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STABILITY OF N-BI-JORDAN HOMOMORPHISMS ON BANACH ALGEBRAS

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Abstract. In this paper, we investigate the stability, in the sense of Hyers-Ulam-Rassias, of n-bi-Jordan homomorphisms on Banach algebras. Also we show that to each approximate bi-Jordan homomorphism φ from a Banach algebra into a semisimple commutative Banach algebra there corresponds a unique bi-ring homomorphism near to φ.

1. Introduction

Let X be real normed space and Y be real Banach space. S. M. Ulam [12] posed the problem: When does a linear mapping near an approximately additive mapping $f : X \longrightarrow Y$ exist?

In 1941, Hyers [6] gave an affirmative answer to the question of Ulam for additive Cauchy equation in Banach space.

Let X and Y be two Banach spaces and let $f : X \longrightarrow Y$ be a mapping satisfying:

$$\|f(x + y) - f(x) - f(y)\| \leq \varepsilon,$$

for all $x, y \in X$ and $\varepsilon > 0$. Then there is a unique additive mapping $F : X \longrightarrow Y$ which satisfies

$$\|F(x) - f(x)\| \leq \varepsilon,$$

for all $x \in X$.

Th. M. Rassias [8] considered a generalized version of the Hyers's result which permitted the Cauchy difference to become unbounded. That is, he proved:

Theorem 1.1. *Let X and Y be two real Banach spaces, $\varepsilon \geq 0$ and $0 \leq p < 1$. If a mapping $f : X \longrightarrow Y$ satisfies*

$$\|f(x + y) - f(x) - f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p),$$

for all $x, y \in X$, then there is a unique additive mapping $F : X \longrightarrow Y$ such that

$$\|F(x) - f(x)\| \leq \frac{2\varepsilon}{|2 - 2^p|}\|x\|^p,$$

for all $x \in X$. If, in addition, for each fixed $x \in X$ the function $t \longmapsto f(tx)$ is continuous in $t \in \mathbb{R}$, then F is linear.

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This result is called the Hyers-Ulam-Rassias stability of the additive Cauchy equation. In [3], Gajda proved that Theorem 1.1 is valid for $p > 1$, which was raised by Rassias [9]. Th. M. Rassias and P. Semrl [10] independently obtained a different example. He also gave an example showing that a similar result to the above does not hold for $p = 1$. If $p < 0$, then $\|x\|^p$ is meaningless for $x = 0$; in this case, if we assume that $\|0\|^p = \infty$, then the proof given in [8] also works for $x \neq 0$. Thus, the Hyers-Ulam-Rassias stability of the additive Cauchy equation holds for $p \in \mathbb{R} \setminus \{1\}$.

An additive mapping $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ between Banach algebras is called n-Jordan homomorphism if

$$\varphi(a^n) = \varphi(a)^n, \qquad a \in \mathcal{A}.$$

If $n = 2$, then φ is called simply a Jordan homomorphism. The concept of n-Jordan homomorphism was studied by Eshaghi Gordji in [4]. See also [13], [14] and [15], for characterization of Jordan and 3-Jordan homomorphism.

Badora [1] proved the Hyers-Ulam-Rassias stability of ring homomorphisms, which generalizes the result of Bourgin [2]. The Hyers-Ulam-Rassias stability of Jordan homomorphisms investigated by Miura et al. [7], and it is extended to n-Jordan homomorphisms in [5] and [11].

Let $\mathcal{A}$ and $\mathcal{B}$ be a two normed (Banach) algebra and set $\mathcal{U} = \mathcal{A} \times \mathcal{B}$. Then $\mathcal{U}$ is a normed (Banach) algebra for the multiplication

$$(a, b)(x, y) = (ax, by), \qquad (a, b), (x, y) \in \mathcal{U},$$

and with norm

$$\|(a, b)\| = \|a\| + \|b\|.$$

Let $\mathcal{D}$ be a normed algebra and let $\varphi : \mathcal{U} \longrightarrow \mathcal{D}$ be a map. Then we say that φ is bi-additive, if

$$\varphi(a + x, b + y) = \varphi(a, b) + \varphi(x, y), \qquad (a, b), (x, y) \in \mathcal{U},$$

and it is called n-bi-multiplicative, if

$$\varphi(x_1x_2...x_n, y_1y_2...y_n) = \varphi(x_1, y_1)\varphi(x_2, y_2)...\varphi(x_n, y_n),$$

for all $(x_i, y_i) \in \mathcal{U}$. If φ is bi-additive and n-bi-multiplicative, then it is called n-bi-ring homomorphism. A 2-bi-ring homomorphism is just a bi-ring homomorphism, in the usual sense. We say that a bi-additive mapping $\varphi : \mathcal{U} \longrightarrow \mathcal{D}$ is an n-bi-Jordan homomorphisms if φ satisfies

$$\varphi(x^n, y^n) = \varphi(x, y)^n, \qquad (x, y) \in \mathcal{U}.$$

A 2-bi-Jordan homomorphism is called simply a bi-Jordan homomorphism. It is obvious that each n-bi-ring homomorphism is an n-bi-Jordan homomorphism, but in general the converse is false.

In this paper, we first prove that each bi-Jordan homomorphism $\varphi : \mathcal{U} \longrightarrow \mathcal{D}$, where $\mathcal{D}$ is semisimple commutative Banach algebra, is a bi-ring homomorphism and then we applying this fact to prove that to each approximate bi-Jordan homomorphism $\varphi : \mathcal{U} \longrightarrow \mathcal{D}$ there corresponds a unique bi-ring homomorphism near to φ. We also obtain the same result for 3-bi-Jordan homomorphisms, with the additional hypothesis that the Banach algebra $\mathcal{U}$ is unital.

2. Main Results

We commence with a characterization of bi-Jordan homomorphism.

Theorem 2.1. *Suppose that $\mathcal{U}$ is a Banach algebra, which need not be commutative and suppose $\mathcal{D}$ is a semisimple commutative Banach algebra. Then each bi-Jordan homomorphism $\varphi : \mathcal{U} \longrightarrow \mathcal{D}$ is a bi-ring homomorphism.*

Proof. We first assume that $\mathcal{D} = \mathbb{C}$ and let $\varphi : \mathcal{U} \longrightarrow \mathbb{C}$ be bi-Jordan homomorphism. Then $\varphi(a^2, b^2) = \varphi(a, b)^2$, for all $(a, b) \in \mathcal{U}$. Replacing a by $a + x$, and b by $b + y$ we get

$$\varphi(ax + xa, by + yb) = 2\varphi(a, b)\varphi(x, y), \qquad (a, b), (x, y) \in \mathcal{U}. \tag{2.1}$$

Take $u = ax + xa$ and $v = by + yb$. Then by (2.1), we have

$$\begin{aligned} 2\varphi(axa, byb) &= \varphi[au + ua, bv + vb] - \varphi[a^2x + xa^2, b^2y + yb^2] \\ &= 2[\varphi(a, b)\varphi(u, v) - \varphi(a^2, b^2)\varphi(x, y)] \\ &= 2[2\varphi(a, b)^2\varphi(x, y) - \varphi(a, b)^2\varphi(x, y)] \\ &= 2\varphi(a, b)^2\varphi(x, y). \end{aligned}$$

Therefore

$$\varphi(axa, byb) = \varphi(a, b)^2\varphi(x, y), \qquad (a, b), (x, y) \in \mathcal{U}. \tag{2.2}$$

Let (a, b) and (x, y) be arbitrarily elements of $\mathcal{U}$, and put

$$2t = \varphi(ax, by) - \varphi(xa, yb). \tag{2.3}$$

It follows from (2.1) and (2.3) that

$$\varphi(ax, by) - t = \varphi(a, b)\varphi(x, y), \qquad \varphi(xa, yb) + t = \varphi(a, b)\varphi(x, y). \tag{2.4}$$

Since φ is bi-additive, we get $2t = \varphi(ax - xa, by - yb)$. So (2.2)-(2.4), gives

$$\begin{aligned} 4t^2 &= \varphi(ax - xa, by - yb)^2 \\ &= \varphi[(ax)^2 + (xa)^2 - ax^2a - xa^2x, (by)^2 + (yb)^2 - by^2b - yb^2y] \\ &= [\varphi(ax, by)^2 + \varphi(xa, yb)^2 - \varphi(a, b)^2\varphi(x^2, y^2) - \varphi(x, y)^2\varphi(a^2, b^2)] \\ &= [t + \varphi(a, b)\varphi(x, y)]^2 + [\varphi(a, b)\varphi(x, y) - t]^2 - [2\varphi(a, b)^2\varphi(x, y)^2] \\ &= 2t^2. \end{aligned}$$

Hence $t = 0$, which proves $\varphi(ax, by) = \varphi(a, b)\varphi(x, y)$, for all $(a, b), (x, y) \in \mathcal{U}$. Thus, φ is bi-ring homomorphism.

Now suppose that $\mathcal{D}$ is semisimple and commutative. Let $\mathfrak{M}(\mathcal{D})$ be the maximal ideal space of $\mathcal{D}$. We associate with each $f \in \mathfrak{M}(\mathcal{D})$ a function $\varphi_f : \mathcal{U} \longrightarrow \mathbb{C}$ defined by

$$\varphi_f(a, b) := f(\varphi(a, b)), \qquad (a, b) \in \mathcal{U}.$$

Pick $f \in \mathfrak{M}(\mathcal{D})$ arbitrary. It is easy to see that φ_f is a bi-Jordan homomorphism, so by the above arqument it is a bi-ring-homomorphism. Thus, by the definition of φ_f we have

$$f(\varphi(ax, by)) = f(\varphi(a, b))f(\varphi(x, y)) = f(\varphi(a, b)\varphi(x, y)).$$

Since $f \in \mathfrak{M}(\mathcal{D})$ was arbitrary and $\mathcal{D}$ is assumed to be semisimple, we obtain

$$\varphi(ax, by) = \varphi(a, b)\varphi(x, y),$$

for all $(a, b), (x, y) \in \mathcal{U}$, and the proof is complete. $\square$

A mapping $\varphi : \mathcal{U} \longrightarrow \mathcal{D}$ is called co-bi-ring homomorphism if φ is bi-additive and satisfies

$$\varphi(ax, by) = -\varphi(a, b)\varphi(x, y), \qquad (a, b), (x, y) \in \mathcal{U},$$

and it is called co-bi-Jordan homomorphism, if

$$\varphi(x^2, y^2) = -\varphi(x, y)^2, \qquad (x, y) \in \mathcal{U}.$$

Note that Theorem 2.1 is also valid for co-bi-Jordan homomorphism. It is obvious to check that, if $\mathcal{U}$ is a unital, and $\varphi : \mathcal{U} \longrightarrow \mathbb{C}$ is a non-zero 3-bi-Jordan homomorphism, then $\varphi(e, e) \neq 0$, where (e, e) is the unit element of $\mathcal{U}$.

Lemma 2.2. *Let $\mathcal{U}$ be a unital Banach algebra and $\varphi : \mathcal{U} \longrightarrow \mathbb{C}$ be a non-zero 3-bi-Jordan homomorphism. Then either φ is bi-Jordan or co-bi-Jordan homomorphism.*

Proof. By assumpition for all $(a, b) \in \mathcal{U}$,

$$\varphi(a^3, b^3) = \varphi(a, b)^3. \tag{2.5}$$

Replace a by $a + e$ and b by $b + e$ in (2.5), to obtain

$$3\varphi(a^2, b^2) + 3\varphi(a, b) + \varphi(e, e) = 3\varphi(a, b)^2\varphi(e, e) + 3\varphi(a, b)\varphi(e, e)^2 + \varphi(e, e)^3. \tag{2.6}$$

Replacing (a, b) by (e, e) in (2.5), we get $\varphi(e, e) = \varphi(e, e)^3$. Since $\varphi(e, e) \neq 0$, thus $\varphi(e, e) = 1$ or $\varphi(e, e) = -1$. If $\varphi(e, e) = 1$, then by (2.6) we get

$$\varphi(a^2, b^2) = \varphi(a, b)^2,$$

hence φ is bi-Jordan. If $\varphi(e, e) = -1$, then by (2.6) we have

$$\varphi(a^2, b^2) = -\varphi(a, b)^2,$$

so φ is co-bi-Jordan homomorphism, as claimed. □

Theorem 2.3. *Suppose that $\mathcal{U}$ is a unital Banach algebra and $\mathcal{D}$ is a semisimple commutative Banach algebra. Then each 3-bi-Jordan homomorphism $\varphi : \mathcal{U} \longrightarrow \mathcal{D}$ is a 3-bi-ring homomorphism.*

Proof. We first let $\mathcal{D} = \mathbb{C}$ and let $\varphi : \mathcal{U} \longrightarrow \mathbb{C}$ be 3-bi-Jordan homomorphism, then by Lemma 2.2, φ is bi-Jordan or co-bi-Jordan homomorphism. If φ is bi-Jordan, then by Theorem 2.1 φ is bi-ring homomorphism and so it is 3-bi-ring homomorphism. If φ is co-bi-Jordan, then by Theorem 2.1 it is co-bi-ring homomorphism. That is,

$$\varphi(ax, by) = -\varphi(a, b)\varphi(x, y), \qquad (a, b), (x, y) \in \mathcal{U}.$$

Hence

$$\varphi(axs, byt) = -\varphi(a, b)\varphi(xs, yt) = -\varphi(a, b)[-\varphi(x, y)\varphi(s, t)] = \varphi(a, b)\varphi(x, y)\varphi(s, t),$$

for all $(a, b), (x, y)(s, t) \in \mathcal{U}$. Therefore, φ is 3-bi-ring homomorphism. The rest of proof is completely similar to the Theorem 2.1. □

Now we consider the Hyers-Ulam-Rassias stability of n-bi-Jordan homomorphisms on Banach algebras.

By a same method of the proofs given in [8] and [3], we can prove the following Theorem.

Theorem 2.4. *Let $\mathcal{U}$ and $\mathcal{D}$ be two Banach spaces, $\epsilon \geq 0$ and $p \in \mathbb{R} \setminus \{1\}$. If a mapping $\varphi : \mathcal{U} \longrightarrow \mathcal{D}$ satisfies*

$$\|\varphi(a+x, b+y) - \varphi(a,b) - \varphi(x,y)\| \leq \varepsilon\big(\|(a,b)\|^p + \|(x,y)\|^p\big),$$

for all $(a,b),(x,y) \in \mathcal{U}$, then there is a unique bi-additive mapping $F : \mathcal{U} \longrightarrow \mathcal{D}$ such that

$$\|F(x,y) - \varphi(x,y)\| \leq \frac{2\varepsilon}{|2-2^p|}\|(x,y)\|^p,$$

for all $(x,y) \in \mathcal{U}$.

Theorem 2.5. *Let $\mathcal{U}$ be a normed algebra, let $\mathcal{D}$ be a Banach algebra, let δ and ε be nonnegative real numbers, and let p,q be a real numbers such that $(p-1)(q-1) > 0$, $q \geq 0$. Assume that $\varphi : \mathcal{U} \longrightarrow \mathcal{D}$ satisfies*

$$\|\varphi(a+x, b+y) - \varphi(a,b) - \varphi(x,y)\| \leq \varepsilon\big(\|(a,b)\|^p + \|(x,y)\|^p\big), \tag{2.7}$$

$$\|\varphi(x^n, y^n) - \varphi(x,y)^n\| \leq \delta\|(x,y)\|^{nq}, \tag{2.8}$$

for all $(a,b),(x,y) \in \mathcal{U}$. Then, there exists a unique n-bi-Jordan homomorphism $F : \mathcal{U} \longrightarrow \mathcal{D}$ such that

$$\|F(x,y) - \varphi(x,y)\| \leq \frac{2\varepsilon}{|2-2^p|}\|(x,y)\|^p, \tag{2.9}$$

for all $(x,y) \in \mathcal{U}$.

Proof. Put $t := -sgn(p-1)$ and

$$F(x,y) = \lim_m \frac{1}{2^{tm}}\varphi(2^{tm}x, 2^{tm}y),$$

for all $(x,y) \in \mathcal{U}$. It follows from Theorem 2.4 that F is bi-additive map satisfies (2.9). We will show that F is n-bi-Jordan homomorphism. We have

$$\begin{aligned}
\lim_m \frac{1}{2^{tmn}}\big(\|\varphi(2^{tmn}x^n, 2^{tmn}y^n) - \varphi(2^{tm}x, 2^{tm}y)^n\|\big)& \\
\leq \lim_m \frac{\delta}{2^{tmn}}\|(2^{tm}x, 2^{tm}y)\|^{nq}& \\
\leq \lim_m \frac{\delta}{2^{tmn}}2^{tmnq}\|(x,y)\|^{nq}& \\
= \lim_m 2^{tmn(q-1)}(\delta\|(x,y)\|^{nq}) = 0.&
\end{aligned}$$

Thus, we get

$$\begin{aligned}
F(x^n, y^n) &= \lim_m \frac{1}{2^{tmn}}\varphi(2^{tmn}x^n, 2^{tmn}y^n) \\
&= \lim_m \frac{1}{2^{tmn}}\{\varphi(2^{tmn}x^n, 2^{tmn}y^n) - \varphi(2^{tm}x, 2^{tm}y)^n + \varphi(2^{tm}x, 2^{tm}y)^n\} \\
&= F(x,y)^n.
\end{aligned}$$

So F is n-bi-Jordan homomorphism. The uniqueness of F follows from preceding Theorem. □

Theorem 2.6. *Let $\mathcal{U}$ be a normed algebra, let $\mathcal{D}$ be a Banach algebra, let δ and ε be nonnegative real numbers, and let p,q be a real numbers such that $(p-1)(q-1) > 0$, and $q < 0$. Assume that $\varphi : \mathcal{U} \longrightarrow \mathcal{D}$ be a mapping with $\varphi(0,0) = 0$, such that* *the inequalities (2.7) and (2.8) are hold. Then, there exists a unique n-bi-Jordan homomorphism $F : \mathcal{U} \longrightarrow \mathcal{D}$ such that*

$$\|F(x,y) - \varphi(x,y)\| \leq \frac{2\varepsilon}{|2-2^p|}\|(x,y)\|^p, \tag{2.10}$$

for all $(x,y) \in \mathcal{U}$.

Proof. It follows from Theorem 2.4 that there exists a bi-additive map $F : \mathcal{U} \longrightarrow \mathcal{D}$ satisfies (2.10), where we assume that $\|(0,0)\|^p = \infty$. We show that

$$F(x^n, y^n) = F(x,y)^n,$$

for all $(x,y) \in \mathcal{U}$. Since F is bi-additive, we have $F(0,0) = 0$. Hence the result is valid for $(x,y) = (0,0)$. Suppose that $(x,y) \in \mathcal{U}\backslash(0,0)$ be arbitrarily. There are now two possibilities. Either $(x^n, y^n) = (0,0)$ or $(x^n, y^n) \neq (0,0)$. If $(x^n, y^n) \neq (0,0)$, then the proof of Theorem 2.5 works well, and so $F(x^n, y^n) = F(x,y)^n$. Now let $(x^n, y^n) = (0,0)$. It follows from (2.8), with the hypothesis $\varphi(0,0) = 0$, that

$$\frac{1}{2^{mn}}\|\varphi(2^m x, 2^m y)^n\| \leq \frac{1}{2^{mn}}\|(2^m x, 2^m y)\|^{nq} = 2^{mn(q-1)}\delta\|(x,y)\|^{nq}. \tag{2.11}$$

Since $(x,y) \neq (0,0)$, and $(q-1) < 0$ we get

$$\lim_m \frac{1}{2^{mn}}\varphi(2^m x, 2^m y)^n = 0. \tag{2.12}$$

On the other hand, we have

$$F(x,y) = \lim_m \frac{1}{2^m}\varphi(2^m x, 2^m y), \quad (x,y) \in \mathcal{U}. \tag{2.13}$$

Thus, by (2.12) and (2.13) we get

$$F(x,y)^n = \lim_m \{\frac{1}{2^{mn}}\varphi(2^m x, 2^m y)^n\} = 0,$$

which implies that $F(x,y)^n = 0 = F(x^n, y^n)$, whenever $(x^n, y^n) = (0,0)$. This complete the proof. ☐

As a consequence of Theorem 2.1, 2.5, and 2.6 we have the following.

Corollary 2.7. *Suppose that $\mathcal{U}$ is a Banach algebra, and suppose $\mathcal{D}$ is a semisimple commutative Banach algebra. Let δ and ε be nonnegative real numbers, and let p, q be a real numbers such that $(p-1)(q-1) > 0$, $q \geq 0$ or $(p-1)(q-1) > 0$, $q < 0$ and $\varphi(0,0) = 0$. Assume that $\varphi : \mathcal{U} \longrightarrow \mathcal{D}$ satisfies (2.7) and (2.8). Then, there exists a unique bi-ring homomorphism $F : \mathcal{U} \longrightarrow \mathcal{D}$ such that*

$$\|F(x,y) - \varphi(x,y)\| \leq \frac{2\varepsilon}{|2-2^p|}\|(x,y)\|^p, \tag{2.14}$$

for all $(x,y) \in \mathcal{U}$.

From Theorem 2.3, 2.5, and 2.6 we get the next result.

Corollary 2.8. *By hypotheses of above Corollary if $\mathcal{U}$ is unital, then there exists a unique 3-bi-ring homomorphism $F : \mathcal{U} \longrightarrow \mathcal{D}$ such that satisfies (2.14).*

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