ON THE VOLKENBORN INTEGRAL OF THE $q$-EXTENSION OF THE $p$-ADIC GAMMA FUNCTION

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Abstract. In the present work we consider the $q$-extension of the $p$-adic gamma function. We derive the Volkenborn integral of the $q$-extension of the $p$-adic gamma function by using its Mahler expansion. Moreover, we give a new representation for the $q$-extension of the $p$-adic Euler constant.

1. Introduction

The $q$-Calculus appeared in the 18th century and it continues to develop rapidly. The $q$-calculus has a great interest and has been studied by Euler, Gauss who discovered $q$-binomial formula and others. The systematic development of the $q$-calculus began with FH Jackson in the early 20th century. Although the $q$-calculus has been studied for over a century, $q$-analogue of special numbers and polynomials are still of interest [1].

The letter $q$ stands for ‘quantum’, and the $q$-binomial coefficients play an important role in ‘quantum calculus’ similar to that of the ordinary binomial coefficients in ordinary calculus. Also, the binomial coefficients are also known combinations or combinatorial numbers. In constructing the properties and identities of some special numbers, binomial coefficients have great interest. The $q$-analogue of the binomial coefficients an important role play in developing the theory of the $q$-analogue of these special numbers [5].

The $q$-binomial coefficients or Gaussian polynomials appear in many identities on $q$-series. In addition, they are studied in several combinatorial environments as partitions of integers. For an easy handling of the $q$-binomial coefficients in combinatory it is essential to be familiar with the basic combinatorial structures that admit those coefficients as generating polynomials [9].

The $p$–adic numbers introduced by the German mathematician Kurt Hensel (1861–1941), are widely used in mathematics: in number theory, algebraic geometry, representation theory, algebraic and arithmetical dynamics, and cryptography. The $p$–adic numbers have been used applying fields with successfully applying in
superfield theory of $p$–adic numbers by V.S. Vladimirov and I. V. Volovich. In addition, the $p$–adic model of the universe, the $p$–adic quantum theory, the $p$–adic string theory such as areas occurred in physics (for detail see [24],[25]).

Throughout this paper, $p$ is a fixed odd prime number and by $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ we denote the ring of $p$–adic integers, the field of $p$–adic numbers and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively.

The $q$–analogue of $n$ called the following notation,

$$ (n)_q = \frac{q^n - 1}{q - 1} = q^{n-1} + q^{n-2} + ... + 1, \quad q \neq 1 $$

for any nonnegative integer $n$. As $q \to 1; (n)_q \to n$.

The $q$–binomial coefficient is a $q$–analogue for the binomial coefficient, also called a Gaussian coefficient or a Gaussian polynomial. A $q$-binomial coefficient, $\binom{n}{k}_q$, is defined by

$$ \binom{n}{k}_q = \frac{(n)_q!}{(n-k)_q! (k)_q!} $$

where $(k)_q!$ denotes the $q$–factorial which given by $(0)_q! = 1$ and $(k)_q! = (k)_q (k-1)_q ... (2)_q (1)_q$ when $k > 0$.

Another expression of the $q$-binomial coefficient for nonnegative integers $n, k$ with $n \geq k$ is defined by

$$ \binom{n}{k}_q = \frac{(q^n-1)(q^{n-1}-1)...(q^{n-k+1}-1)}{(q^k-1)(q^{k-1}-1)...(q-1)}. \quad (1.1) $$

It is clearly that when $q \to 1; \binom{n}{k}_q \to \binom{n}{k}$.

The $q$-binomial coefficients have $q$-Pascal rule;

$$ \binom{n}{k}_q = q^{n-k} \binom{n-1}{k-1}_q + \binom{n-1}{k}_q \quad (1.2) $$

where $1 \leq k \leq n - 1$ ([6], [11]).

In 1975 Y. Morita [17] defined the $p$–adic gamma function $\Gamma_p$ by the formula

$$ \Gamma_p(x) = \lim_{n \to x} (-1)^n \prod_{1 \leq j < n \atop (p,j)=1} j \quad (p \in \mathbb{Z}_p), $$

for $x \in \mathbb{Z}_p$, where $n$ approaches $x$ through positive integers.

The $p$–adic gamma function $\Gamma_p(x)$ has a great interest and has been studied by J. Diamond (1977) [7], D. Barsky (1977) [2], M. Boyarsky (1980) [3], B. Dwork (1983) [8], T. Kim (1997) [12] and others. The relationship between some special functions and the $p$–adic gamma function $\Gamma_p(x)$ was investigated by B. Gross and N. Koblitz (1979) [10], H. Cohen and E. Friedman (2008) [11] and I. Shapiro (2012) [21].

The $q$–extension of the $p$–adic gamma function $\Gamma_{p,q}(x)$ is defined by N. Koblitz [14] as follows:
Let \( q \in \mathbb{C}_p, |q - 1|_p < 1, q \neq 1 \). The \( q \)-extension of the \( p \)-adic gamma function \( \Gamma_{p,q}(x) \) is defined by formula
\[
\Gamma_{p,q}(x) = \lim_{n \to x} \left( -1 \right)^n \prod_{j<n, (p,j)=1} \frac{1-q^j}{1-q}.
\]
for \( x \in \mathbb{Z}_p \), where \( n \) approaches \( x \) through positive integers. We recall that \( \lim_{q \to 1} \Gamma_{p,q} = \Gamma_p \).

N. Koblitz determined the relation between the derivative of \( \Gamma_{p,q}(x) \) with the \( q \)-extension of the \( p \)-adic Euler constant \( \gamma_{p,q} \) is defined by the formula
\[
\Gamma'_{p,q}(0) = -\left( \frac{1}{1-p} \right) \gamma_{p,q}. \tag{1.3}
\]

The \( q \)-extension of the \( p \)-adic gamma function \( \Gamma_{p,q}(x) \) was studied by N. Koblitz (1980, 1982) \[14\], \[15\], H. Nakazato (1988) \[18\], Y. S. Kim (1998) \[13\] and others.

Conrad studied a \( q \)-analogue of Mahler expansions for continuous functions in \( p \)-adic analysis, replacing binomial coefficient polynomials \( \binom{n}{k} \) with a \( q \)-analogue \( \binom{n}{k}_q \) for a \( p \)-adic variable \( q \in \mathbb{C}_p \) with \( |q - 1|_p < 1 \) \[6\].

The Mahler coefficients \( \Gamma_{p,q} \) are determined by the following proposition:

**Proposition 1.1.** \[6\] Let
\[
\Gamma_{p,q}(x + 1) = \sum_{n=0}^{\infty} \tau_{p,q}(n) \binom{x}{n}_q \quad (x \in \mathbb{Z}_p) \quad \tag{1.4}
\]
be the Mahler series of \( \Gamma_{p,q} \) with \( |q - 1|_p < p^{-\frac{1}{p-1}} \) and \( q \in \mathbb{C}_p \) where \( \tau_{p,q}(n) \) be \( n \)th \( q \)-Mahler coefficient of the sequence \( \Gamma_{p,q}(n+1) \). The following equality
\[
\sum_{n \geq 0} (-1)^n \tau_{p,q}(n) \frac{X^n}{(n)_q}! = \frac{1-X^p}{1-X} \mathcal{E}_{1\setminus q}(x) \mathcal{E}_{q^p} \left( \frac{x^p}{(p)_q} \right) \quad \tag{1.5}
\]
and
\[
\mathcal{E}_{1\setminus q}(x) \mathcal{E}_{q^p} \left( \frac{x^p}{(p)_q} \right) = \sum_{n \geq 0} b_{p,q}(n) \frac{X^n}{(n)_q}!
\]
where \( \mathcal{E}_q(X) \) is the \( q \)-extension of the exponential series and also may denote a slightly different series holds.

The Volkenborn integral was introduced by A. Volkenborn in his PhD dissertation and subsequently in the set of twin papers \[22\], \[23\]; a more recent treatment of the subject can be found in \[20\].

The indefinite sum of a continuous function \( f : \mathbb{Z}_p \to \mathbb{C}_p \) is the continuous function \( Sf \) interpolating \( n \to \sum_{j=0}^{n-1} f(j) \ (n \in \mathbb{N}) \). Instead of \( Sf(x) \ (x \in \mathbb{Z}_p) \) we can write \( \sum_{j=0}^{x-1} f(j) \equiv \lim_{n \to x} \sum_{j=0}^{n-1} f(j) \) see \[19\] and \[20\]. The infinite sum of continuous function has below property
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\[
S_f(x+1) - S_f(x) = f(x).
\]  

(1.6)

Let \( f \) be a function from \( C^1(\mathbb{Z}_p \to \mathbb{C}_p) \). The Volkenborn integral of \( f \) on \( \mathbb{Z}_p \) is defined by the formula

\[
\int_{\mathbb{Z}_p} f(x) \, dx := \lim_{n \to \infty} p^{-n} \sum_{j=0}^{p^n-1} f(j) = (Sf)'(0).
\]

(1.7)

For any \( f \in C^1(\mathbb{Z}_p \to \mathbb{C}_p) \), the relation holds:

\[
\int_{\mathbb{Z}_p} f(x+1) \, dx - \int_{\mathbb{Z}_p} f(x) \, dx = f'(0).
\]

(1.8)

2. Main Results

To evaluate Volkenborn integral of the \( q \)-extension of the \( p \)-adic gamma function, the following equalities can be expressed:

**Lemma 2.1.** Let \(|q - 1|_p < 1\) with \( q \in \mathbb{C}_p \), \( n \in \mathbb{N} \) and \( f \in C(\mathbb{Z}_p \to \mathbb{C}_p) \) such that the \( q \)-Mahler expansion of \( f \), \( f(x) = \sum_{n=0}^{\infty} \tau_{p,q}(n) \binom{x}{n}_q \). Then, the indefinite of \( f \) is given by

\[
S_f(x) = \sum_{n=0}^{\infty} q^{n-x} \tau_{p,q}(n) \binom{x}{n+1}_q.
\]

**Proof.** Let \(|q - 1|_p < 1\) with \( q \in \mathbb{C}_p \), \( n \in \mathbb{N} \), \( x \in \mathbb{Z}_p \). Assume that

\[
S_f(x) = \sum_{n=0}^{\infty} \sigma_{p,q}(n) \binom{x}{n}_q.
\]

From (1.6), we get

\[
\sum_{n=0}^{\infty} \sigma_{p,q}(n) \binom{x+1}{n}_q - \sum_{n=0}^{\infty} \sigma_{p,q}(n) \binom{x}{n}_q = \sum_{n=0}^{\infty} \tau_{p,q}(n) \binom{x}{n+1}_q.
\]

By using (1.2), we have

\[
\sum_{n=0}^{\infty} \sigma_{p,q}(n) q^{x+n-1} \binom{x}{n-1}_q + \sum_{n=0}^{\infty} \sigma_{p,q}(n) \binom{x}{n}_q - \sum_{n=0}^{\infty} \tau_{p,q}(n) \binom{x}{n+1}_q = \sum_{n=0}^{\infty} \tau_{p,q}(n) \binom{x}{n}_q.
\]

or

\[
\sum_{n=0}^{\infty} \sigma_{p,q}(n) q^{x+n} \binom{x}{n-1}_q = \sum_{n=0}^{\infty} \tau_{p,q}(n) \binom{x}{n}_q.
\]

So we have

\[
\sum_{n=0}^{\infty} \sigma_{p,q}(n+1) q^{x+n} \binom{x}{n}_q = \sum_{n=0}^{\infty} \tau_{p,q}(n) \binom{x}{n}_q.
\]

We know that \( n \in \mathbb{N} \). Thus, we can write

\[
\sum_{n=0}^{\infty} \sigma_{p,q}(n+1) q^{x+n} \binom{x}{n}_q = \sum_{n=0}^{\infty} \tau_{p,q}(n) \binom{x}{n}_q.
\]
Hence, we have
\[ \sigma_{p,q}(n + 1)q^{x-n} = \tau_{p,q}(n) \quad n \in \mathbb{N} \]
or
\[ \sigma_{p,q}(n) = q^{-1}x\tau_{p,q}(n - 1) \quad n \in \mathbb{N}. \]
Then, we can write
\[ S f(x) = \sum_{n=0}^{\infty} q^{-n-x} \tau_{p,q}(n - 1) x \frac{n}{q}. \]
By some computing, we obtain
\[ S f(x) = \sum_{n=0}^{\infty} q^{-n-x} \tau_{p,q}(n) x \frac{n+1}{q}. \]

If we take \( f(x) = \left( \frac{x}{n} \right)_q \), we obtain the following results:

**Corollary 2.2.** If \( x \in \mathbb{Z}_p \) and \( q \in \mathbb{C}_p \) with \( |q - 1|_p < 1 \) then
\[ S \left( \frac{x}{n} \right)_q = q^{-n-x} \left( \frac{x}{n+1} \right)_q. \]

**Lemma 2.3.** For \( x, s \in \mathbb{Z}_p, n \in \mathbb{N} \) and \( q \in \mathbb{C}_p \) with \( |q - 1|_p < 1 \), the following identity:
\[ \left( \frac{x+s}{n+1} \right)_q = q^{x+s} \ln q \frac{\sum_{j=1}^{n-1} q^{-j} \left( (q^{x+s} - 1) \ldots (q^{x+s-1} - 1) \right) q}{(q^{n+1} - 1) \ldots (q - 1)} \]
is valid.

**Proof.** From \([1.1]\), we have
\[ \left( \frac{x+s}{n+1} \right)_q' = \left( \frac{(q^{x+s} - 1) \ldots (q^{x+s-n} - 1)}{(q^{n+1} - 1) \ldots (q - 1)} \right)'. \]
By applying the definition of the derivative, we can write
\[ \left( \frac{x+s}{n+1} \right)_q = \frac{\ln q^{x+s} (q^{x+s} - 1) \ldots (q^{x+s-n} - 1) + \ldots + \ln q^{x+s-n} (q^{x+s} - 1) \ldots (q^{x+s-n+1} - 1)}{(q^{n+1} - 1) \ldots (q - 1)}. \]
We easily obtain the following relation with a little rearranging
\[ \left( \frac{x+s}{n+1} \right)_q = \frac{\ln q}{(q^{n+1} - 1)} \left[ \sum_{j=1}^{n-1} q^{x+s-j} \left( \frac{(q^{x+s-j} - 1) \ldots (q^{x+s-n-j+1} - 1)}{(q^{n-j} - 1)} \right) q \right]. \]

**Theorem 2.4.** For \( x, s \in \mathbb{Z}_p, n \in \mathbb{N} \) and \( q \in \mathbb{C}_p \) with \( |q - 1|_p < 1 \), the Volkenborn integral of \( q \)-binomial coefficient is
\[ \int_{\mathbb{Z}_p} \left( \frac{x+s}{n} \right)_q dx = \ln q^{n-s} \left( \frac{s-1}{n} \right)_q \left[ 1 + \sum_{j=1}^{n} q^{s-j} (q^{s-j} - 1) \right]. \]
Using (1.1) we easily discover Corollary 2.5. From Proposition 1.1 and Corollary 2.5 we have

$$\int_{\mathbb{Z}_p} \left( \frac{x+s}{n} \right)_q dx = \left( q^{n-x-s} \left( \frac{x+s}{n+1} \right)_q \right) (0)$$

or

$$\int_{\mathbb{Z}_p} \left( \frac{x+s}{n} \right)_q dx = \left( -\ln q q^{n-x-s} \left( \frac{x+s}{n+1} \right)_q + q^{n-x-s} \left( \frac{x+s}{n+1} \right)_q \right) (0) \quad (2.1)$$

By using Lemma 2.3 and (1.7) and we can rewrite (2.1)

$$\int_{\mathbb{Z}_p} \left( \frac{x+s}{n} \right)_q dx = \ln q q^{n-s} \left[ \sum_{j=1}^{n-1} \frac{q^{s-j} (q^{s-k-1})}{(q^{n+1} - 1) (q^{n-k} - 1)} \left( s-j-1 \right)_q + \frac{q^s}{(q^{n+1} - 1)} \left( s-1 \right)_q \right]$$

or

$$\int_{\mathbb{Z}_p} \left( \frac{x+s}{n} \right)_q dx = \ln q q^{n-s} \left[ \sum_{j=1}^{n-1} \frac{q^{s-j} (q^{s-k-1})}{(q^{n+1} - 1) (q^{n-k} - 1)} \left( s-j-1 \right)_q \right]$$

Using (1.1) we easily discover

$$\int_{\mathbb{Z}_p} \left( \frac{x+s}{n} \right)_q dx = \ln q q^{n-s} \left[ \sum_{j=1}^{n-1} \frac{q^{s-j} (q^{s-k-1})}{(q^{n+1} - 1) (q^{n-k} - 1)} \left( s-j-1 \right)_q \right]$$

or

$$\int_{\mathbb{Z}_p} \left( \frac{x+s}{n} \right)_q dx = \ln q q^{n-s} \left[ \sum_{j=1}^{n-1} \frac{q^{s-j} (q^{s-k-1})}{(q^{n+1} - 1) (q^{n-k} - 1)} \left( s-j-1 \right)_q \right].$$

In the case $s = 0$ in Lemma 2.4, we obtain following corollary::

Corollary 2.5. For $x \in \mathbb{Z}_p, n \in \mathbb{N}$ and $q \in \mathbb{C}_p$ with $|q-1|_p < 1$, the relation holds:

$$\int_{\mathbb{Z}_p} \left( \frac{x}{n} \right)_q dx = \frac{(-1)^n \ln q}{q^{\frac{n(n-1)}{2}} (q^{n+1} - 1)}$$

Theorem 2.6. For all $x \in \mathbb{Z}_p, n \in \mathbb{N}$ and $|q-1|_p < p^{-\frac{1}{n-1}}$ with $q \in \mathbb{C}_p$, the following identity holds:

$$\int_{\mathbb{Z}_p} \Gamma_{p,q}(x+1)dx = \sum_{n=0}^{\infty} \tau_{p,q}(n) \frac{(-1)^n \ln q}{q^{\frac{n(n-1)}{2}} (q^{n+1} - 1)}$$

where $\tau_{p,q}(n)$ is defined by Proposition 1.1.

Proof. From Proposition 1.1 and Corollary 2.5 we have

$$\int_{\mathbb{Z}_p} \Gamma_{p,q}(x+1)dx = \int_{\mathbb{Z}_p} \sum_{n=0}^{\infty} \tau_{p,q}(n) \left( \frac{x}{n} \right)_q dx = \sum_{n=0}^{\infty} \tau_{p,q}(n) \int_{\mathbb{Z}_p} \left( \frac{x}{n} \right)_q dx \quad (2.2)$$

or

$$\int_{\mathbb{Z}_p} \Gamma_{p,q}(x+1)dx = \sum_{n=0}^{\infty} \tau_{p,q}(n) \frac{(-1)^n \ln q}{q^{\frac{n(n-1)}{2}} (q^{n+1} - 1)}.$$
Theorem 2.7. Let $x, s \in \mathbb{Z}_p$. For $|q - 1|_p < p^{-\frac{1}{r}}$ with $q \in \mathbb{C}_p$ the following identity:

$$\int_{\mathbb{Z}_p} \Gamma_{p,q}(x + s)dx = \sum_{n=0}^{\infty} \tau_{p,q}(n) \frac{\ln q^{n-s+1}}{q^{n+1} - 1} \left( s - 2 \right) \binom{n}{q} \left[ 1 + \sum_{j=1}^{n} \frac{q^{s+1-j}}{(q^{s+1-j} - 1)} \right]$$

is true where $\tau_{p,q}(n)$ is defined by Proposition 1.1.

Proof. Using Proposition 1.1 we have

$$\int_{\mathbb{Z}_p} \Gamma_{p,q}(x + s)dx = \int_{\mathbb{Z}_p} \sum_{n=0}^{\infty} \tau_{p,q}(n) \left( x + s \right) \frac{n}{n+1} \binom{n}{q} dx = \sum_{n=0}^{\infty} \tau_{p,q}(n) \int_{\mathbb{Z}_p} \left( x + s \right) \binom{n}{q} dx$$

By Theorem 2.4 we can write

$$\int_{\mathbb{Z}_p} \Gamma_{p,q}(x + s)dx = \sum_{n=0}^{\infty} \tau_{p,q}(n) \frac{\ln q^{n-s+1}}{q^{n+1} - 1} \left( s - 2 \right) \binom{n}{q} \left[ 1 + \sum_{j=1}^{n} \frac{q^{s+1-j}}{(q^{s+1-j} - 1)} \right]$$

As a result, we indicate the $q$-extension of the $p$-adic Euler constant with $q$-Mahler coefficient of the sequence $\Gamma_{p,q}$.

Theorem 2.8. The equality holds:

$$\gamma_{p,q} = \sum_{n=0}^{\infty} \tau_{p,q}(n) \frac{p(-1)^n \ln q}{(1-p)} \left( \frac{1}{q^{n+1} - 1} - \frac{1}{q} \left( 1 + \sum_{j=1}^{n} \frac{q^{1-j}}{(q^{1-j} - 1)} \right) \right)$$

for $x \in \mathbb{Z}_p$, $|q - 1|_p < p^{-\frac{1}{r}}$ with $q \in \mathbb{C}_p$ where $\tau_{p,q}(n)$ is defined by Proposition 1.1.

Proof. From (1.8) we have

$$\int_{\mathbb{Z}_p} \Gamma_{p,q}(x + 1)dx - \int_{\mathbb{Z}_p} \Gamma_{p,q}(x)dx = \Gamma'_{p,q}(0). \quad (2.3)$$

In the case $s = 0$ in Theorem 2.7 we get

$$\int_{\mathbb{Z}_p} \Gamma_{p,q}(x)dx = \sum_{n=0}^{\infty} \tau_{p,q}(n) \frac{\ln q^n}{q^{n+1} - 1} \left( -2 \right) \binom{n}{q} \left[ 1 + \sum_{j=1}^{n} \frac{q^{1-j}}{(q^{1-j} - 1)} \right]$$

or

$$\int_{\mathbb{Z}_p} \Gamma_{p,q}(x)dx = \sum_{n=0}^{\infty} \tau_{p,q}(n) \frac{\ln q^n}{q^{n+1} - 1} \left( -1 \right)^n \frac{(q^n - 1)}{q^{n+1} - 1} \left( 1 + \sum_{j=1}^{n} \frac{q^{1-j}}{(q^{1-j} - 1)} \right)$$

or

$$\int_{\mathbb{Z}_p} \Gamma_{p,q}(x)dx = \sum_{n=0}^{\infty} \tau_{p,q}(n) \frac{(-1)^n \ln q^n}{q^{n+1} - 1} \left( 1 + \sum_{j=1}^{n} \frac{q^{1-j}}{(q^{1-j} - 1)} \right) \quad (2.4)$$
Then, we rewrite the equality (2.3), using the equality (2.4) and Theorem 2.6.

\[ \Gamma'_{p,q}(0) = \sum_{n=0}^{\infty} \tau_{p,q}(n) (-1)^n q^{n+1} \ln q \left[ \frac{1}{q^{n(n-1)/2}(q^{n+1} - 1)} - \frac{q^{n+1}}{q^{n(n-1)/2}} (1 + \sum_{j=1}^{n} q^{1-j} (q-1)) \right]. \]

From the equality (1.3), we obtain a new representation for the \(q\)-extension of the \(p\)-adic Euler constant \(\gamma_{p,q}\) with the coefficients of the power series.

\[ \gamma_{p,q} = \sum_{n=0}^{\infty} \tau_{p,q}(n) \frac{p(-1)^n \ln q}{1-p} \left[ \frac{1}{q^{n(n-1)/2}(q^n - 1)} - \frac{1}{q^{n(n-1)/2}} (1 + \sum_{j=1}^{n} q^{1-j} (q-1)) \right]. \]

\[ \square \]

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