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A REMARKS ON THE PAPER " SOME FIXED POINT THEOREMS FOR GENERALIZED CONTRACTIVE MAPPINGS IN COMPLETE METRIC SPACES "

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Abstract. In this short paper we show that the results obtained by [N. Hussain, V. Parvaneh, B. Samet and C. Vetro, *Some fixed point theorems for generalized contractive mappings in complete metric spaces*, Fixed Point Theory and Applications (2015) 2015:185] can be obtained without the continuity assumption for the self mapping.

1. Introduction and preliminaries.

The Banach contraction mapping principle is the first important result on fixed points for contractive-type mappings. This well-known theorem, becomes an essential tool in many branches of mathematical analysis. According to its importance and simplicity, several authors have obtained many interesting extensions and generalizations of the Banach contraction principle. Some of such generalizations are obtained by contraction conditions described by rational expressions (see [[3]-[16]]).

Definition 1.1. Let X be a nonempty set and $d : X \times X \to [0, \infty)$ be a mapping such that for all $x, y, z \in X$ the following conditions hold:

(1) $d(x, y) = 0$ iff $x = y$;
(2) $d(x, y) = d(y, x)$
(3) $d(x, y) \leq d(x, z) + d(z, y)$ (triangle inequality).

Then (X, d) is called metric space.

In [1] Jleli and Samet introduced a new type of contractive mappings by using a class of functions was called Θ and established a new fixed point theorem for such contractive mappings in the setting of generalized metric spaces. Consistent with [1], we denote by Θ the set of all functions $\theta : (0, \infty) \to (1, \infty)$ satisfying the following conditions:

(1) (θ_1) θ is nondecreasing;
(2) (θ_2) for each sequence $\{t_n\} \subseteq (0, \infty)$, $\lim_{n\to\infty} \theta(t_n) = 1$ if and only if $\lim_{n\to\infty} t_n = 0$;

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(3) (θ_3) there exist $r \in (0,1)$ and $L \in (0,\infty]$ such that $\lim_{t\to 0^+} \frac{\theta(t)-1}{t^r} = L$.

Since every metric space is a generalized metric space, they established the following result as a generalization of Banach Contraction Principle.

Theorem 1.2. *([1], Corollary 2.1) Let (X,d) be a complete metric space and $f : X \to X$ be a mapping. Suppose that there exist $\theta \in \Theta$ and $k \in (0,1)$ such that*

$$x, y \in X, \quad d(fx, fy) \neq 0 \Longrightarrow \theta(d(fx, fy)) \leq [\theta(d(x,y))]^k . \tag{1.1}$$

Then f has a unique fixed point.

In 2015, Hussain et al. [2] modified the above family of functions Θ by adding a new condition with enriching the domain and the range. To be consistent with Hussain et al. [2], we denote by Ψ the set of all functions $\psi : [0,\infty) \to [1,\infty)$ satisfying the conditions $(\psi_1 - \psi_4)$.

(1) (ψ_1) ψ is nondecreasing and $\psi(t) = 1$ if and only if $t = 0$;
(2) (ψ_2) for each sequence $\{t_n\} \subseteq (0,\infty)$, $\lim_{n\to\infty} \psi(t_n) = 1$ if and only if $\lim_{n\to\infty} t_n = 0$;
(3) (ψ_3) there exist $r \in (0,1)$ and $L \in (0,\infty]$ such that $\lim_{t\to 0^+} \frac{\psi(t)-1}{t^r} = L$;
(4) (ψ_4) $\psi(a+b) \leq \psi(a)\psi(b)$ for all $a, b > 0$.

Also, they modified the contraction (1.1) to be as follows:

Definition 1.3. [2] Let (X,d) be a metric space, and let $f : X \to X$ be a self mapping. Then f is said to be a JS-contraction whenever there exist a function $\psi \in \Psi$ and positive real numbers k_1, k_2, k_3, k_4 with $0 \leq k_1 + k_2 + k_3 + 2k_4 < 1$ such that

$$\begin{aligned} \psi(d(fx, fy)) \quad \leq \quad & [\psi(d(x,y))]^{k_1} [\psi(d(x,fx))]^{k_2} [\psi(d(y,fy))]^{k_3} \\ & \times [\psi(d(x,fy) + d(y,fx))]^{k_4} , \end{aligned} \tag{1.2}$$

for all $x, y \in X$.

Furthermore, they proved the following result.

Theorem 1.4. *([2], Theorem 2.3) Let (X,d) be a complete metric space and $f : X \to X$ be a continuous JS-contraction. Then f has a unique fixed point.*

From the above Theorem they give the following results:

Theorem 1.5. *([2], Theorem 2.4) Let (X,d) be a complete -metric space and $f : X \to X$ be a continuous mapping. Suppose that there exist positive real numbers k_1, k_2, k_3, k_4 with $0 \leq k_1 + k_2 + k_3 + 2k_4 < 1$ such that*

$$\sqrt{d(fx, fy)} \leq k_1 \sqrt{d(x,y)} + k_2 \sqrt{d(x,fx)} + k_3 \sqrt{d(y,fy)} + k_4 \sqrt{d(x,fy) + d(y,fx)} \tag{1.3}$$

for all $x, y \in X$. Then f has a unique fixed point.

Theorem 1.6. *([2], Theorem 2.6) Let (X,d) be a complete -metric space and $f : X \to X$ be a continuous mapping. Suppose that there exist positive real numbers k_2, k_3 with $0 \leq k_2 + k_3 < 1$ such that*

$$d(fx, fy) \leq k_2^2 d(x,fx) + k_3^2 d(y,fy) + 2k_2 k_3 \sqrt{d(x,fx)\, d(y,fy)} \tag{1.4}$$

for all $x, y \in X$. Then f has a unique fixed point.

Theorem 1.7. *([2], Theorem 2.7) Let (X,d) be a complete -metric space and $f : X \to X$ be a continuous mapping. Suppose that there exist positive real numbers $k_4 \in [0, \frac{1}{2})$ such that*

$$d\,(fx, fy) \leq k_4^2 \left[d\,(x, fy) + d\,(y, fx)\right] \tag{1.5}$$

for all $x, y \in X$. Then f has a unique fixed point.

Theorem 1.8. *([2], Theorem 2.8) Let (X,d) be a complete -metric space and $f : X \to X$ be a continuous mapping. Suppose that there exist positive real numbers k_1, k_2, k_3 with $0 \leq k_1 + k_2 + k_3 < 1$ such that*

$$\begin{aligned} d\,(fx, fy) \quad \leq \quad & k_1^2 d\,(x, y) + k_2^2 d\,(x, fx) + k_3^2 d\,(y, fy) + 2k_1k_2\sqrt{d\,(x,y)\,d\,(x, fx)} \\ & +2k_1k_3\sqrt{d\,(x,y)\,d\,(y,fy)} + 2k_2k_3\sqrt{d\,(x,fx)\,d\,(y,fy)} \end{aligned} \tag{1.6}$$

for all $x, y \in X$. Then f has a unique fixed point.

Theorem 1.9. *([2], Corollary 2.9) Let (X,d) be a complete -metric space and $f : X \to X$ be a continuous mapping. Suppose that there exist positive real numbers k_1, k_2, k_3, k_4 with $0 \leq k_1 + k_2 + k_3 + 2k_4 < 1$ such that*

$$\sqrt[n]{d\,(fx, fy)} \leq k_1 \sqrt[n]{d\,(x,y)} + k_2 \sqrt[n]{d\,(x,fx)} + k_3 \sqrt[n]{d\,(y,fy)} + k_4 \sqrt[n]{d\,(x,fy) + d\,(y,fx)} \tag{1.7}$$

for all $x, y \in X$. Then f has a unique fixed point.

By tracing the proof of Theorem 2.3 in [2], we found that the continuity has only been used in proving the existence of the fixed point. In this short paper, we will show that the above results are still valid without the continuity assumption of the self mapping f.

2. Main Results

Theorem 2.1. *Let (X,d) be a complete metric space and $f : X \to X$ be a JS-contraction. Then f has a unique fixed point.*

Proof. Let $x_0 \in X$ be arbitrary. For $x_0 \in X$, we define the sequence $\{x_n\}$ by $x_n = f^n x_0 = fx_{n-1}$. Also, if there exist $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$, then x_{n_0} is a fixed point of f, and we have nothing to prove. Thus, we suppose that $x_n \neq x_{n+1}$, i.e., $d\,(x_n, x_{n+1}) > 0$ for all $n \in \mathbb{N} \cup \{0\}$.

Following the proof of Theorem 2.3 in [2] we get that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0 \text{ and } \lim_{n\to\infty} x_n = x^* \tag{2.1}$$

Now we will show that x^* is a fixed point of f without using the continuity.

Using triangle inequality we get,

$$d(x^*, fx^*) \quad \leq \quad d(x^*, x_{n+1}) + d(x_{n+1}, fx^*) \tag{2.2}$$

and

$$d\,(x_n, fx^*) \leq (d\,(x_n, x^*)) + (d\,(x^*, fx^*)) \tag{2.3}$$

hence, by the properties of ψ we get that,

$$\psi(d(x^*, fx^*)) \leq \psi(d(x^*, x_{n+1}))\psi(d(x_{n+1}, fx^*)) \tag{2.4}$$

$$\psi(d\,(x_n, fx^*)) \leq \psi(d\,(x_n, x^*))\psi(d\,(x^*, fx^*)) \tag{2.5}$$

But by, using (1.2), (ψ_4) and (2.5) we have

$$\begin{aligned}
\psi\left(d\left(x_{n+1}, fx^*\right)\right) &= \psi\left(d\left(fx_n, fx^*\right)\right) \\
&\leq \left[\psi\left(d\left(x_n, x^*\right)\right)\right]^{k_1}\left[\psi\left(d\left(x_n, fx_n\right)\right)\right]^{k_2}\left[\psi\left(d\left(x^*, fx^*\right)\right)\right]^{k_3} \\
&\quad \times\left[\psi\left(d\left(x_n, fx^*\right)+d\left(x^*, fx_n\right)\right)\right]^{k_4} \\
&\leq \left[\psi\left(d\left(x_n, x^*\right)\right)\right]^{k_1}\left[\psi\left(d\left(x_n, fx_n\right)\right)\right]^{k_2}\left[\psi\left(d\left(x^*, fx^*\right)\right)\right]^{k_3} \\
&\quad \times\left[\psi\left(d\left(x_n, fx^*\right)\right)\right]^{k_4}\left[\psi\left(d\left(x^*, fx_n\right)\right)\right]^{k_4} \\
&= \left[\psi\left(d\left(x_n, x^*\right)\right)\right]^{k_1}\left[\psi\left(d\left(x_n, fx_n\right)\right)\right]^{k_2}\left[\psi\left(d\left(x^*, fx^*\right)\right)\right]^{k_3} \\
&\quad \times\left[\psi(d\left(x_n, x^*\right))\right]^{k_4}\left[\psi(d\left(x^*, fx^*\right))\right]^{k_4}\left[\psi\left(d\left(x^*, fx_n\right)\right)\right]^{k_4} \\
&= \left[\psi\left(d\left(x_n, x^*\right)\right)\right]^{k_1+k_4}\left[\psi(d\left(x^*, fx^*\right))\right]^{k_3+k_4} \\
&\quad \times\left[\psi\left(d\left(x_n, x_{n+1}\right)\right)\right]^{k_2}\left[\psi\left(d\left(x^*, x_{n+1}\right)\right)\right]^{k_4}
\end{aligned} \tag{2.6}$$

Now substituting (2.6) in (2.4) we get

$$\begin{aligned}
\psi(d(x^*, fx^*)) &\leq \psi(d(x^*, x_{n+1}))\left[\psi\left(d\left(x_n, x^*\right)\right)\right]^{k_1+k_4}\left[\psi(d\left(x^*, fx^*\right))\right]^{k_3+k_4} \\
&\quad \times\left[\psi\left(d\left(x_n, x_{n+1}\right)\right)\right]^{k_2}\left[\psi\left(d\left(x^*, x_{n+1}\right)\right)\right]^{k_4}
\end{aligned} \tag{2.7}$$

Hence,

$$\begin{aligned}
1 \leq \left[\psi(d(x^*, fx^*))\right]^{1-k_3-k_4} &\leq \left[\psi(d(x^*, x_{n+1}))\right]^{1+k_4}\left[\psi\left(d\left(x_n, x^*\right)\right)\right]^{k_1+k_4} \\
&\quad \times\left[\psi\left(d\left(x_n, x_{n+1}\right)\right)\right]^{k_2}
\end{aligned} \tag{2.8}$$

By taking the limit as $n \to \infty$ and using, (2.1) ψ_1, ψ_2, in the above equation we get

$$\psi(d(x^*, fx^*)) = 1 \tag{2.9}$$

which implies (from ψ_1) that, $d(x^*, fx^*) = 0$ and so, $fx^* = x^*$. Thus, x^* is a fixed point of f. The uniqueness follows word by word as in proof of Theorem 2.3 in [2]. □

Now the following example illustrate Theorem (2.1) which shows that the function f need not be continuous on X.

Example 2.2. Let $X = [0, \infty)$, $d(x, y) = |x - y|$, $f : X \to X$ be defined by

$$f(x) = \begin{cases} \frac{x}{81}, & \text{if } x \in [0, 1] \\ \\ \frac{x}{64}, & \text{if } x \in (1, \infty), \end{cases} \tag{2.10}$$

$\psi(t) = e^{\sqrt{t}}$ and $k_i = \frac{1}{7}$, for $i = 1, 2, 3, 4$.

Let $x, y \in X$, we consider the following three cases:

Case(1): If $x, y \in [0, 1]$, then $e^{\sqrt{d(fx,fy)}} = \left(e^{\sqrt{|x-y|}}\right)^{\frac{1}{9}} \leq \left(e^{\sqrt{|x-y|}}\right)^{\frac{1}{7}} = \left(e^{\sqrt{d(x,y)}}\right)^{1/7}$

Case(2): If $x, y \in (1, \infty)$, then $e^{\sqrt{d(fx,fy)}} = \left(e^{\sqrt{|x-y|}}\right)^{\frac{1}{8}} \leq \left(e^{\sqrt{|x-y|}}\right)^{\frac{1}{7}} = \left(e^{\sqrt{d(x,y)}}\right)^{\frac{1}{7}}$

Case(3): If one of $x, y \in [0, \infty)$, say x, and the other one $y \in (1, \infty)$, then
$e^{\sqrt{d(fx,fy)}} = \left(e^{\sqrt{|\frac{x}{81} - \frac{y}{64}|}}\right) \leq e^{\sqrt{\frac{y}{64}}} < \left(e^{\sqrt{\frac{63y}{64}}}\right)^{\frac{1}{7}} = \left(e^{\sqrt{|y - \frac{y}{64}|}}\right)^{\frac{1}{7}} = \left(e^{\sqrt{d(y,fy)}}\right)^{\frac{1}{7}}$
Note that in all three cases

$$\begin{aligned} e^{\sqrt{d(fx,fy)}} \quad &\leq \quad \left[e^{\sqrt{d(x,y)}}\right]^{\frac{1}{7}} \left[e^{\sqrt{d(x,fx)}}\right]^{\frac{1}{7}} \left[e^{\sqrt{d(y,fy)}}\right]^{\frac{1}{7}} \\ &\times \left[e^{\sqrt{d(x,fy)+d(y,fx)}}\right]^{\frac{1}{7}}. \end{aligned}$$

One can notice that f is not continuous but satisfies the contractive condition (1.2), and $x = 0$ is the unique fixed point of f.

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