Journal of Mathematical Analysis
ISSN: 2217-3412, URL: www.ilirias.com/jma
Volume 8 Issue 2 (2017), Pages 64-72.

ON THE VOLKENBORN INTEGRAL OF THE q-EXTENSION OF THE p-ADIC GAMMA FUNCTION

ÖZGE ÇOLAKOĞLU HAVARE, HAMZA MENKEN

Abstract. In the present work we consider the q-extension of the p-adic gamma function. We derive the Volkenborn integral of the q-extension of the p-adic gamma function by using its Mahler expansion. Moreover, we give a new representation for the q-extension of the p-adic Euler constant.

1. Introduction

The q-Calculus appeared in the 18th century and it continues to develop rapidly. The q-calculus has a great interest and has been studied by Euler, Gauss who discovered q-binomial formula and others. The systematic development of the q-calculus began with FH Jackson in the early 20th century. Although the q-calculus has been studied for over a century, q-analogue of special numbers and polynomials are still of interest [1].

The letter q stands for 'quantum', and the q-binomial coefficients play an important role in 'quantum calculus' similar to that of the ordinary binomial coefficients in ordinary calculus. Also, the binomial coefficients are also known combinations or combinatorial numbers. In constructing the properties and identities of some special numbers, binomial coefficients have great interest. The q-analogue of the binomial coefficients an important role play in developing the theory of the q-analogue of these special numbers [5].

The q-binomial coefficients or Gaussian polynomials appear in many identities on q-series. In addition, they are studied in several combinatorial environments as partitions of integers. For an easy handling of the q-binomial coefficients in combinatory it is essential to be familiar with the basic combinatorial structures that admit those coefficients as generating polynomials [9].

The $p-$adic numbers introduced by the German mathematician Kurt Hensel (1861–1941), are widely used in mathematics: in number theory, algebraic geometry, representation theory, algebraic and arithmetical dynamics, and cryptography. The $p-$adic numbers have been used applying fields with successfully applying in

---

2000 *Mathematics Subject Classification.* 11S80,11E95, 11S23.
Key words and phrases. q-extension of p-adic gamma function; Volkenborn integral; q-extension of p-adic Euler constant; q-binomial coefficient.

Submitted November 25, 2016. Published April 1, 2017.
Communicated by Farrukh Mukhamedov.

superfield theory of $p-$adic numbers by V.S. Vladimirov and I. V. Volovich. In addition, the $p-$adic model of the universe, the $p-$adic quantum theory, the $p-$adic string theory such as areas occurred in physics (for detail see [24],[25]).

Throughout this paper, p is a fixed odd prime number and by $\mathbb{Z}_p, \mathbb{Q}_p$ and $\mathbb{C}_p$ we denote the ring of p-adic integers, the field of p-adic numbers and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively.

The $q-$analogue of n called the following notation,

$$(n)_q = \frac{q^n - 1}{q - 1} = q^{n-1} + q^{n-2} + ... + 1, \qquad q \neq 1$$

for any nonnegative integer n. As $q \to 1; (n)_q \to n$.

The $q-$binomial coefficient is a $q-$analogue for the binomial coefficient, also called a Gaussian coefficient or a Gaussian polynomial. A q-binomial coefficient, $\binom{n}{k}_q$, is defined by

$$\binom{n}{k}_q = \frac{(n)_q!}{(n-k)_q!\,(k)_q!}$$

where $(k)_q!$ denotes the $q-$factorial which given by $(0)_q! = 1$ and $(k)_q! = (k)_q\,(k-1)_q \ldots (2)_q\,(1)_q$ when $k > 0$.

Another expression of the q-binomial coefficient for nonnegative integers n, k with $n \geq k$ is defined by

$$\binom{n}{k}_q = \frac{(q^n - 1)(q^{n-1} - 1)...(q^{n-k+1} - 1)}{(q^k - 1)(q^{k-1} - 1)...(q - 1)}. \tag{1.1}$$

It is clearly that when $q \to 1$; the $q-$binomial coefficients reduce to the usual binomial coefficients.

The q-binomial coefficients have q-Pascal rule;

$$\binom{n}{k}_q = q^{n-k}\binom{n-1}{k-1}_q + \binom{n-1}{k}_q \tag{1.2}$$

where $1 \leq k \leq n - 1$ ([6], [11]).

In 1975 Y. Morita [17] defined the p-adic gamma function Γ_p by the formula

$$\Gamma_p\,(x) = \lim_{n \to x} (-1)^n \prod_{\substack{1 \leq j < n \\ (p,j)=1}} j$$

for $x \in \mathbb{Z}_p$, where n approaches x through positive integers.

The p-adic gamma function $\Gamma_p(x)$ has a great interest and has been studied by J. Diamond (1977) [7], D. Barsky (1977) [2], M. Boyarsky (1980) [3], B. Dwork (1983) [8], T. Kim (1997) [12] and others. The relationship between some special functions and the p-adic gamma function $\Gamma_p(x)$ was investigated by B. Gross and N. Koblitz (1979) [10], H. Cohen and E. Friedman (2008) [4] and I. Shapiro (2012) [21].

The q-extension of the p-adic gamma function $\Gamma_{p,q}(x)$ is defined by N. Koblitz [14] as follows:

Let $q \in \mathbb{C}_p, |q-1|_p < 1, q \neq 1$.The q-extension of the p-adic gamma function $\Gamma_{p,q}(x)$ is defined by formula

$$\Gamma_{p,q}(x) = \lim_{n\to x}(-1)^n \prod_{\substack{j<n \\ (p,j)=1}} \frac{1-q^j}{1-q}.$$

for $x \in \mathbb{Z}_p$, where n approaches x through positive integers. We recall that $\lim\limits_{q\to 1}\Gamma_{p,q} = \Gamma_p$.

N. Koblitz determined the relation between the derivative of $\Gamma_{p,q}(x)$ with the q-extension of the p-adic Euler constant $\gamma_{p,q}$ is defined by the formula

$$\Gamma'_{p,q}(0) = -\left(1-\frac{1}{p}\right)\gamma_{p,q}. \tag{1.3}$$

The q-extension of the p-adic gamma function $\Gamma_{p,q}(x)$ was studied by N. Koblitz (1980, 1982) [14], [15], H. Nakazato (1988) [18],Y. S. Kim (1998) [13] and others.

In 1958, Mahler introduced an expansion for continuous functions of a $p-$adic variable using special polynomials as binomial coefficient polynomial [16].

Conrad studied a $q-$analogue of Mahler expansions for continuous functions in $p-$adic analysis, replacing binomial coefficient polynomials $\binom{n}{k}$ with a $q-$analogue $\binom{n}{k}_q$ for a $p-$adic variable $q \in \mathbb{C}_p$ with $|q-1|_p < 1$ [6]

The Mahler coefficients $\Gamma_{p,q}$ are determined by the following proposition:

Proposition 1.1. [6]*Let*

$$\Gamma_{p,q}(x+1) = \sum_{n=0}^{\infty}\tau_{p,q}(n)\binom{x}{n}_q \qquad (x \in \mathbb{Z}_p) \tag{1.4}$$

be the Mahler series of $\Gamma_{p,q}$ *with* $|q-1|_p < p^{-\frac{1}{p-1}}$ *and* $q \in \mathbb{C}_p$ *where* $\tau_{p,q}(n)$ *be nth q-Mahler coefficient of the sequence* $\Gamma_{p,q}(n+1)$. *The following equality*

$$\sum_{n\geq 0}(-1)^n\tau_{p,q}(n)\frac{X^n}{(n)_q!} = \frac{1-X^p}{1-X}E_{1\backslash q}(x)\,E_{q^p}\left(\frac{x^p}{(p)_q}\right) \tag{1.5}$$

and

$$E_{1\backslash q}(x)\,E_{q^p}\left(\frac{x^p}{(p)_q}\right) = \sum_{n\geq 0}b_{p,q}(n)\frac{X^n}{(n)_q!}$$

where $E_q(X)$ *is the q-extension of the exponential series and also may denote a slightly different series holds.*

The Volkenborn integral was introduced by A. Volkenborn in his PhD dissertation and subsequently in the set of twin papers [22], [23]; a more recent treatment of the subject can be found in [20].

The *indefinite sum* of a continuous function $f : \mathbb{Z}_p \to \mathbb{C}_p$ is the continuous function Sf interpolating $n \to \sum\limits_{j=0}^{n-1} f(j)$ $(n \in \mathbb{N})$. Instead of $Sf(x)$ $(x \in \mathbb{Z}_p)$ we can write $\sum\limits_{j=0}^{x-1} f(j) \equiv \lim\limits_{n\to x}\sum\limits_{j=0}^{n-1} f(j)$ see [19] and [20]. The infinte sum of continuous function has below property

$$Sf(x+1) - Sf(x) = f\left(x\right). \tag{1.6}$$

Let f be a function from $C^1(\mathbb{Z}_p \to \mathbb{C}_p)$. The Volkenborn integral of f on $\mathbb{Z}_p$ is defined by the formula

$$\int\limits_{\mathbb{Z}_p} f(x)dx := \lim_{n\to\infty} p^{-n} \sum_{j=0}^{p^n-1} f(j) = (Sf)'\left(0\right). \tag{1.7}$$

For any $f \in C^1(\mathbb{Z}_p \to \mathbb{C}_p)$, the relation holds:

$$\int\limits_{\mathbb{Z}_p} f(x+1)dx - \int\limits_{\mathbb{Z}_p} f(x)dx = f^{'}(0). \tag{1.8}$$

2. Main Results

To evaluate Volkenborn integral of the q-extension of the p-adic gamma function, the following equalities can be expressed:

Lemma 2.1. *Let $|q-1|_p < 1$ with $q \in \mathbb{C}_p$, $n \in \mathbb{N}$ and $f \in C\left(\mathbb{Z}_p \to \mathbb{C}_p\right)$ such that the q-Mahler expansion of f, $f(x) = \sum\limits_{n=0}^{\infty} \tau_{p,q}(n) \binom{x}{n}_q$. Then, the indefinite of f is given by*

$$Sf(x) = \sum_{n=0}^{\infty} q^{n-x} \tau_{p,q}(n) \binom{x}{n+1}_q$$

Proof. Let Let $|q-1|_p < 1$ with $q \in \mathbb{C}_p$, $n \in \mathbb{N}, x \in \mathbb{Z}_p$. Assume that

$$Sf(x) = \sum_{n=0}^{\infty} \sigma_{p,q}(n) \binom{x}{n}_q .$$

From (1.6), we get

$$\sum_{n=0}^{\infty} \sigma_{p,q}(n) \binom{x+1}{n}_q - \sum_{n=0}^{\infty} \sigma_{p,q}(n) \binom{x}{n}_q = \sum_{n=0}^{\infty} \tau_{p,q}(n) \binom{x}{n}_q .$$

By using (1.2), we have

$$\sum_{n=0}^{\infty} \sigma_{p,q}(n) \left(q^{x+1-n} \binom{x}{n-1}_q + \binom{x}{n}_q - \binom{x}{n}_q \right) = \sum_{n=0}^{\infty} \tau_{p,q}(n) \binom{x}{n}_q$$

or

$$\sum_{n=0}^{\infty} \sigma_{p,q}(n) q^{x+1-n} \binom{x}{n-1}_q = \sum_{n=0}^{\infty} \tau_{p,q}(n) \binom{x}{n}_q .$$

So we have

$$\sum_{n=-1}^{\infty} \sigma_{p,q}(n+1) q^{x-n} \binom{x}{n}_q = \sum_{n=0}^{\infty} \tau_{p,q}(n) \binom{x}{n}_q .$$

We know that $n \in \mathbb{N}$. Thus, we can write

$$\sum_{n=0}^{\infty} \sigma_{p,q}(n+1) q^{x-n} \binom{x}{n}_q = \sum_{n=0}^{\infty} \tau_{p,q}(n) \binom{x}{n}_q .$$

Hence, we have

$$\sigma_{p,q}(n+1)q^{x-n} = \tau_{p,q}(n) \qquad n \in \mathbb{N}$$

or

$$\sigma_{p,q}(n) = q^{n-1-x}\tau_{p,q}(n-1) \qquad n \in \mathbb{N}.$$

Then, we can write

$$Sf(x) = \sum_{n=0}^{\infty} q^{n-x-1}\tau_{p,q}(n-1)\binom{x}{n}_q.$$

By some computing, we obtain

$$Sf(x) = \sum_{n=0}^{\infty} q^{n-x}\tau_{p,q}(n)\binom{x}{n+1}_q.$$

□

If we take $f(x) = \binom{x}{n}_q$, we obtain the following results;

Corollary 2.2. *If $x \in \mathbb{Z}_p$ and $q \in \mathbb{C}_p$ with $|q-1|_p < 1$ then*

$$S\binom{x}{n}_q = q^{n-x}\binom{x}{n+1}_q.$$

Lemma 2.3. *For $x, s \in \mathbb{Z}_p, n \in \mathbb{N}$ and $q \in \mathbb{C}_p$ with $|q-1|_p < 1$, the following identity:*

$$\binom{x+s}{n+1}_q' = \frac{q^{x+s}\ln q}{(q^{n+1}-1)}\left[\sum_{j=1}^{n}\prod_{k=0}^{j-1}\frac{q^{-j}\left(q^{x+s-k}-1\right)}{(q^{n-k}-1)}\binom{x+s-j-1}{n-j}_q + \binom{x+s-1}{n}_q\right]$$

is valid.

Proof. From (1.1), we have

$$\binom{x+s}{n+1}_q' = \left[\frac{(q^{x+s}-1)...(q^{x+s-n}-1)}{(q^{n+1}-1)...(q-1)}\right]'.$$

By applying the definition of the derivative, we can write

$$\binom{x+s}{n+1}_q' = \frac{\ln q q^{x+s}(q^{x+s-1}-1)...(q^{x+s-n}-1) + ... + \ln q q^{x+s-n}(q^{x+s}-1)...(q^{x+s-n+1}-1)}{(q^{n+1}-1)...(q-1)}.$$

We easily obtain the following relation with a little rearranging

$$\binom{x+s}{n+1}_q' = \frac{\ln q}{(q^{n+1}-1)}\left[\sum_{j=1}^{n}\prod_{k=0}^{j-1}\frac{q^{x+s-j}\left(q^{x+s-k}-1\right)}{(q^{n-k}-1)}\binom{x+s-j-1}{n-j}_q + q^{x+s}\binom{x+s-1}{n}_q\right].$$

□

Theorem 2.4. *For $x, s \in \mathbb{Z}_p, n \in \mathbb{N}$ and $q \in \mathbb{C}_p$ with $|q-1|_p < 1$, the Volkenborn integral of q-binomial coefficient is*

$$\int_{\mathbb{Z}_p}\binom{x+s}{n}_q dx = \frac{\ln q q^{n-s}}{q^{n+1}-1}\binom{s-1}{n}_q\left[1 + \sum_{j=1}^{n}\frac{q^{s-j}\left(q^{s}-1\right)}{(q^{s-j}-1)}\right]$$

Proof. From (1.7) and Corollary 2.2, we easily obtain the following:

$$\int_{\mathbb{Z}_p} \binom{x+s}{n}_q dx = \left(q^{n-x-s} \binom{x+s}{n+1}_q \right)' (0)$$

or

$$\int_{\mathbb{Z}_p} \binom{x+s}{n}_q dx = \left(-\ln q q^{n-x-s} \binom{x+s}{n+1}_q + q^{n-x-s} \binom{x+s}{n+1}_q' \right) (0) \tag{2.1}$$

By using Lemma 2.3 and (1.7) and we can rewrite (2.1)

$$\int_{\mathbb{Z}_p} \binom{x+s}{n}_q dx = \ln q q^{n-s} \left[\sum_{j=1}^{n} \prod_{k=0}^{j-1} \frac{q^{s-j}\left(q^{s-k}-1\right)}{\left(q^{n+1}-1\right)\left(q^{n-k}-1\right)} \binom{s-j-1}{n-j}_q + \frac{q^s}{\left(q^{n+1}-1\right)} \binom{s-1}{n}_q - \binom{s}{n+1}_q \right]$$

Using (1.1) we easily discover

$$\int_{\mathbb{Z}_p} \binom{x+s}{n}_q dx = \frac{\ln q q^{n-s}}{q^{n+1}-1} \left[\binom{s-1}{n}_q + \sum_{j=1}^{n} \prod_{k=0}^{j-1} \frac{q^{s-j}\left(q^{s-k}-1\right)}{\left(q^{n-k}-1\right)} \binom{s-j-1}{n-j}_q \right]$$

or

$$\int_{\mathbb{Z}_p} \binom{x+s}{n}_q dx = \frac{\ln q q^{n-s}}{q^{n+1}-1} \left[\binom{s-1}{n}_q + \sum_{j=1}^{n} \frac{q^{s-j}\left(q^{s}-1\right)}{\left(q^{s-j}-1\right)} \binom{s-1}{n}_q \right]$$

□

In the case $s = 0$ in Lemma 2.4, we obtain following corollary::

Corollary 2.5. *For $x \in \mathbb{Z}_p, n \in \mathbb{N}$ and $q \in \mathbb{C}_p$ with $|q-1|_p < 1$, the relation holds:*

$$\int_{\mathbb{Z}_p} \binom{x}{n}_q dx = \frac{(-1)^n \ln q}{q^{\frac{n(n-1)}{2}}\left(q^{n+1}-1\right)}$$

Theorem 2.6. *For all $x \in \mathbb{Z}_p, n \in \mathbb{N}$ and $|q-1|_p < p^{-\frac{1}{p-1}}$ with $q \in \mathbb{C}_p$, the following identity holds:*

$$\int_{\mathbb{Z}_p} \Gamma_{p,q}(x+1) dx = \sum_{n=0}^{\infty} \tau_{p,q}(n) \frac{(-1)^n \ln q}{q^{\frac{n(n-1)}{2}}\left(q^{n+1}-1\right)}$$

where $\tau_{p,q}(n)$ is defined by Proposition 1.1.

Proof. From Proposition 1.1 and Corollary 2.5 we have

$$\int_{\mathbb{Z}_p} \Gamma_{p,q}(x+1) dx = \int_{\mathbb{Z}_p} \sum_{n=0}^{\infty} \tau_{p,q}(n) \binom{x}{n}_q dx = \sum_{n=0}^{\infty} \tau_{p,q}(n) \int_{\mathbb{Z}_p} \binom{x}{n}_q dx \tag{2.2}$$

or

$$\int_{\mathbb{Z}_p} \Gamma_{p,q}(x+1) dx = \sum_{n=0}^{\infty} \tau_{p,q}(n) \frac{(-1)^n \ln q}{q^{\frac{n(n-1)}{2}}\left(q^{n+1}-1\right)}.$$

□

Theorem 2.7. *Let $x, s \in \mathbb{Z}_p$. For $|q-1|_p < p^{-\frac{1}{p-1}}$ with $q \in \mathbb{C}_p$ the following identity:*

$$\int_{\mathbb{Z}_p} \Gamma_{p,q}(x+s)dx = \sum_{n=0}^{\infty} \tau_{p,q}(n) \frac{\ln q q^{n-s+1}}{q^{n+1}-1} \binom{s-2}{n}_q \left[1 + \sum_{j=1}^{n} \frac{q^{s+1-j}\left(q^{s+1}-1\right)}{\left(q^{s+1-j}-1\right)}\right]$$

is true where $\tau_{p,q}(n)$ is defined by Proposition 1.1.

Proof. Using Proposition 1.1, we have

$$\int_{\mathbb{Z}_p} \Gamma_{p,q}(x+s)dx = \int_{\mathbb{Z}_p} \sum_{n=0}^{\infty} \tau_{p,q}(n) \binom{x+s-1}{n}_q dx = \sum_{n=0}^{\infty} \tau_{p,q}(n) \int_{\mathbb{Z}_p} \binom{x+s-1}{n}_q dx$$

By Theorem 2.4, we can write

$$\int_{\mathbb{Z}_p} \Gamma_{p,q}(x+s)dx = \sum_{n=0}^{\infty} \tau_{p,q}(n) \frac{\ln q q^{n-s+1}}{q^{n+1}-1} \binom{s-2}{n}_q \left[1 + \sum_{j=1}^{n} \frac{q^{s+1-j}\left(q^{s+1}-1\right)}{\left(q^{s+1-j}-1\right)}\right]$$

□

As a result, we indicate the q-extension of the p-adic Euler constant with q-Mahler coefficient of the sequence $\Gamma_{p,q}$.

Theorem 2.8. *The equality holds:*

$$\gamma_{p,q} = \sum_{n=0}^{\infty} \tau_{p,q}(n) \frac{p(-1)^n \ln q}{(1-p)} \left(\frac{1}{q^{\frac{n(n-1)}{2}}\left(q^{n+1}-1\right)} - \frac{1}{q^{\frac{(n+2)(n-1)}{2}}(q-1)} \left(1 + \sum_{j=1}^{n} \frac{q^{1-j}(q-1)}{\left(q^{1-j}-1\right)}\right)\right)$$

for $x \in \mathbb{Z}_p$, $|q-1|_p < p^{-\frac{1}{p-1}}$ with $q \in \mathbb{C}_p$ where $\tau_{p,q}(n)$ is defined by Proposition 1.1.

Proof. From (1.8) we have

$$\int_{\mathbb{Z}_p} \Gamma_{p,q}(x+1)dx - \int_{\mathbb{Z}_p} \Gamma_{p,q}(x)dx = \Gamma'_{p,q}(0). \tag{2.3}$$

In the case $s=0$ in Theorem 2.7, we get

$$\int_{\mathbb{Z}_p} \Gamma_{p,q}(x)dx = \sum_{n=0}^{\infty} \tau_{p,q}(n) \frac{\ln q q^{n+1}}{q^{n+1}-1} \binom{-2}{n}_q \left[1 + \sum_{j=1}^{n} \frac{q^{1-j}(q-1)}{\left(q^{1-j}-1\right)}\right]$$

or

$$\int_{\mathbb{Z}_p} \Gamma_{p,q}(x)dx = \sum_{n=0}^{\infty} \tau_{p,q}(n) \frac{\ln q q^{n+1}}{q^{n+1}-1} \frac{(-1)^n\left(q^{n+1}-1\right)}{q^{\frac{n(n+3)}{2}}(q-1)} \left(1 + \sum_{j=1}^{n} \frac{q^{1-j}(q-1)}{\left(q^{1-j}-1\right)}\right)$$

or

$$\int_{\mathbb{Z}_p} \Gamma_{p,q}(x)dx = \sum_{n=0}^{\infty} \tau_{p,q}(n) \frac{(-1)^n \ln q q^{n+1}}{q^{\frac{n(n+3)}{2}}(q-1)} \left(1 + \sum_{j=1}^{n} \frac{q^{1-j}(q-1)}{\left(q^{1-j}-1\right)}\right) \tag{2.4}$$

Then, we rewrite the equality (2.3) using the equality (2.4) and Theorem 2.6

$$\Gamma'_{p,q}(0)=\sum_{n=0}^{\infty}\tau_{p,q}(n)(-1)^n\ln q\left[\frac{1}{q^{\frac{n(n-1)}{2}}(q^{n+1}-1)}-\frac{q^{n+1}}{q^{\frac{n(n+3)}{2}}(q-1)}\left(1+\sum_{j=1}^{n}\frac{q^{1-j}(q-1)}{(q^{1-j}-1)}\right)\right].$$

From the equality (1.3), we obtain a new representation for the q-extension of the p-adic Euler constant $\gamma_{p,q}$ with the coefficients of the power series.

$$\gamma_{p,q}=\sum_{n=0}^{\infty}\tau_{p,q}(n)\frac{p(-1)^n\ln q}{(1-p)}\left(\frac{1}{q^{\frac{n(n-1)}{2}}(q^{n+1}-1)}-\frac{1}{q^{\frac{(n+2)(n-1)}{2}}(q-1)}\left(1+\sum_{j=1}^{n}\frac{q^{1-j}(q-1)}{(q^{1-j}-1)}\right)\right)$$

□

Acknowledgments. The authors would like to thank the anonymous reviewer for his/her careful reading and many valuable suggestions that helped us improve this article.

References

[1] S. Araci and M. Açıkgöz, *A note on the values of weighted q-Bernstein polynomials and weighted q-Genocchi numbers*, Advances in Differences Equations, (2015) 1-9.

[2] D. Barsky, *On Morita's -adic Gamma Function*, Groupe d'Etude d'Analyse Ultramétrique, 5e année (1977/78), **3**, (1978),1-6.

[3] M. Boyarsky, *p-adic Gamma Functions and Dwork Cohomology*, Trans. Amer. Math. Soc. **257(2)**, (1980) 359-369.

[4] H. Cohen and E. Friedman, *Raabe's formula for p-adic gamma and zeta functions,* Ann. Inst. Fourier (Grenoble) **58(1)**, (2008), 363-376.

[5] RB. Corcino, *On $p;q$ -Binomial Coefficients*, Integers: Electronic Journal of Combinatorial Number Theory, **8**, (2008), #29.

[6] K. Conrad, *A q-Analogue of Mahler Expansions I*, Advances in Mathematics, **153(2)**, (2000)185-230.

[7] J. Diamond, *The p-adic log gamma function and p-adic Euler constant*, Trans. Amer. Math. Soc. **233**, (1977) 321-337 . Diamond J (1977).

[8] B. Dwork, *A note on p-adic gamma function*, Groupe de travail d'analyse ultrametrique, **9(3)**, (1983),J1-J10.

[9] D. Foata and G. Han, The q-series in combinotorics; Permütation Statistics. (2004) Lecture Note

[10] Gross, B. H. and Koblitz, N., Gauss Sums and the p-adic Γ-function, The Annals of Mathematics, Second Series, **109(3)**, (1979), pp. 569-581.

[11] V. Kac and P. Cheung, *Quantum Calculus,* Springer, 1943.

[12] T. Kim, *A note on analogue of gamma functions*, Proc.Confer. on 5th Transcendental Number Theory 5, No. **1,** Gakushin Univ. Tokyo Japan, (1997), 111-118.

[13] Y. S. Kim, *q-analogues of p-adic gamma functions and p-adic Euler constants*, Communications of the Korean Mathematical Society, **13 (4)**, (1998), 735.741.

[14] N. Koblitz, *q-extension of the p-adic gamma function*, Transactions of the American Mathematical Society, **260(2)**, (1980), 449-457.

[15] N. Koblitz, *q-extension of the p-adic gamma function II*, Transactions of the American Mathematical Society, **273(1)**, (1982), 111–129.

[16] K. Mahler, *An Interpolation Series for Continuous Functions of a p-adic Variable*, Journal für die reine und angewandte Mathematik, **199**, (1958), 23-34.

[17] Y. Morita, *A p-adic analogue of the Γ-function*, J. Fac. Science Univ., Tokyo, **22(2)**, (1975) 225- 266 .

[18] H. Nakazato, *The q-analogue of the p-adic gamma function*, Kodai Math. J.*,* **11(1)**, (1988), 141–153 .

[19] A. M. Robert, *A Course in p-adic Analysis*, Graduate Texts in Mathematics 198, Springer-Verlag New York, 2000.
[20] W. H. Schikhof, *Ultrametric Calculus: An Introduction to p-adic Analysis*, Cambridge University Pres, 1984.
[21] I. Shapiro, *Frobenius map and the p-adic gamma function,* J. Number Theory, **132(8)**, (2012), 1770–1779
[22] A. Volkenborn, *Ein p-adisches Integral und seine Anwendungen. I*, Manuscripta Math. **7(4),** (1972), 341–373.
[23] A. Volkenborn, *Ein p-adisches Integral und seine Anwendungen II*, Manuscripta Math. **12**, (1974), 17-46.
[24] I. V. Volovich, Number theory as the ultimate physical theory, Preprint No. TH 4781/87, CERN, Geneva, (1987).
[25] V. S Vladimirov and I. V. Volovich, Superanalysis. I. Differential calculus, Theor. Math. Phys. **59**,(1984) 317–335.

Özge Çolakoğlu Havare
Mersin University, Science and Arts Faculty, Mathematics Department, 33343, Mersin-Turkey
E-mail address: ozgecolakoglu@mersin.edu.tr

Hamza Menken
Mersin University, Science and Arts Faculty, Mathematics Department, 33343, Mersin-Turkey
E-mail address: hmenken@mersin.edu.tr