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COMMON FIXED POINT THEOREMS IN G-METRIC SPACES WITH Ω-DISTANCE

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Abstract. In this paper we prove some common fixed point theorems for self mappings of cyclic form using the concept of generalized Ω-distance mappings that was introduced by K. Abodaya *et.al.* [1] in a complete G_b-metric space in sense of A. Aghanjani *et.al.* [2].

1. Introduction and Preliminary

In 1989, Bakhtin [3] introduced the concept of b-metric spaces as a generalization of the usual concept of metric spaces, while the result of Bakhtin became known more by Czerwik [4]. In 2006 Z. Mustafa and B. Sims [5] generalized metric spaces in a different way and called this generalization a G-metric space. After that many authors studied many fixed and common fixed point results in G-metric spaces, see [7]–[36]. In 2010 R. Saadati *et.al.* [15] introduced the concept of Ω-distance as a generalization of ω-distance [16] and proved some of the fixed point results. Then after, many fixed point results were studied using Ω-distance mappings, see [17]–[21].

On the other hand, A. Aghanjani *et.al.* [2] used the concept of G-metric spaces and the concept of b-metric spaces introduced the concept of G_b-metric spaces.

Recently, many authors proved many fixed and common fixed point theorems for mappings of cyclic form in different metric spaces, see [22]-[36].

The concept of G_b-metric spaces is defined as follows:

Definition 1.1. [2] *Let X be a nonempty set and $s \geq 1$ be a given real number. Suppose that a mapping $G : X \times X \times X \to \mathbb{R}^+$ is a function satisfies:*
(G_b1) $G(x, y, z) = 0 \; if \; x = y = z$;
(G_b2) $G(x, x, y) > 0 \; for \; all \; x, y \in X, \; with \; x \neq y$;
(G_b3) $G(x, y, y) \leq G(x, y, z) \, for \, all \, x, y, z \in X \, , with \, y \neq z$;

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(G_b4) *$G(x,y,z) = G(p\{x,y,z\})$, where p is a permutation of x,y,z (symmetry);*

(G_b5) *$G(x,y,z) \leq s[G(x,a,a) + G(a,y,z)]\ \forall\ x,y,z,a \in X$ (rectangle inequality).*

The function G is called a generalized b-metric and the pair (X,G) is called generalized b-metric space or G_b-metric space.

It is clear that the class of G_b-metric spaces is larger than the G-metric spaces.

Example 1.1. [2] *Let (X,G) be a G-metric space, and $G_*(x,y,z) = G(x,y,z)^p$, where $p > 1$ is a real number.*

Aghanjani *et.al.* defined the G_b-convergence and G_b-Cauchy sequences as follows:

Definition 1.2. [2] *Let X be a G_b-metric space. A sequence (x_n) in X is said to be*

(1) G_b-convergent to $x \in X$ if for any $\epsilon > 0$, there exists $k \in \mathbb{N}$ such that for all $n,m \geq k$, $G(x,x_n,x_m) < \epsilon$, .

(2) G_b-Cauchy sequence if for any $\epsilon > 0$, there exists $k \in \mathbb{N}$ such that for all $n,m,l \geq k$, $G(x_n,x_m,x_l) < \epsilon$.

Proposition 1.1. [2] *Let X be a G_b-metric space. Then the following are equivalent:*

(1) the sequence (x_n) is G_b-convergent to x.

(2) $G(x_n,x_n,x) \to 0$ as $n \to \infty$.

(3) $G(x_n,x,x) \to 0$ as $n \to \infty$.

Proposition 1.2. [2] *Let X be a G_b-metric space. Then the following are equivalent:*

(1) the sequence (x_n) is G_b-Cauchy .

(2) for any $\epsilon > 0$, there exists $k \in \mathbb{N}$ such that for all $n,m \geq k$, $G(x_n,x_m,x_m) < \epsilon$.

Definition 1.3. *A G_b-metric space X is called G_b-complete or complete G_b-metric space if every G_b-Cauchy sequence is G_b-convergent in X.*

Definition 1.4. [1] *Let X be a G_b-metric space. Then a mapping $\Omega : X \times X \times X \to [0,\infty)$ is called a generalized Ω-distance mapping or Ω_b-distance mapping on X if the following conditions satisfied:*

(1) $\Omega(x,y,z) \leq s\ [\Omega(x,a,a) + \Omega(a,y,z)]\,\forall\, x,y,z,a \in X$ and $s \geq 1$,

(2) for any $x,y \in X, \Omega(x,y,.), \Omega(x,.,y) : X \to X$ are lower semi continuous,

(3) for every $\epsilon > 0$, there is a $\delta > 0$ such that $\Omega(x,a,a) \leq \delta$ and $\Omega(a,y,z) \leq \delta$ imply $G_b(x,y,z) \leq \epsilon$.

Example 1.2. [1] *Let $X = \mathbb{R}$ and Consider the G_b-metric G defined by $G(x,y,z) = (|x-y|+|y-z|+|x-z|)^2\ \forall\ x,y,z \in \mathbb{R}$. Define $\Omega : X \times X \times X \to [0,\infty)$ by $\Omega(x,y,z) = (|x-y|+|x-z|)^2\ \forall\ x,y,z \in \mathbb{R}$. Then Ω is a generalized Ω-distance with $s = 2$.*

Definition 1.5. [1] *Let (X,G) be a G_b-metric space and Ω be an Ω_b-distance on X. Then we say that X is Ω-bounded if there exists $M > 0$ such that $\Omega(x,y,z) \leq M$ for all $x,y,x \in X$.*

Lemma 1.1. [1] *Let X be a G_b-metric space and Ω_b be a generalized Ω-distance on X. Let (x_n),(y_n) be sequences in X, (α_n),(β_n) be sequences in $[0,\infty)$ converging to zero and let $x,y,z,a \in X$. Then we have the following:*

(1)If $\Omega_b(y_n,x_n,x_n) \leq \alpha_n$ and $\Omega_b(x_n,y_m,z) \leq \beta_n$ for any $m > n \in \mathbb{N}$, then $G(y_n,y_m,z) \to 0$ and hence $y_n \to z$.

(2)If $\Omega_b(y,x_n,x_n) \leq \alpha_n$ and $\Omega_b(x_n,y,z) \leq \beta_n$ for $n \in \mathbb{N}$, then $G(y,y,z) < \epsilon$ and hence $y = z$.

(3)If $\Omega_b(x_n,x_m,x_l) \leq \alpha_n$ for any $m,n,l \in \mathbb{N}$ with $n \leq m \leq l$, then (x_n) is a G_b-Cauchy sequence.

(4)If $\Omega_b(x_n,a,a) \leq \alpha_n$ for any $n \in \mathbb{N}$, then (x_n) is a G_b-Cauchy sequence.

2. Main Result

In this section, the concept of the cyclic mappings is given as follows:

Definition 2.1. *Let A and B be two nonempty subsets of a space X. A mapping $T : A \cup B \to A \cup B$ is called cyclic if $T(A) \subseteq B$ and $T(B) \subseteq A$.*

Now, we state and prove our main result.

Definition 2.2. *Let (X,G) be a complete G_b-metric space and Ω be a generalized Ω-distance on X with a constant $s > 1$ such that X is Ω-bounded. Let A and B be two nonempty closed subsets of X with respect to the topology induced by G with $X = A \cup B$ and $A \cap B \neq \phi$. Two mappings $f,g : A \cup B \to A \cup B$ are said to be Ω_b-cyclic Kannan contraction if $f(A) \subseteq B$ and $g(B) \subseteq A$ and there exists $r \in [0, \frac{1}{1+s})$ such that the following conditions hold true*

$$\Omega(fx,gfx,gy) \leq r\,[\,\Omega(x,fx,fx) + \Omega(y,gy,gy)\,]\ \forall x \in A \text{ and } \forall y \in B, \tag{2.1}$$

$$\Omega(gx,fgx,fy) \leq r\,[\,\Omega(x,gx,gx) + \Omega(y,fy,fy)\,]\ \forall y \in A \text{ and } \forall x \in B, \tag{2.2}$$

$$\Omega(fx,gfx,fy) \leq r\,[\,\Omega(x,fx,fx) + \Omega(y,fy,fy)\,]\ \forall x,y \in A, \tag{2.3}$$

and

$$\Omega(gx,fgx,gy) \leq r\,[\,\Omega(x,gx,gx) + \Omega(y,gy,gy)\,]\ \forall x,y \in B. \tag{2.4}$$

Theorem 2.1. *Let (X,G) be a complete G_b-metric space and Ω be a generalized Ω-distance on X such that X is Ω-bounded. Let A and B be two nonempty closed subsets of X with respect to the topology induced by G with $X = A \cup B$ and $A \cap B \neq \phi$. Suppose that f and g are Ω_b-cyclic Kannan contraction. Also, $\forall u \in X$ if $fu \neq u$ or $gu \neq u$, then* (b) $\inf\{\Omega(fx,gfx,u) : x \in X\} > 0$.
If f or g is continuous, then f and g have a unique common fixed point in $A \cap B$.

Proof. Let $x_0 \in A$. Since $f(A) \subseteq B$, then $fx_0 = x_1 \in B$. Also, since $g(B) \subseteq A$, then $gx_1 = x_2 \in A$. Continuing this process we obtain a sequence (x_n) in X such that $fx_{2n} = x_{2n+1}$,

$x_{2n} \in A,\ \ gx_{2n+1} = x_{2n+2}$ and $x_{2n+1} \in B,\ n = 0, 1, 2, \cdots$.

First, since X is Ω-bounded, then there exists $M \geq 0$ such that

$$\Omega(x, y, z) \leq M \ \ \forall x, y, z \in X.$$

Now, our claim is to show that $\Omega(x_n, x_{n+1}, x_{n+s}) \leq q^{n-1}\ M\ \forall n, s \in \mathbb{N}$, where $q = \dfrac{r}{1-r}$.

Let $n, s \in \mathbb{N}$. Then we have four cases:

Case (1): n is even and s is even. Therefore $n = 2t$ for some $t \in \mathbb{N}$. By (2.4), we have

$$\begin{aligned} \Omega(x_n, x_{n+1}, x_{n+s}) &= \Omega(x_{2t}, x_{2t+1}, x_{2t+s}) \\ &= \Omega(gx_{2t-1}, fgx_{2t-1}, gx_{2t+s-1}) \\ &\leq r\ [\,\Omega(x_{2t-1}, x_{2t}, x_{2t}) + \Omega(x_{2t+s-1}, x_{2t+s}, x_{2t+s})\,]. \end{aligned} \tag{2.5}$$

Also, by (2.1), we get

$$\begin{aligned} &\Omega(x_{2t-1}, x_{2t}, x_{2t}) + \Omega(x_{2t+s-1}, x_{2t+s}, x_{2t+s}) \\ &=\Omega(fx_{2t-2}, gfx_{2t-2}, gx_{2t-1}) + \Omega(fx_{2t+s-2}, gfx_{2t+s-2}, gx_{2t+s-1}) \\ &\leq r\,[\Omega(x_{2t-2}, x_{2t-1}, x_{2t-1}) + \Omega(x_{2t-1}, x_{2t}, x_{2t})] \\ &+r\ [\Omega(x_{2t+s-2}, x_{2t+s-1}, x_{2t+s-1}) + \Omega(x_{2t+s-1}, x_{2t+s}, x_{2t+s})]. \end{aligned}$$

Therefore

$$\begin{aligned} &\Omega(x_{2t-1}, x_{2t}, x_{2t}) + \Omega(x_{2t+s-1}, x_{2t+s}, x_{2t+s}) \\ &\leq \frac{r}{1-r}\,[\Omega(x_{2t-2}, x_{2t-1}, x_{2t-1}) + \Omega(x_{2t+s-2}, x_{2t+s-1}, x_{2t+s-1})\,] \\ &\leq q\ [\,\Omega(x_{2t-2}, x_{2t-1}, x_{2t-1}) + \Omega(x_{2t+s-2}, x_{2t+s-1}, x_{2t+s-1})\,]. \end{aligned}$$

Hence by applying the previous steps repeatedly we get

$\Omega(x_{2t-1}, x_{2t}, x_{2t}) + \Omega(x_{2t+s-1}, x_{2t+s}, x_{2t+s}) \leq q^{n-1}\,[\Omega(x_0, x_1, x_1) + \Omega(x_s, x_{s+1}, x_{s+1})]$.

Since X in Ω-bounded, then $\Omega(x_{2t-1}, x_{2t}, x_{2t}) + \Omega(x_{2t+s-1}, x_{2t+s}, x_{2t+s}) \leq 2q^{n-1}M$.

Since $r < \frac{1}{2}$, then inequality (2.5) becomes

$$\Omega(x_n, x_{n+1}, x_{n+s}) \leq q^{n-1}M. \tag{2.6}$$

Case (2): n is odd, s is even. Therefore n=2t+1 for some $t \in \mathbb{N} \cup \{0\}$. By (2.3), we get

$$\begin{aligned} \Omega(x_n, x_{n+1}, x_{n+s}) &= \Omega(x_{2t+1}, x_{2t+2}, x_{2t+s+1}) \\ &= \Omega(fx_{2t}, gfx_{2t}, fx_{2t+s}) \\ &\leq r\,[\,\Omega(x_{2t}, x_{2t+1}, x_{2t+1}) + \Omega(x_{2t+s}, x_{2t+s+1}, x_{2t+s+1})\,]. \end{aligned} \tag{2.7}$$

By (2.2), we get

$$
\begin{aligned}
&\Omega(x_{2t},x_{2t+1},x_{2t+1})+\Omega(x_{2t+s},x_{2t+s+1},x_{2t+s+1})\\
&=\Omega(gx_{2t-1},fgx_{2t-1},fx_{2t})+\Omega(gx_{2t+s-1},fgx_{2t+s-1},fx_{2t+s})\\
&\leq r\,[\Omega(x_{2t-1},x_{2t},x_{2t})+\Omega(x_{2t},x_{2t+1},x_{2t+1})]\\
&+r\,[\Omega(x_{2t+s-1},x_{2t+s},x_{2t+s})+\Omega(x_{2t+s},x_{2t+s+1},x_{2t+s+1})].
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\Omega(x_{2t},x_{2t+1},x_{2t+1})+\Omega(x_{2t+s},x_{2t+s+1},x_{2t+s+1})\\
&\leq \frac{r}{1-r}\,[\Omega(x_{2t-1},x_{2t},x_{2t})+\Omega(x_{2t+s-1},x_{2t+s},x_{2t+s})\,]\\
&\leq q\,[\,\Omega(x_{2t-1},x_{2t},x_{2t})+\Omega(x_{2t+s-1},x_{2t+s},x_{2t+s})\,].
\end{aligned}
$$

Hence by applying the previous steps repeatedly we get

$\Omega(x_{2t},x_{2t+1},x_{2t+1})+\Omega(x_{2t+s},x_{2t+s+1},x_{2t+s+1})\leq q^{n-1}\,[\Omega(x_0,x_1,x_1)+\Omega(x_s,x_{s+1},x_{s+1})]$.

Since X in Ω-bounded then $\Omega(x_{2t},x_{2t+1},x_{2t+1})+\Omega(x_{2t+s},x_{2t+s+1},x_{2t+s+1})\leq 2q^{n-1}M$.

Since $r<\frac{1}{2}$, then inequality (2.7) becomes

$$\Omega(x_n,x_{n+1},x_{n+s})\leq q^{n-1}M. \tag{2.8}$$

Case (3): n is even, and s is odd. Therefore $n=2t$ for some $t\in\mathbb{N}$. By (2.2), we have

$$
\begin{aligned}
\Omega(x_n,x_{n+1},x_{n+s}) &= \Omega(x_{2t},x_{2t+1},x_{2t+s})\\
&= \Omega(gx_{2t-1},fgx_{2t-1},fx_{2t+s-1})\\
&\leq r\,[\,\Omega(x_{2t-1},x_{2t},x_{2t})+\Omega(x_{2t+s-1},x_{2t+s},x_{2t+s})\,].
\end{aligned} \tag{2.9}
$$

By (2.1) and (2.2), we have

$$
\begin{aligned}
&\Omega(x_{2t-1},x_{2t},x_{2t})+\Omega(x_{2t+s-1},x_{2t+s},x_{2t+s})\\
&=\Omega(fx_{2t-2},gfx_{2t-2},gx_{2t-1})+\Omega(gx_{2t+s-2},fgx_{2t+s-2},fx_{2t+s-1})\\
&\leq r\,[\Omega(x_{2t-2},x_{2t-1},x_{2t-1})+\Omega(x_{2t-1},x_{2t},x_{2t})]\\
&+r\,[\Omega(x_{2t+s-2},x_{2t+s-1},x_{2t+s-1})+\Omega(x_{2t+s-1},x_{2t+s},x_{2t+s})].
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\Omega(x_{2t-1},x_{2t},x_{2t})+\Omega(x_{2t+s-1},x_{2t+s},x_{2t+s})\\
&\leq \frac{r}{1-r}\,[\Omega(x_{2t-2},x_{2t-1},x_{2t-1})+\Omega(x_{2t+s-2},x_{2t+s-1},x_{2t+s-1})\,]\\
&\leq q\,[\,\Omega(x_{2t-2},x_{2t-1},x_{2t-1})+\Omega(x_{2t+s-2},x_{2t+s-1},x_{2t+s-1})\,].
\end{aligned}
$$

Hence by applying the previous steps repeatedly we get

$\Omega(x_{2t-1},x_{2t},x_{2t})+\Omega(x_{2t+s-1},x_{2t+s},x_{2t+s})\leq q^{n-1}\,[\Omega(x_0,x_1,x_1)+\Omega(x_s,x_{s+1},x_{s+1})]$

Since X in Ω-bounded, then $\Omega(x_{2t-1}, x_{2t}, x_{2t}) + \Omega(x_{2t+s-1}, x_{2t+s}, x_{2t+s}) \leq 2q^{n-1}M$.
Since $r < \frac{1}{2}$, then inequality (2.9) becomes

$$\Omega(x_n, x_{n+1}, x_{n+s}) \leq q^{n-1}M. \tag{2.10}$$

Case (4): n is odd, s is odd. Therefore n=2t+1 for some $t \in \mathbb{N} \cup \{0\}$. By (2.1), we get

$$\begin{aligned}
\Omega(x_n, x_{n+1}, x_{n+s}) &= \Omega(x_{2t+1}, x_{2t+2}, x_{2t+s+1}) \\
&= \Omega(fx_{2t}, gfx_{2t}, gx_{2t+s}) \\
&\leq r\,[\,\Omega(x_{2t}, x_{2t+1}, x_{2t+1}) + \Omega(x_{2t+s}, x_{2t+s+1}, x_{2t+s+1})\,].
\end{aligned} \tag{2.11}$$

By (2.1) and (2.2), we have

$\Omega(x_{2t}, x_{2t+1}, x_{2t+1}) + \Omega(x_{2t+s}, x_{2t+s+1}, x_{2t+s+1})$
$=\Omega(gx_{2t-1}, fgx_{2t-1}, fx_{2t}) + \Omega(fx_{2t+s-1}, gfx_{2t+s-1}, gx_{2t+s})$
$\leq r\,[\Omega(x_{2t-1}, x_{2t}, x_{2t}) + \Omega(x_{2t}, x_{2t+1}, x_{2t+1})]$
$+r\,[\Omega(x_{2t+s-1}, x_{2t+s}, x_{2t+s}) + \Omega(x_{2t+s}, x_{2t+s+1}, x_{2t+s+1})]$.

So

$\Omega(x_{2t}, x_{2t+1}, x_{2t+1}) + \Omega(x_{2t+s}, x_{2t+s+1}, x_{2t+s+1})$
$\leq \dfrac{r}{1-r}\,[\Omega(x_{2t-1}, x_{2t}, x_{2t}) + \Omega(x_{2t+s-1}, x_{2t+s}, x_{2t+s})\,]$
$\leq q\,[\,\Omega(x_{2t-1}, x_{2t}, x_{2t}) + \Omega(x_{2t+s-1}, x_{2t+s}, x_{2t+s})\,]$.

Hence by applying the previous steps repeatedly we get
$\Omega(x_{2t}, x_{2t+1}, x_{2t+1}) + \Omega(x_{2t+s}, x_{2t+s+1}, x_{2t+s+1}) \leq q^{n-1}\,[\Omega(x_0, x_1, x_1) + \Omega(x_s, x_{s+1}, x_{s+1})]$.
Since X in Ω-bounded then $\Omega(x_{2t}, x_{2t+1}, x_{2t+1}) + \Omega(x_{2t+s}, x_{2t+s+1}, x_{2t+s+1}) \leq 2q^{n-1}M$.
Since $r < \frac{1}{2}$, then inequality (2.11) becomes

$$\Omega(x_n, x_{n+1}, x_{n+s}) \leq q^{n-1}M. \tag{2.12}$$

Thus in all cases we have

$$\Omega(x_n, x_{n+1}, x_{n+s}) \leq q^{n-1}M,\ \forall n, s\ \in \mathbb{N}. \tag{2.13}$$

Now, $\forall\, l \geq m \geq n$, we have

$$\begin{aligned}
\Omega(x_n,x_m,x_l) &\leq s\Omega(x_n,x_{n+1},x_{n+1})+s\Omega(x_{n+1},x_m,x_l)\\
&\leq s\Omega(x_n,x_{n+1},x_{n+1})+s^2\Omega(x_{n+1},x_{n+2},x_{n+2})+s^2\Omega(x_{n+2},x_m,x_l)\\
&\vdots\\
&\leq s\Omega(x_n,x_{n+1},x_{n+1})+s^2\Omega(x_{n+1},x_{n+2},x_{n+2})+\cdots\\
&+s^{m-n-1}\Omega(x_{m-2},x_{m-1},x_{m-1})+s^{m-n-1}\Omega(x_{m-1},x_m,x_l)\\
&\leq sq^{n-1}M+s^2q^nM+\cdots+s^{m-n-1}q^{m-3}M+s^{m-n-1}q^{m-2}M\\
&\leq sq^{n-1}M+s^2q^nM+s^3q^{n+1}M+\cdots\\
&= sq^{n-1}M(1+sq+(sq)^2+\cdots)\\
&= sq^{n-1}M\frac{1}{1-sq}.
\end{aligned}$$

Thus by Lemma (1.1) (x_n) is a G_b-Cauchy sequence. Therefore there exists $u\in X$ such that (x_n) is G-convergent to u. Since (x_n) G_b-converges to u, then each subsequence of (x_n) also G_b-converges to u. So the subsequences $(x_{2n+1})=(fx_{2n})$ and $(x_{2n+2})=(gx_{2n+1})$ are G_b-converge to u.

First, (without lose of generality) suppose that f is continuous. Then $\lim\limits_{n\to\infty} fx_{2n}=fu$ and $\lim\limits_{n\to\infty} x_{2n+1}=u$, by uniqueness of the limit we have $fu=u$.

Let $\epsilon>0$. By the lower semi continuity of Ω, we have

$\Omega(x_n,x_m,u)\leq \lim\limits_{p\to\infty}\inf\Omega(x_n,x_m,x_p)\leq\epsilon$, $\forall m\geq n$.

Suppose that $gu\neq u$. Then by (b), we get

$$\begin{aligned}
0 &< \inf\{\Omega(fx,gfx,u): x\in X\}\\
&\leq \inf\{\Omega(x_n,x_{n+1},u): n\, odd\}\leq\epsilon
\end{aligned}$$

for each $\epsilon>0$ a contradiction. Hence $u=gu$.

Since $(x_{2n})\subseteq A$ and A is closed, then $u\in A$. Also, since $(x_{2n+1})\subseteq B$ and B is closed, then $u\in B$. Hence u is a common fixed point for f and g in $A\cap B$.

Now, we prove the uniqueness. First, we show that if $w=fw=gw$, then $\Omega(w,w,w)=0$.

By (2.1), we have

$$\begin{aligned}
\Omega(w,w,w)=\Omega(fw,gfw,gw) &\leq r\,[\Omega(w,w,w)+\Omega(w,w,w)]\\
&\leq 2r\Omega(w,w,w).
\end{aligned}$$

Since $r<\frac{1}{2}$, then $\Omega(w,w,w)=0$.

Now, let $v\in X$ be another common fixed point for f and g. Then by (2.1), we get $\Omega(v,v,u)=\Omega(fv,gfv,gu)\leq r\,[\,\Omega(v,v,v)+\Omega(u,u,u)\,]$. Since $v=fv=gv$ and $u=fu=gu$, then $\Omega(v,v,v)=\Omega(u,u,u)=0$. Therefore $\Omega(v,v,u)=0$. Thus by the definition of Ω-distance we have $G(v,v,u)=0$. Hence $u=v$. □

If we choose $X=A=B$ in Theorem (2.1), then we have the following result

Corollary 2.1. *Let (X, G) be a complete G_b-metric space and Ω be a generalized Ω-distance on X with constant $s > 1$ such that X is Ω-bounded. Suppose that $f, g : X \to X$ be two self mappings. Also, assume that there exists $r \in [0, \frac{1}{1+s})$ such that the following conditions hold true:*

$$\Omega(fx, gfx, gy) \leq r\,[\,\Omega(x, fx, fx) + \Omega(y, gy, gy)\,]\ \forall x, y \in X, \tag{2.14}$$

$$\Omega(gx, fgx, fy) \leq r\,[\,\Omega(x, gx, gx) + \Omega(y, fy, fy)\,]\ \forall x, y \in X, \tag{2.15}$$

$$\Omega(fx, gfx, fy) \leq r\,[\,\Omega(x, fx, fx) + \Omega(y, fy, fy)\,]\ \forall x, y \in X, \tag{2.16}$$

and

$$\Omega(gx, fgx, gy) \leq r\,[\,\Omega(x, gx, gx) + \Omega(y, gy, gy)\,]\ \forall x, y \in X. \tag{2.17}$$

Also, $\forall u \in X$ if $fu \neq u$ or $gu \neq u$, then (b) $\inf\{\Omega(fx, gfx, u) : x \in X\} > 0$.
If f or g is continuous, then f and g have a unique common fixed point in X.

If we replace g by f in Theorem (2.1), then condition (b) can be dropped. So we have the following theorem

Theorem 2.2. *Let (X, G) be a complete G_b-metric space and Ω be a generalized Ω-distance on X with constant $s > 1$ such that X is Ω-bounded. Let A and B be two nonempty closed subsets of X with respect to the topology induced by G with $X = A \cup B$ and $A \cap B \neq \phi$. Suppose that $f : A \cup B \to A \cup B$ be a cyclic mapping. Also, assume that there exists $r \in [0, \frac{1}{1+s})$ such that the following conditions hold true*

$$\Omega(fx, f^2x, fy) \leq r\,[\,\Omega(x, fx, fx) + \Omega(y, fy, fy)\,]\ \forall x \in A \text{ and } \forall y \in B, \tag{2.18}$$

$$\Omega(fx, f^2x, fy) \leq r\,[\,\Omega(x, fx, fx) + \Omega(y, fy, fy)\,]\ \forall y \in A \text{ and } \forall x \in B, \tag{2.19}$$

$$\Omega(fx, f^2x, fy) \leq r\,[\,\Omega(x, fx, fx) + \Omega(y, fy, fy)\,]\ \forall x, y \in A, \tag{2.20}$$

and

$$\Omega(fx, f^2x, fy) \leq r\,[\,\Omega(x, fx, fx) + \Omega(y, fy, fy)\,]\ \forall x, y \in B. \tag{2.21}$$

If f is continuous, then f has a unique fixed point in $A \cap B$.

Proof. By following the proof of Theorem (2.1) step by step, we get the result. □

By choosing $X = A = B$ in Theorem (2.2), we get the following result

Corollary 2.2. *Let (X, G) be a complete G_b-metric space and Ω be a generalized Ω-distance on X with constant $s > 1$ such that X is Ω-bounded. Let $f : X \to X$ be a self mapping. Also, assume that there exists $r \in [0, \frac{1}{1+s})$ such that the following condition holds true:*

$$\Omega(fx, f^2x, fy) \leq r\,[\,\Omega(x, fx, fx) + \Omega(y, fy, fy)\,]\ \forall x, y \in X. \tag{2.22}$$

If f is continuous, then f has a unique fixed point in X.

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