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VARIATIONAL ITERATION METHOD FOR FRACTIONAL WAVE-LIKE AND HEAT-LIKE EQUATIONS IN LARGE DOMAINS

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Abstract. In this paper, we introduced a novel variational iteration method (VIM) for solving fractional wave–like and heat–like equations in a large domain which is improved by adding an auxiliary parameter. The suggested method is very effective and we can control the convergence region very easily. By solving some fractional wave-like and heat-like equations with the proposed method and comparing with standard VIM, it was concluded that complete reliability, efficiency, and accuracy of this method is guaranteed.

1. Introduction

In the past years, fractional calculus has emerged in physical phenomena. Fractional derivatives prepared a great tool for the elucidation of memory and patrimonial confidants of disparate materials and processes [1, 2]. Fractional differential equations (FDE) have been attracted the attention of many researchers because there are many applications in science and engineering such as acoustics, control, viscoelasticity, edge detection, signal processing and a lot of other matters [3, 4, 5]. Recently, fractional diffusion equations are considered by Adomian decomposition method and series expansion method in [6, 7]. The fractional Maxwell fluid within fractional Caputo–Fabrizio derivative operator by an analytical method in [8]. Numerous excellent books and papers explaining the state-of-the-art extant in the literature, testify to the maturity of the theory of fractional order [9, 10, 11]. There are prepared the solutions method of differential equations of optional real order and applications of the demonstrated methods in several fields and gave a systematic presentation of the applications, methods, and ideas of the fractional calculus. Their book played a significant role in the expansion of the theory of fractional order [12, 13, 14].

The main feature of using fractional calculus in most usages is its non-local attribute. It is well known that the integer order differential operators and the integer order integral operators are local, while, the fractional order differential operators and the fractional order integral operators are nonlocal. This means that the next

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situation of a system depends not only on its current situation but also upon all of its historical situation [15, 16]. Problems in fractional PDEs are not only important but also quite challenging which generally involves hard mathematical solution method. Riemann-Liouville and Caputo are two more routines used in fractional calculus. The order of evaluation is the distinction between two definitions [17]. As long as there is no exact solution for FDE, the most efforts have supplied numerical and analytically methods for solving these equations. Indeed, many powerful methods have been developed lately, e.g., Adomian decomposition method, homotopy analysis method, homotopy perturbation method, collocation method, finite difference method and Tau method [18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33]. The variational iteration method (VIM) proffered by He [34] has been recognized to be a reliable and effective algorithm for solving various ordinary differential equations, delay differential equations, boundary differential equations partial differential equations, and nonlinear problems arising in engineering [35, 36, 37, 38, 39].

To improve the convergence speed and enlarge the interval of convergence for VIM series solutions some modifications were propounded [40, 41, 42]. There are many modifications of VIM, which are exclusively appropriate for the fractional differential equation. For example, Odibat and Momani used the VIM to fractional differential equations in fluid mechanics [43]. Wu solved the time-fractional heat equation by using the Laplace transform in the determination of the Lagrange multiplier in VIM [44]. Hristov applied the VIM with a new multiplier to a fractional Bernoulli equation that happens during transient conduction with a nonlinear heat flux at the boundary [45]. Other modified VIMs are variational iteration-Pade method, variational iteration-Adomian method, VIM with an auxiliary parameter [46, 47, 48, 49]. The optimal method as introduced in [49] is for the first time applied to the fractional type partial differential equations with great success. The standard VIM applied to solve fractional wave-like and heat-like equations in [50], these models are an integral part of applied sciences and appear in several physical phenomena.

In this paper, we applied VIM indeed an auxiliary parameter for solving fractional wave-like and heat-like equations in a large domain which introduced in [50] and we make a comparison with standard VIM. In the proposed method by using β_i's functions (that we'll explain that later) we determine the appropriate value for the auxiliary parameter value. Indeed some theorems proved in this topic. The proposed approach minimizes the norm of the β_i's function at each step of VIM, which contains an unknown auxiliary parameter.

In this paper, we consider FDE as the form

$$\frac{\partial u^{\alpha}}{\partial t^{\alpha}} = f(x,y,z)u_{xx} + g(x,y,z)u_{yy} + h(x,y,z)u_{zz}, \tag{1.1}$$

subject to the Neumann boundary conditions

$$\begin{aligned} &u_x(0,y,z,t) = f_1(y,z,t), u_x(a,y,z,t) = f_2(y,z,t),\\ &u_y(x,0,z,t) = g_1(x,z,t), u_y(x,b,z,t) = g_2(x,z,t),\\ &u_z(x,y,0,t) = h_1(x,y,t), u_z(x,y,c,t) = h_2(x,y,t), \end{aligned} \tag{1.2}$$

and the initial conditions

$$u(x,y,z,0) = \psi(x,y,z), u_t(x,y,z,0) = \theta(x.y.z), \tag{1.3}$$

where a constant α describes the fractional derivative. This type of equations are obtained from the usual heat like or wave like equation, the difference is that the first- or second-order time derivatives term has become to a fractional derivative of order $\alpha > 0$.

2. Fractional calculus

Here, we give some definitions of fractional calculus and their properties.

Definition 2.1 A real function $f(x),\ x > 0$, is said to be in space $C_\mu,\ \mu \in \mathbb{R}$, if it exists a real number $p > \mu$, such that $f(t) = t^p f_1(t)$, where $f_1(t) \in C(0, \infty)$, and it is said to be in the space C_μ^n if and only if $f_n \in C_\mu,\ n \in \mathbb{N}$.

Definition 2.2 The Riemann-Liouville fractional integral operator of order $\alpha > 0$, of a function $f \in C_\mu, \mu > 0$, is defined as

$$\begin{aligned} I^\alpha &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \qquad \alpha > 0, \\ I^0 &= f(x). \end{aligned} \tag{2.1}$$

Definition 2.3 The fractional derivative of $f(t)$ in the Caputo sense is defined as

$$D^\alpha f(t) = I^{m-\alpha} D^m f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-s)^{m-\alpha-1} f^{(m)}(s) ds, \tag{2.2}$$

for $m - 1 < \alpha \le m, m \in \mathbb{N}, t > 0$ and $f \in C_{-1}^m$.

Lemma 2.4 [51] If $m - 1 < \alpha \le m, m \in \mathbb{N}$, and $f \in C_\mu^m, \mu > -1$, then

$$D^\alpha I^\alpha f(t) = f(t),$$

and

$$I^\alpha D^\alpha f(t) = f(t) + \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{t^k}{k!}, \qquad t > 0. \tag{2.3}$$

Here we consider the Caputo fractional derivative because by this definition we can use traditional initial and boundary conditions which are included in the formulation of the problem. Also, by this definition we can see that the singularity is removed, [52]. Here, the multi-dimensional time fractional heat- and wave-like equation (1.1) is considered, where the unknown function u is vanishing for $t < 0$, i.e., is a causal function of time, [53].

3. Variational iteration method with an auxiliary parameter

In this section, we describe the variational iteration method with an auxiliary parameter. Consider the following nonlinear equation:

$$Hu = Lu + Nu + Ru + g(x,t) = 0, \tag{3.1}$$

where L shows the highest order derivative that is supposed to be easily invertible, R demonstrated a linear differential operator of order less than L, Nu illustrates the nonlinear terms, and g represents the source inhomogeneous term. Ji-Huan He proposed the variational iteration method in which a correction functional for (3.1), can be written as:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\tau) Hu_n(x,\tau) d\tau. \tag{3.2}$$

In the mentioned equation λ is a Lagrange multiplier which is obtained by optimally via variational theory, u_n is the n-th approximate solution, and $\tilde{u}_n$ interprets a restricted variation, i.e., $\partial \tilde{u}_n = 0$. The approximations $u_{n+1}(x,t),\ n \ge 0$, of the solution $u(x,t)$, will be readily obtained upon using the determined Lagrangian multiplier and any chosen function $u_0(x,t)$, providing that $Lu_0(x,t) = 0$. The correction functional (3.2) will give several approximations such as follows,

$$u(x,t) = \lim_{n\to\infty} u_n(x,t). \tag{3.3}$$

The following variational iteration algorithm for (3.1) is summarized as follows:

$$\begin{cases} u_0(x,t) \textit{ is an arbitrary function}, \\ u_{n+1}\left(x,t\right) = u_n\left(x,t\right) + \int_0^t \lambda\left(\tau\right) Hu_n\left(x,\tau\right) d\tau, \qquad n \geq 0. \end{cases} \tag{3.4}$$

The novel proposed method can offer as follows:

$$\begin{cases} u_0(x,t) \textit{ is an arbitrary function}, \\ u_1\left(x,t,h\right) = u_0\left(x,t\right) + h\int_0^t \lambda\left(\tau\right) Hu_n\left(x,\tau\right) d\tau, \\ u_{n+1}\left(x,t,h\right) = u_n\left(x,t,h\right) + h\int_0^t \lambda\left(\tau\right) Hu_n\left(x,\tau,h\right) d\tau, \qquad n \geq 1, \end{cases} \tag{3.5}$$

where in an unknown auxiliary parameter is entered into the variational iteration formula (3.4). The sequential approximate solutions $u_{n+1}\left(x,t,h\right),\ n \geq 1$ contain the auxiliary parameter h. The auxiliary parameter in such a way that is ensured that the veracity of the method and that approximation $u_{n+1}\left(x,t,h\right),\ n \geq 1$ converges to the exact solution. Of course, the parameter h of how to ensure convergence depends to select the appropriate value of h that at the end of the next section will explain.

4. Convergence analysis

In this section, at first we present the alternative approach of the variational iteration method with an auxiliary parameter and after that, the convergence of this method will be examined. This approach can be performed in a trustworthy and effective way and also can handle the fractional differential equation (3.1). When the variational iteration method with an auxiliary parameter is applied to solve the fractional heat-like and wave-like equations, the linear operator L is defined as $L = \dfrac{\partial^\alpha}{\partial t^\alpha}$, and Lagrange multiplier λ identified optimally via variational theory as

$$\lambda(t,\tau) = \frac{-1}{\Gamma(\alpha)}(t-\tau)^{\alpha-1}. \tag{4.1}$$

Now, we define the operator A and the ingredients v_n, s_n, $n \geq 0$, as follows:

$$Au(\mathbf{x},t,h) = h\int_0^t \lambda\left(t,\tau\right) Hu(\mathbf{x},\tau,h)d\tau, \tag{4.2}$$

$$\begin{cases} v_0(\mathbf{x},t) = u_0(\mathbf{x},t), \\ s_0(\mathbf{x},t) = v_0(\mathbf{x},t), \end{cases}$$

$$\begin{cases} v_1(\mathbf{x},t,h) = As_0(\mathbf{x},t), \\ s_1(\mathbf{x},t,h) = s_0(\mathbf{x},t) + v_1(\mathbf{x},t,h), \end{cases}$$

and in general for $n \geq 1$,

$$\begin{cases} v_{n+1}(\mathbf{x},t,h) = As_n(\mathbf{x},t,h), \\ s_{n+1}(\mathbf{x},t,h) = s_n(\mathbf{x},t,h) + v_{n+1}(\mathbf{x},t,h), \end{cases} \tag{4.3}$$

as a result, we have,

$$u(\mathbf{x},t,h)=\lim_{n\to\infty}s_n(\mathbf{x},t,h)=v_0(\mathbf{x},t)+\sum_{n=1}^{\infty}v_n(\mathbf{x},t,h). \tag{4.4}$$

The initial approximation $u_0(\mathbf{x},t)$ can be freely selected, while $Lu_0(\mathbf{x},t)=0$ and it satisfies the initial conditions of the problem. For the approximation purpose, we approximate the solution $u(\mathbf{x},t,h)=v_0(\mathbf{x},t)+\sum_{n=1}^{\infty}v_n(\mathbf{x},t,h)$, by the Nth-order truncated series $u_N(\mathbf{x},t,h)=v_0(\mathbf{x},t)+\sum_{n=1}^{N}v_n(\mathbf{x},t,h)$. The approximate solutions $u_N\left(t,h\right)$, contain the auxiliary parameter h. It is the auxiliary parameter that ensures that the convergence can be satisfied by means of the minimize of norm 2 of the β function (that we'll explain that later). The optimal value for the auxiliary parameter can be found by minimizing the residual of the total error, see [54] for more details.

The sufficient conditions for convergence of the method and the error estimate will be presented below. The main results are proposed in the following theorems [55].

Theorem 4.1 Let A, defined in (4.2), be an operator from a Hilbert space H to H. If $\exists\tilde{h}\neq 0, 0<\gamma<1$, such that,

$$\begin{cases} \parallel As_0(\mathbf{x},t)\parallel \ \leq\gamma\parallel s_0(\mathbf{x},t)\parallel, & \\ \parallel As_1(\mathbf{x},t,\tilde{h})\parallel\leq\gamma\parallel As_0(\mathbf{x},t)\parallel, & \\ \parallel As_n(\mathbf{x},t,\tilde{h})\parallel\leq\gamma\parallel As_{n-1}(\mathbf{x},t,\tilde{h})\parallel, & n=2,3,4,\cdots, \end{cases}$$

then the series solution defined in (4.4),

$$u(\mathbf{x},t)=\lim_{n\to\infty}s_n(\mathbf{x},t,\tilde{h})=v_0(\mathbf{x},t)+\sum_{n=1}^{\infty}v_n(\mathbf{x},t,\tilde{h}),$$

converges [56, 57].

Lemma 4.2 Let L, defined in (3.1), be as, $L=\dfrac{\partial^\alpha}{\partial t^\alpha}$, and λ identified optimally via variational theory in (4.1). If k be a function from a Hilbert space H to H, then:

$$L\left\{\int_0^t\lambda\left(t,\tau\right)k(\mathbf{x},\tau)d\tau\right\}=-k(\mathbf{x},t).$$

Proof. Suppose that L, defined in (3.1), be as, $L=\dfrac{\partial^\alpha}{\partial t^\alpha}$, and λ be as (4.1). Thus,

$$\begin{aligned} L\left\{\textstyle\int_0^t\lambda\left(\tau\right)k(\mathbf{x},\tau)d\tau\right\} &= \frac{\partial^\alpha}{\partial t^\alpha}\textstyle\int_0^t\frac{-1}{\Gamma(\alpha)}(t-\tau)^{\alpha-1}k(\mathbf{x},\tau)d\tau \\ &= -D^\alpha I^\alpha k(\mathbf{x},t)=-k(\mathbf{x},t). \end{aligned}$$ □

Theorem 4.3 [56, 57] Let L, defined in (3.1), be as follow as, $L=\dfrac{\partial^\alpha}{\partial t^\alpha}$, according to Lemma 4.2, if we have $u(\mathbf{x},t)=v_0(\mathbf{x},t)+\sum_{n=1}^{\infty}v_n(\mathbf{x},t,\tilde{h})$, then $u(\mathbf{x},t)$, is an exact solution of the nonlinear problem (3.1).

Theorem 4.4 [56, 57] Suppose that the series solution $u(\mathbf{x},t)=v_0(\mathbf{x},t)+\sum_{n=1}^{\infty}v_n(\mathbf{x},t,\tilde{h})$ defined in (4.4) is convergent to exact solution of the nonlinear problem (3.1). If the truncated series $u_N(\mathbf{x},t)=v_0(\mathbf{x},t)+\sum_{n=1}^{N}v_n(\mathbf{x},t,\tilde{h})$ is used as an approximate solution, then the maximum error is estimated as

$$\| u(\mathbf{x},t) - u_N(\mathbf{x},t) \| \leq \frac{1}{1-\gamma}\gamma^{N+1} \| v_0 \| .$$

If we want to summarize what was said above, we can define,

$$\beta_i = \begin{cases} \dfrac{\| v_{i+1} \|}{\| v_i \|}, & \| v_i \| \neq 0, \\ 0, & \| v_i \| = 0, \qquad\qquad i = 0,1,2,\cdots . \end{cases} \tag{4.5}$$

Now, if $0 < \beta_i < 1$ for $i = 0,1,2,\cdots$, then the series solution $v_0(\mathbf{x},t)+\sum_{n=1}^{\infty} v_n(\mathbf{x},t,\tilde{h})$, of problem (3.1) converges to the exact solution, $u(\mathbf{x},t)$. Moreover, as stated in Theorem 4.4, the maximum absolute truncation error is estimated to be,

$$\| u(\mathbf{x},t) - u_N(\mathbf{x},t) \| \leq \frac{1}{1-\beta}\beta^{N+1} \| v_0 \|,$$

where $\beta = max\{\beta_i, i = 0,1,2,\cdots\}$.

Regard that, the first finite terms do not affect the convergence of series solution. In fact, if the first finite β_i's, $i = 0,1,2,\cdots,l$, are not less than one and $\beta_i < 1$, for $i > l$, then, of course the series solution $v_0(\mathbf{x},t) + \sum_{n=1}^{\infty} v_n(\mathbf{x},t,\tilde{h})$, of problem (3.1), converges to an exact solution [58]. Now we choose a proper value of h as the β_i's, $i = l+1, l+2, l+3,\cdots,N$, are less than 1, When you select this h convergence of the series solution $v_0(\mathbf{x},t)+\sum_{n=1}^{\infty} v_n(\mathbf{x},t,h)$, is guaranteed. In fact, the proposed method, ensures convergence the variational iteration method with an auxiliary parameter and is capable the use of this method in large intervals with high precision. It should be noted that $v_{i+1} = u_{i+1} - u_i$.

5. Numerical Examples

In this section, to demonstrate the proposed method, three examples of fractional wave-like and heat-like equations are chosen from [50]. According to the numerical results of suggested method, the standard VIM is not suitable for a large interval. In fact, the comparison between the solutions obtained by the VIM with an auxiliary parameter and the standard VIM shows that large interval has no effect on the accuracy of solutions of the proposed method.

Example 5.1 Consider the following one-dimensional fractional heat-like equation [50]:

$$\begin{cases} D_t^{\alpha} u = \dfrac{1}{2}x^2 u_{xx}, & 0 \leq x \leq 10, \quad 0 < \alpha \leq 1, \quad t > 0, \\ u(x,0) = x, \quad u_t(x,0) = x^2, & 0 \leq x \leq 10, \end{cases}$$

where the exact solution is $u(x,t) = x + x^2\sinh(t)$, when $\alpha = 1$. Take $(x,t) \in [0,10]\times[0,10]$. According to the recursive formula (3.5), we will have:

$$\begin{aligned} &u_{n+1}(x,t,h) = u_n(x,t,h) \\ &+\frac{h}{\Gamma(\alpha)}\int_0^t (s-t)^{\alpha-1}\left\{\frac{\partial^{\alpha} u_n(x,s,h)}{\partial s^{\alpha}} - \frac{1}{2}x^2\frac{\partial^2 u_n(x,s,h)}{\partial x^2}\right\} ds, \quad n \geqslant 1. \end{aligned} \tag{5.1}$$

Beginning with $u_0(x,t) = u(x,0) + tu_t(x,0) = x + x^2 t$, the solution procedure is stopped at $u_{10}(x,t,h)$. It is noteworthy that by letting $h = 1$ in (5.1) we have the

TABLE 1. Comparison of absolute errors for 22th-order approximation by present method with $h = 1.20$ and standard VIM, when $\alpha = 1$ in Example 5.1.

x	t	Absolute error in present method	Absolute error in standard VIM
1	1	$4.73546953 \times 10^{-17}$	$4.03604836 \times 10^{-23}$
2	2	$2.26802629 \times 10^{-15}$	$1.41547605 \times 10^{-15}$
3	3	$4.15368839 \times 10^{-13}$	$3.74267852 \times 10^{-11}$
4	4	$1.34772860 \times 10^{-12}$	$5.218125114 \times 10^{-8}$
5	5	$8.71728028 \times 10^{-10}$	$1.45232207 \times 10^{-5}$
6	6	$2.04026218 \times 10^{-8}$	$1.46017192 \times 10^{-3}$
7	7	$3.77319357 \times 10^{-7}$	$7.27748367 \times 10^{-2}$
8	8	$1.24245609 \times 10^{-5}$	2.17210382
9	9	$5.57649610 \times 10^{-4}$	4.38625622×10^{1}
10	10	$1.76516888 \times 10^{-2}$	6.51403668×10^{2}

solutions of standard VIM. For detecting a appropriate value of h, we define the following function

$$\beta_i(h) = \frac{\| v_{i+1}(x,t,h) \|}{\| v_i(x,t,h) \|}, \ i = 0, 1, 2, \cdots, N-1.$$

Where, as mentioned,

$$v_{i+1}(x,t,h) = u_{i+1}(x,t,h) - u_i(x,t,h),$$

and

$$\| v_i \|^2 = \int_0^{10} \int_0^{10} |v_i(x,t,h)|^2 \, dt \, dx.$$

We apply a numerical integration to approximate $\|v_i\|$. For obtaining an optimal value of h, we minimized $\beta_i(h)$ by using Maple software. Figure 1 shows the two-dimensional variation of 10th-order approximate solution with respect to $t = 10$ and x for different values of α. Figure 2 describes absolute error for the 22th-order approximation by present method for $u(x,t)$, when $\alpha = 1$. The comparison between absolute errors for 22th-order approximation by present method and standard VIM for $\alpha = 1$ is given in Table 1. The numerical results in Table 1 shows that present method works very well in large domain, while standard VIM is ineffective. Table 2 shows that in the proposed approach, the values of β_is close to zero compared to the standard VIM. So, the present method has higher convergence speed to the exact solution to meet the standard VIM.

Example 5.2 Consider the following two-dimensional fractional wave-like equation [50]:

$$\begin{cases} D_{tt}^{\alpha} u = \dfrac{1}{12}\left(x^2 u_{xx} + y^2 u_{yy}\right), & 0 \le x, y \le 20, \quad 1 < \alpha \le 2, \quad t > 0, \\ u(x,y,0) = x^4, \qquad u_t(x,y,0) = y^4, & 0 \le x, y \le 20. \end{cases} \tag{5.2}$$

TABLE 2. Values of β_i defended in (**16**) for present method and standard VIM in Example 5.1.

	Present method	Standard VIM	Present method	Standard VIM	Present method	Standard VIM	Present method	Standard VIM
β_i	$\alpha = 0.7$ h=2.0263	$\alpha = 0.7$	$\alpha = 0.8$ h=1.8026	$\alpha = 0.8$	$\alpha = 0.9$ h=1.6153	$\alpha = 0.9$	$\alpha = 1$ h=1.5335	$\alpha = 1$
β_0	2.70×10^3	1.60×10^2	3.27×10^3	3.10×10^2	4.02×10^3	5.91×10^2	6.11×10^3	1.10×10^3
β_1	5.76×10^2	7.15×10^1	6.46×10^2	1.08×10^2	6.94×10^2	1.58×10^2	8.54×10^2	2.22×10^2
β_2	1.79×10^2	3.4×10^1	1.75×10^2	4.34×10^1	1.64×10^2	5.28×10^1	1.69×10^2	6.18×10^1
β_3	6.27×10^1	1.85×10^1	5.53×10^1	2.09×10^1	4.73×10^1	2.24×10^1	4.29×10^1	2.30×10^1
β_4	2.31×10^1	1.12×10^1	1.88×10^1	1.14×10^1	1.51×10^1	1.11×10^1	1.22×10^1	1.03×10^1
β_5	8.46	7.28	6.45	6.88	4.97	6.18	3.55×10^1	5.31
β_6	2.85	5.00	2.05	4.42	1.54	3.72	9.49×10^{-1}	2.99
β_7	7.64×10^{-1}	3.59	5.18×10^{-1}	2.99	3.96×10^{-1}	2.37	1.89×10^{-1}	1.80
β_8	1.04×10^{-1}	2.66	6.48×10^{-2}	2.11	5.75×10^{-2}	1.59	1.31×10^{-2}	1.15
β_9	9.23×10^{-4}	2.03	3.85×10^{-4}	1.54	4.15×10^{-4}	1.11	1.99×10^{-4}	7.68×10^{-1}

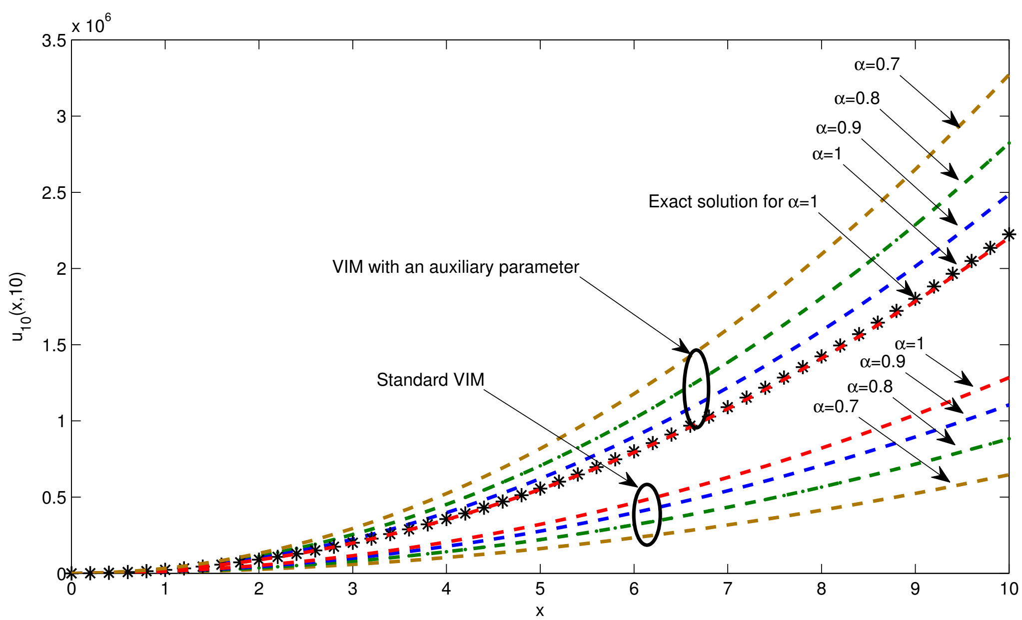

FIGURE 1. Plots of 10th-order approximation solutions by present method and standard VIM at $t = 10$ for different values of α in Example 5.1.

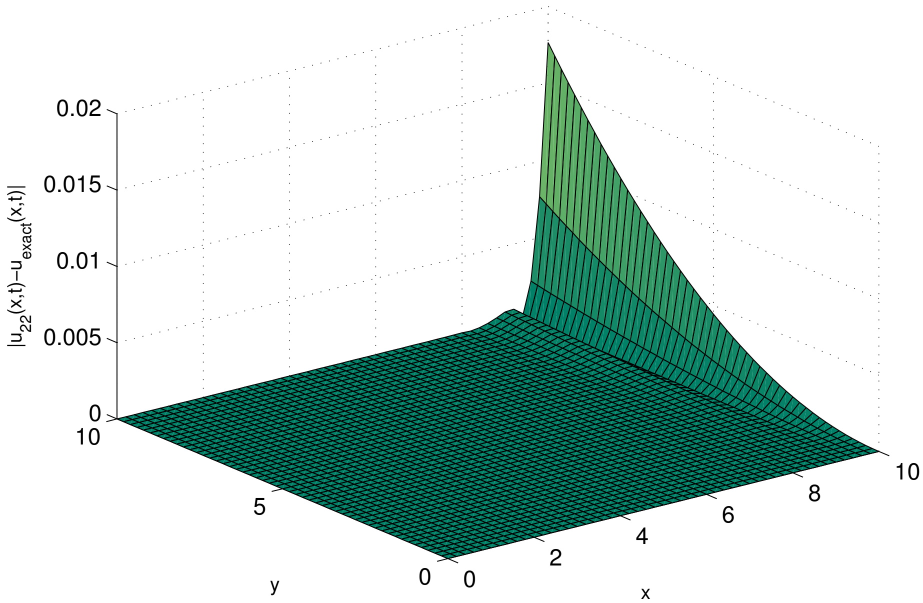

FIGURE 2. Absolute error for the 22th-order approximation by present method for $u(x,t)$ when $\alpha = 1$ and $h = 1.20$, in Example 5.1.

Where $(x, y, t) \in [0, 20] \times [0, 20] \times [0, 20]$, and the exact solution is $u(x, y, t) = x^4 \cosh(t) + y^4 \sinh(t)$, when $\alpha = 2$. Using the iteration scheme (3.5), we successively have

$$u_0(x,y,t) = u(x,y,0) + tu_t(x,y,0) = x^4 + y^4 t,$$
$$u_1(x,y,t) = x^4 + y^4 t + h\left(\tfrac{1}{2}x^4t^2 + \tfrac{1}{6}y^4t^3\right),$$

and in general,

$$u_{n+1}(x,y,t,h) = u_n(x,y,t,h) + \frac{h}{\Gamma(\alpha)}\int_0^t (s-t)^{\alpha-1}\left\{\frac{\partial^\alpha u_n(x,y,s,h)}{\partial s^\alpha} - \frac{1}{12}\left(x^2\frac{\partial^2 u_n(x,y,s,h)}{\partial x^2} + y^2\frac{\partial^2 u_n(x,y,s,h)}{\partial y^2}\right)\right\}ds,\ n \geqslant 1. \tag{5.3}$$

For obtaining an optimal value of auxiliary parameter h we define:

$$\beta_i(h) = \frac{\parallel v_{i+1}(x,y,t,h)\parallel}{\parallel v_i(x,y,t,h)\parallel},\ i = 0,1,2,\cdots,N-1.$$

The value of h is obtained by minimizing $\beta_i(h)$ where

$$\parallel v_i \parallel^2 = \int_0^{20}\int_0^{20}\int_0^{20} |v_i(x,y,t,h)|^2\, dt\, dx\, dy. \tag{5.4}$$

The plot of $u_{10}(x, 20, 20)$, by standard VIM and VIM with an auxiliary parameter, is shown in the Figure 3 with different values of $1 \leq \alpha \leq 2$, also Figure 4 illustrate absolute error for the $u_{28}(x, 20, 20)$, by VIM with an auxiliary parameter. Table 3 describes the comparison between absolute errors for 27th-order approximation by present method and standard VIM for $\alpha = 2$. Table 4 shows the proposed method has more convergence rate compared to standard VIM.

TABLE 3. Comparison of absolute errors for 27th-order approximation by present method with $h = 1.0501$ and standard VIM, when $\alpha = 2$ in Example 5.2.

x	y	t	Absolute error in present method	Absolute error in standard VIM
2	2	2	$7.76564115 \times 10^{-32}$	$1.68047000 \times 10^{-57}$
4	4	4	$6.90870467 \times 10^{-27}$	$2.01042234 \times 10^{-39}$
6	6	6	$9.16676450 \times 10^{-23}$	$7.68083085 \times 10^{-29}$
8	8	8	$1.13489578 \times 10^{-19}$	$2.50617878 \times 10^{-21}$
10	10	10	$1.70831555 \times 10^{-16}$	$1.70441238 \times 10^{-15}$
12	12	12	$4.88799595 \times 10^{-13}$	$1.00264396 \times 10^{-10}$
14	14	14	$3.00317174 \times 10^{-10}$	$1.09004956 \times 10^{-6}$
16	16	16	$2.80880695 \times 10^{-7}$	$3.44600930 \times 10^{-3}$
18	18	18	$6.97570499 \times 10^{-5}$	4.24382026
20	20	20	$9.21663628 \times 10^{-3}$	2.48608161×10^{3}

TABLE 4. Values of β_i defended in (**16**) for present method and standard VIM in Example 5.2.

	Present method	Standard VIM	Present method	Standard VIM	Present method	Standard VIM	Present method	Standard VIM
β_i	$\alpha = 1.7$ h=1.5517	$\alpha = 1.7$	$\alpha = 1.8$ h=1.4682	$\alpha = 1.8$	$\alpha = 1.9$ h=1.3984	$\alpha = 1.9$	$\alpha = 2$ h=1.3397	$\alpha = 2$
β_0	1.00×10^{13}	3.00×10^{11}	2.46×10^{13}	1.14×10^{12}	6.21×10^{13}	4.24×10^{12}	1.61×10^{14}	1.55×10^{13}
β_1	1.58×10^{10}	5.91×10^{8}	2.21×10^{10}	1.23×10^{9}	3.11×10^{10}	2.47×10^{9}	4.39×10^{10}	4.78×10^{9}
β_2	1.50×10^{8}	7.94×10^{6}	1.51×10^{8}	1.15×10^{7}	1.52×10^{8}	1.60×10^{7}	1.54×10^{8}	2.15×10^{7}
β_3	3.41×10^{6}	2.97×10^{5}	2.76×10^{6}	3.34×10^{5}	2.22×10^{6}	3.61×10^{5}	1.79×10^{6}	3.75×10^{5}
β_4	1.20×10^{5}	2.08×10^{4}	8.29×10^{4}	1.93×10^{4}	5.66×10^{4}	1.71×10^{4}	3.84×10^{4}	1.46×10^{4}
β_5	5.04×10^{3}	2.24×10^{3}	3.03×10^{3}	1.77×10^{3}	1.81×10^{3}	1.34×10^{3}	1.08×10^{3}	9.76×10^{3}
β_6	1.89×10^{2}	3.28×10^{2}	1.01×10^{2}	2.27×10^{2}	5.44×10^{1}	1.50×10^{2}	2.89×10^{1}	9.56×10^{1}
β_7	3.91	6.06×10^{1}	1.89	3.74×10^{1}	9.16×10^{-1}	2.20×10^{1}	4.39×10^{-1}	1.24×10^{1}
β_8	7.52×10^{-3}	1.34×10^{1}	3.20×10^{-3}	7.50	1.36×10^{-3}	4.00	5.77×10^{-4}	2.04
β_9	1.74×10^{-3}	3.45	7.60×10^{-4}	1.75	3.31×10^{-4}	8.57×10^{-1}	1.43×10^{-4}	4.00×10^{-1}

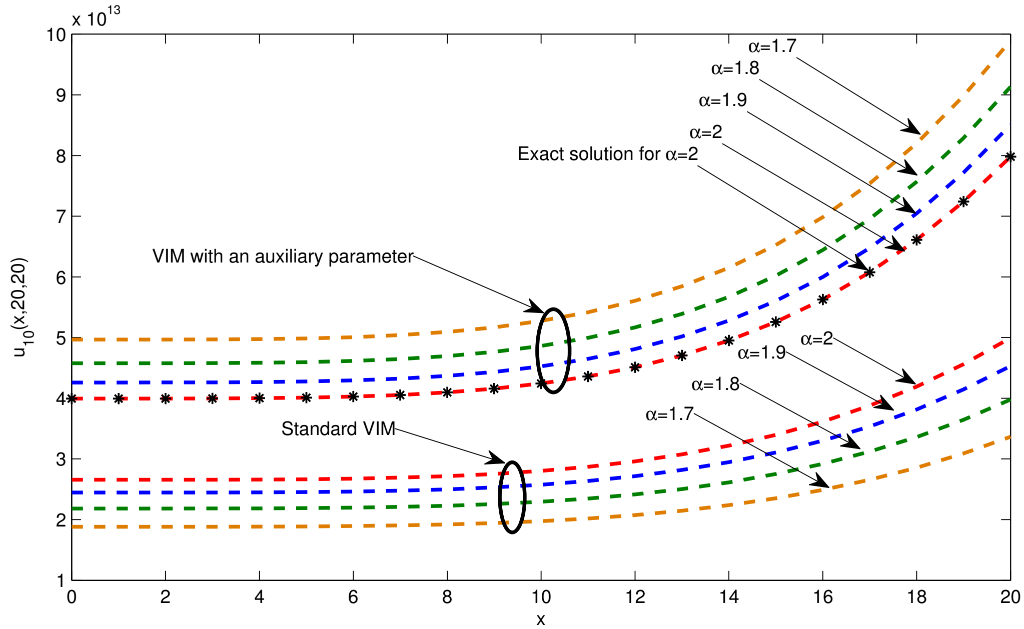

FIGURE 3. Plots of 10th-order approximation solutions by present method and standard VIM at $t = 20$ and $y = 20$ for different values of α in Example 5.2.

Example 5.3 Consider the following three-dimensional fractional heat-like equation [50]:

$$\begin{cases} D_t^{\alpha} u = x^4 y^4 z^4 + \dfrac{1}{36}\left(x^2 u_{xx} + y^2 u_{yy} + z^2 u_{zz}\right), & 0 \le x, y, z \le 10, \quad 0 < \alpha \le 1, \quad t > 0, \\ u\left(x, y, z, 0\right) = 0, & 0 \le x, y, z \le 10. \end{cases} \tag{5.5}$$

Where $(x, y, z, t) \in [0, 10] \times [0, 10] \times [0, 10] \times [0, 10]$, and the exact solution is $u(x, y, z, t) = x^4 y^4 z^4 \left(e^t - 1\right)$, when $\alpha = 1$. According to the the recursive scheme (3.5), we successively have:

$$\begin{aligned} &u_0\left(x, y, z, t\right) = u\left(x, y, z, 0\right) = 0, \\ &u_1\left(x, y, t\right) = h x^4 y^4 z^4 t, \end{aligned}$$

and in general,

$$\begin{aligned} u_{n+1}\left(x, y, z, t, h\right) = u_n\left(x, y, z, t, h\right) - \frac{h}{\Gamma(\alpha)} \int_0^t \Bigg\{ \frac{\partial^{\alpha} u_n(x, y, z, s, h)}{\partial s^{\alpha}} - x^4 y^4 z^4 \\ - \frac{1}{36}\left(x^2 \frac{\partial^2 u_n(x, y, z, s, h)}{\partial x^2} + y^2 \frac{\partial^2 u_n(x, y, z, s, h)}{\partial y^2} + z^2 \frac{\partial^2 u_n(x, y, z, s, h)}{\partial z^2} \right) \Bigg\} ds. \end{aligned} \tag{5.6}$$

We stop the solution procedure at $u_{10}\left(x, y, z, t\right)$. Here, too, such as before in order to find a suitable value of h, we define the following functions:

$$\beta_i(h) = \frac{\| v_{i+1}(x, y, z, t, h) \|}{\| v_i(x, y, z, t, h) \|}, \; i = 0, 1, 2, \cdots, N - 1.$$

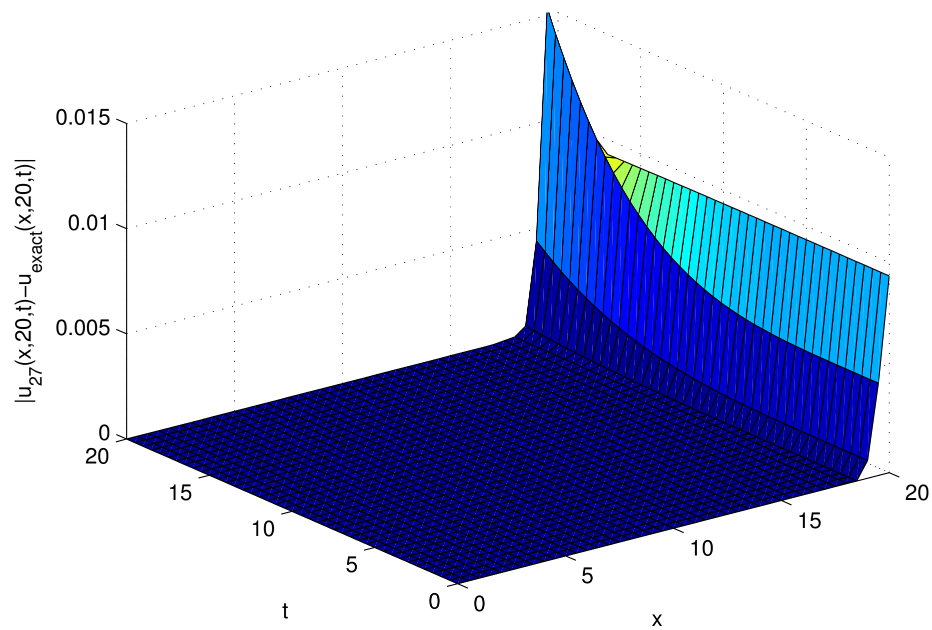

FIGURE 4. Absolute error for the 27th-order approximation by present method for $u(x, 20, t)$ when $\alpha = 2$ and $h = 1.0501$, in Example 5.2.

and

$$\| v_i \|^2 = \int_0^{10} \int_0^{10} \int_0^{10} \int_0^{10} |v_i\,(x, y, z, t, h)|^2\, dt\, dx\, dy\, dz.$$

Figure 5 shows the plot of $u_{10}(x, 10, 10, 10)$, by standard VIM and VIM with an auxiliary parameter, with different values of $0 \leq \alpha \leq 1$, also Figure 6 indicates absolute error of the $u_{34}(x, 10, 10, t)$, for presented method. Table 5 gives the comparison between absolute errors for 34th-order approximation by present method and standard VIM for $\alpha = 1$. Table 6 shows that approximate solution by present approach with regard to values β convergent to the exact solution. In actuality, the accuracy is significant rectified by the optimal choice of h.

6. Discussion and Conclusions

The variational iteration method has been successfully used for solving many application problems. However, difficulties may arise in dealing with obtaining suitable accuracy in large domains. To overcome these difficulties the modified VIM is proposed using VIM with an auxiliary parameter and is applied for solving fractional wave-like and heat-like equations. Figures and numerical results were presented to determine higher accuracy and simplicity of proposed method. Actually, our method is easy to implement and capable of approximate solution more accurate in a bigger interval compared to the original VIM. Also, it should be mentioned that the proposed method can be easily generalized for more complicated

TABLE 5. Comparison of absolute errors for 34th-order approximation by present method with $h = 1.169$ and standard VIM, when $\alpha = 1$ in Example 5.3.

x	y	z	t	Absolute error in present method	Absolute error in standard VIM
1	1	1	1	$4.64732923 \times 10^{-29}$	$2.76283711 \times 10^{-42}$
2	2	2	2	$8.21047259 \times 10^{-24}$	$7.99836516 \times 10^{-28}$
3	3	3	3	$3.25142628 \times 10^{-20}$	$2.33302687 \times 10^{-19}$
4	4	4	4	$1.75146139 \times 10^{-16}$	$2.38707738 \times 10^{-13}$
5	5	5	5	$9.52323734 \times 10^{-14}$	$1.10357676 \times 10^{-8}$
6	6	6	6	$3.13809518 \times 10^{-11}$	$7.19690799 \times 10^{-5}$
7	7	7	7	$3.99933415 \times 10^{-10}$	121519317×10^{-1}
8	8	8	8	$1.38290909 \times 10^{-6}$	7.63388404×10^{1}
9	9	9	9	$1.33673238 \times 10^{-4}$	2.25430185×10^{4}
10	10	10	10	$3.26400849 \times 10^{-3}$	3.67115833×10^{6}

TABLE 6. Values of β_i defended in (**16**) for present method and standard VIM in Example 5.3.

	Present method	Standard VIM	Present method	Standard VIM	Present method	Standard VIM	Present method	Standard VIM
β_i	$\alpha = 0.7$ h=2.0414	$\alpha = 0.7$	$\alpha = 0.8$ h=1.8053	$\alpha = 0.8$	$\alpha = 0.9$ h=1.6296	$\alpha = 0.9$	$\alpha = 1$ h=1.4949	$\alpha = 1$
β_0	4.92×10^{10}	5.41×10^{5}	1.39×10^{11}	1.09×10^{7}	4.43×10^{11}	1.79×10^{8}	1.52×10^{12}	2.44×10^{9}
β_1	6.86×10^{8}	4.37×10^{5}	8.19×10^{8}	1.56×10^{6}	1.00×10^{9}	5.02×10^{6}	1.25×10^{9}	1.46×10^{7}
β_2	1.52×10^{7}	5.07×10^{4}	1.13×10^{7}	1.03×10^{5}	8.52×10^{6}	1.83×10^{5}	6.30×10^{6}	2.82×10^{5}
β_3	4.14×10^{5}	8.25×10^{3}	2.13×10^{5}	1.09×10^{4}	1.09×10^{5}	1.22×10^{4}	5.45×10^{4}	1.15×10^{4}
β_4	1.14×10^{4}	1.72×10^{3}	4.34×10^{3}	1.60×10^{3}	1.62×10^{3}	1.23×10^{3}	5.90×10^{2}	7.99×10^{2}
β_5	2.58×10^{2}	4.32×10^{2}	7.52×10^{1}	3.00×10^{2}	2.15×10^{1}	1.70×10^{2}	5.97	8.01×10^{1}
β_6	3.41	1.26×10^{2}	7.76×10^{-1}	6.81×10^{1}	1.73×10^{-1}	2.97×10^{1}	3.77×10^{-2}	1.06×10^{1}
β_7	1.18×10^{-2}	4.15×10^{1}	2.09×10^{-3}	1.79×10^{1}	3.66×10^{-4}	6.25	6.26×10^{-5}	1.76
β_8	4.55×10^{-7}	1.50×10^{1}	5.38×10^{-8}	5.36	6.44×10^{-9}	1.52	7.76×10^{-10}	3.48×10^{-1}
β_9	1.09×10^{-7}	5.91	1.31×10^{-8}	1.77	1.59×10^{-9}	4.20×10^{-1}	1.94×10^{-10}	7.98×10^{-2}

fractional problems in large domains.

Acknowledgements

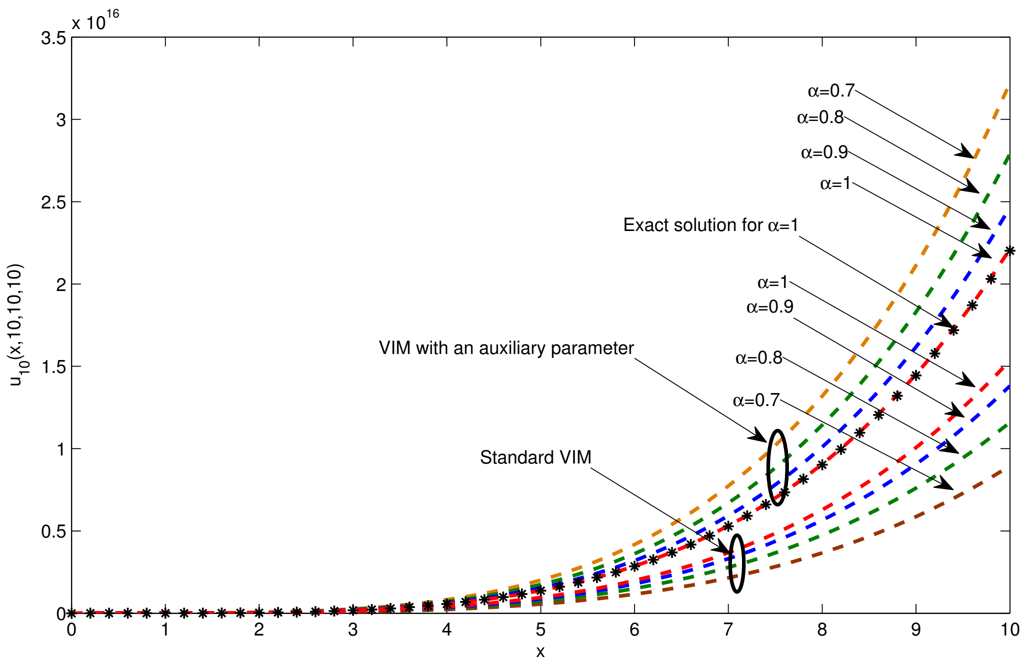

FIGURE 5. Plots of 10th-order approximation solutions by present method and standard VIM at $t = 10$, $y = 10$ and $z = 10$ for different values of α in Example 5.3.

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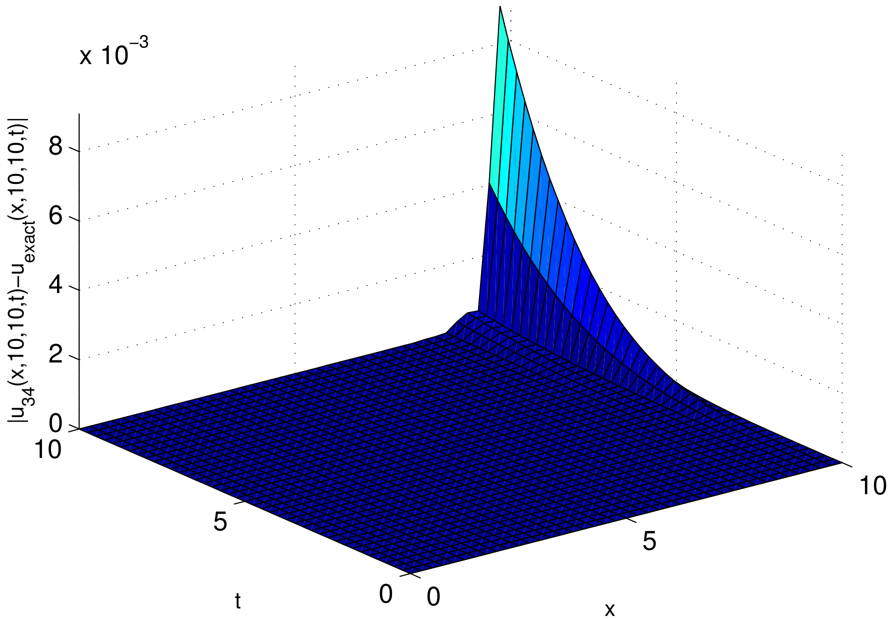

FIGURE 6. Absolute error for the 34th-order approximation by VIM with an auxiliary parameter for $u(x, 10, 10, t)$ when $\alpha = 1$ and $h = 1.169$, in Example 5.3.

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