $\{n_k\}$ of $\{n\}$ such that

$$\frac{1}{2}d(x_{2n_k}, Sx_{2n_k}) \leq d(x_{2n_k}, z) \quad \text{or} \quad \frac{1}{2}d(x_{2n_k+1}, Tx_{2n_k+1}) \leq d(x_{2n_k+1}, z).$$

We cosider the following two cases.

Case i. If $\frac{1}{2}d(x_{2n_k}, Sx_{2n_k}) \leq d(x_{2n_k}, z)$, then, by (2.1), we get

$$\psi(d(x_{2n_k+1}, Tz)) = \psi(d(Sx_{2n_k}, Tz)) \leq \psi(m(x_{2n_k}, z)) - \varphi(m(x_{2n_k}, z)), \tag{2.12}$$

where

$$\begin{aligned} m(x_{2n_k}, z) &= \max\{d(x_{2n_k}, z), d(x_{2n_k}, Sx_{2n_k}), d(z, Tz), \frac{d(x_{2n_k}, Tz) + d(Sx_{2n_k}, z)}{2}\} \\ &= \max\{d(x_{2n_k}, z), d(x_{2n_k}, x_{2n_k+1}), d(z, Tz), \frac{d(x_{2n_k}, Tz) + d(x_{2n_k+1}, z)}{2}\}. \end{aligned}$$

Then

$$\lim_{k\to\infty} m(x_{2n_k}, z) = \max\{d(z, z), 0, d(z, Tz), \frac{d(z, Tz) + d(z, z)}{2}\} = d(z, Tz).$$

Taking upper limits as $k \to \infty$ in (2.12),

$$\psi(d(z, Tz)) \leq \psi(d(z, Tz)) - \varphi(d(z, Tz)),$$

which implies $d(z, Tz) = 0$ and $Tz = z$.

Notice that

$$\frac{1}{2}d(z, Tz) = \frac{1}{2}d(z, z) = 0 \leq d(z, z).$$

Using (2.1), we get

$$\begin{aligned} \psi(d(Sz, z)) = \psi(d(Sz, Tz)) &\leq \psi(m(z, z)) - \varphi(m(z, z)) \\ &= \psi(d(z, Sz)) - \varphi(d(z, Sz)), \end{aligned}$$

which implies $d(z, Sz) = 0$ and $Sz = z$. Thus, in this case, we obtain that z is a common fixed point of T and S.

Case ii. If $\frac{1}{2}d(x_{2n_k+1}, Tx_{2n_k+1}) \leq d(x_{2n_k+1}, z)$, then, by (2.1), we get

$$\psi(d(x_{2n_k+2}, Sz)) = \psi(d(Sz, Tx_{2n_k+1})) \leq \psi(m(z, x_{2n_k+1})) - \varphi(m(z, x_{2n_k+1})), \tag{2.13}$$

where

$$\begin{aligned} m(z, x_{2n_k+1}) &= \max\{d(z, x_{2n_k+1}), d(z, Sz), d(x_{2n_k+1}, Tx_{2n_k+1}), \\ &\qquad \frac{d(z, Tx_{2n_k+1}) + d(x_{2n_k+1}, Sz)}{2}\} \\ &= \max\{d(z, x_{2n_k+1}), d(z, Sz), d(x_{2n_k+1}, x_{2n_k+2}), \\ &\qquad \frac{d(z, x_{2n_k+2}) + d(x_{2n_k+2}, Sz)}{2}\}. \end{aligned}$$

Similar to the proof of Case i, we conclude that $Tz = Sz = z$.

Step 5. We prove the uniqueness of the common fixed point z.

Suppose that y is another common fixed point of T and S. Then

$$\frac{1}{2}d(z,Tz) = 0 \leq d(y,z)$$

implies

$$\begin{aligned}\psi(d(y,z)) &= \psi(d(Sy,Tz)) \leq \psi(m(y,z)) - \varphi(m(y,z)) \\ &= \psi(d(y,z)) - \varphi(d(y,z)).\end{aligned}$$

This leads to $d(y,z) = 0$ and $y = z$.

This completes the proof of Theorem 2.1. □

If we take $S = T$ in Theorem 2.1, we obtain the following result.

Corollary 2.2. [19, Theorem 2.1] *Let X be a complete metric space and $T : X \to X$ such that for every $x, y \in X$,*

$$\frac{1}{2}d(x,Tx) \leq d(x,y) \text{ implies } \psi(d(Tx,Ty)) \leq \psi(m(x,y)) - \varphi(m(x,y)),$$

where ψ and φ are defined as in Theorem 2.1, and

$$m(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty) + d(y,Tx)}{2}\}.$$

Then T has a unique fixed point.

From Theorem 2.1, we obtain the following result which is a generalization of [11, Theorem 2.2].

Corollary 2.3. *Let X be a complete metric space and $S, T : X \to X$ such that for every $x, y \in X$,*

$$\psi(d(Sx,Ty)) \leq \psi(m(x,y)) - \varphi(m(x,y)), \tag{2.14}$$

where ψ, φ and $m(x,y)$ are defined as in Theorem 2.1. Then S and T have a unique common fixed point.

The next is an example which can apply Theorem 2.1 but not Corollary 2.3.

Example 2.4. *Let $X = \{(1,1,1), (4,1,0), (1,4,0), (4,5,1), (5,4,1)\}$ be endowed with the usual metric. Set*

$$a = (1,1,1), b = (4,1,0), c = (1,4,0), d = (4,5,1), e = (5,4,1).$$

Define $S : X \to X$ and $T : X \to X$ by

$$Sa = Sb = Sc = a,\ Sd = b,\ Se = c;\ Ta = Tb = Tc = a,\ Td = c,\ Te = b.$$

Then T does not satisfy the condition (2.14) of Corollary 2.3 at $x = d, y = d$ and $x = e, y = e$. Choose $\psi(t) = \frac{5}{6}t$ and $\varphi(t) = \frac{1}{6}t$. It is easily verified that T satisfies all all conditions of Theorem 2.1. In fact, $a = (1,1,1)$ is a unique common fixed point of S and T.

Acknowledgments. The authors would like to thank the anonymous referee for his/her comments that helped us improve this article.

References

[1] A. Al-Rawashdeh, H. Aydi, A. Felhi, S. Sahmim, W. Shatanawi, *On common fixed points for $\alpha - F$-contractions and applications*, J. Nonlinear Sci. Appl. **9** (5) (2016), 3445–3458.

[2] H. Aydi, *α-implicit contractive pair of mappings on quasi b-metric spaces and an application to integral equations*, Journal of Nonlinear and Convex Analysis, **17** (12) (2016), 2417–2433.

[3] H. Aydi, *On common fixed point theorems for (ψ, φ)-generalized f-weakly contractive mappings*, Miskolc Mathematical Notes, **14** (1) (2013), 19–30.

[4] H. Aydi, M. Jellali, E. Karapinar, *On fixed point results for α-implicit contractions in quasi-metric spaces and consequences*, Nonlinear Analysis: Modelling and Control, **21** (1) (2016), 40–56.

[5] H. Aydi, E. Karapinar, C. Vetro, *On Ekeland's variational principle in partial metric spaces*, Applied Mathematics and Information Sciences, **9** (1) (2015), 1–6.

[6] H. Aydi, E. Karapinar, D. Zhang, *On common fixed points in the context of Brianciari metric spaces*, Results of Mathematics, **71** (1) (2017), 73–92.

[7] H. Aydi, D. Zhang, *Refinements of Caristi's fixed point theorem*, J. Nonlinear Sci. Appl. **9** (6) (2016), 4084–4097.

[8] Lj. B. Ćirić, *A generalization of Banach's contraction principle*, Proc. Amer. Math. Soc. **45** (1974) 267–273.

[9] K. M. Das and K. V. Naik, *Common fixed point theorems for commuting maps on a metric space*, Proc. Amer. Math. Soc. **77** (1979) 369–373.

[10] C. Di Bari and P. Vetro, *Nonlinear quasi-contractions of Ćirić type*, Fixed Point Theory. **13** (2012) 453–459.

[11] D. Dorić, *Common fixed point for generalized (ψ, φ)-weak contraction*, Appl. Math. Letters **22** (2009) 1896-1900.

[12] D. Dorić and R. Lazović, *Some Suzuki-type fixed point theorems for generalized multivalued mappings and applications*, Fixed Point Theory Appl. **2011** (2011) 13 pages.

[13] A. Felhi, H. Aydi, D. Zhang, *Fixed points for α-admissible contractive mappings via simulation functions*, J. Nonlinear Sci. Appl. **9** 10 (2016), 5544–5560.

[14] F. He, *Common fixed points for nonlinear quasi-contractions of Ćirić type*, Abstract Appl. Anal. **2014** (2014) 9 pages.

[15] F. He and X. Y. Nan, *A unified view on common fixed point theorems for Ćirić quasi-contraction maps*, Fixed Point Theory Appl. **2015** (2015) 17 pages.

[16] D. Ilić and V. Rakočević, *Common fixed points for maps on metric space with w-distance*, Appl. Math. Comput. **199** (2008) 599-610.

[17] K.P.R. Rao, K.P.K. Rao, H. Aydi, *A Suzuki type unique common fixed point theorem for hybrid pairs of maps under a new condition in partial metric spaces*, Mathematical Sciences, **7** 2013: 51.

[18] S. L. Singh, Renu Chugh and Raj Kamal, *Suzuki type common fixed point theorems and applications*, Fixed Point Theory **14**(2) (2013) 497–506.

[19] S. L. Singh, Raj Kamal, M. De la Sen and Renu Chugh, *A fixed point theorem for generalized weak contractions*, Filomat **29**(7) (2015) 1481-1490.

[20] T. Suzuki, *A generalized Banach contraction principle that Characterizes metric completeness*, Proc. Amer. Math. Soc. **136** (2008) 1861–1869.

[21] T. Suzuki, *A new type of fixed point theorem in metric spaces*, Nonlinear Anal. **71**(11) (2009) 5313-5317.

[22] J. S. Ume, *Fixed point theorems related to Ćirić contraction principle*, J. Math. Anal. Appl. **225** (1998) 630–640.

[23] F. Vetro and S. Radenović, *Nonlinear ψ-quasi-contractions of Ćirić-type in Partial metric space*, Appl. Math. Comput. **219** (2012) 1594-1600.

[24] Q. Zhang and Y. Song, *Fixed point theory for generalized φ-weak contractions*, Appl. Math. Letters **22** (2009) 75–78.

Fei He*

School of Mathematical Sciences, Inner Mongolia University, Hohhot 010021, China

E-mail address: * Corresponding author: hefei@imu.edu.cn

Yu-Qi Sun
School of Mathematical Sciences, Inner Mongolia University, Hohhot 010021, China
E-mail address: sunyuqi00@163.com

Xiao-Yue Zhao
School of Mathematical Sciences, Inner Mongolia University, Hohhot 010021, China
E-mail address: xiaoyuezhao12300@163.com