Journal of Mathematical Analysis
ISSN: 2217-3412, URL: www.ilirias.com/jma
Volume 8 Issue 1(2017), Pages 143-155.

ON STRONG β-I-OPEN SETS AND DECOMPOSITIONS OF CONTINUITY IN IDEAL TOPOLOGICAL SPACES

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Abstract. In this paper, first we give some characterizations and properties of strong β-I-open sets. Then, we obtain a decomposition of semi-I-closed sets by using strong β-I-open sets and t-I-sets. In addition, we give decompositions of open sets in any extremally disconnected space and in any ideal topological space. Moreover, we define AK_I-set and AK_I^*-set and give decompositions of continuity.

1. Introduction and Preliminaries

In the present paper, (X, τ) or (Y, φ) will denote topological spaces on which no separation property is assumed unless explicitly stated. In a topological space (X, τ), the closure and the interior of any subset A of X will be denoted by $Cl(A)$ and $Int(A)$, respectively. An ideal I on X is defined as a nonempty collection of subsets of X satisfying the following two conditions: (1) If $A \in I$ and $B \subset A$, then $B \in I$; (2) If $A \in I$ and $B \in I$, then $A \cup B \in I$. Let (X, τ) be a topological space and I an ideal on X. An ideal topological space is a topological space (X, τ) with an ideal I on X and is denoted by (X, τ, I). For a subset $A \subset X$, $A^*(I, \tau) = \{x \in X \mid U \cap A \notin I$ for each neighborhood U of $x\}$ is called the local function of A with respect to I and τ [22]. It is obvious that $(.)^* : \wp(X) \longrightarrow \wp(X)$ is a set operator. Throughout this paper, we use A^* instead of $A^*(I, \tau)$. Besides, in [15], authors introduced a new Kuratowski closure operator $Cl^*(.)$ defined by $Cl^*(A) = A \cup A^*(I, \tau)$ and obtained a new topology on X which is called an $*$-topology. This topology is denoted by τ^* (I) which is finer than τ.

We start with recalling two lemmas and two definitions which are necessary for this study in the sequel.

Lemma 1.1. *([15]). Let (X, τ) be a topological space and I an ideal on X . For every subset A of X, the following property holds: $A^* \subset Cl(A)$.*

Lemma 1.2. *([13]). Let A be a subset of an ideal topological space (X, τ, I) and U be an open set. Then, $U \cap Cl^*(A) \subseteq Cl^*(U \cap A)$.*

2000 *Mathematics Subject Classification.* 54A05, 54A10, 54C08, 54C10.
Key words and phrases. strong β-I-open set, α-I-open set, AK_I-set, AB_I^*-set, AK_I^*-set, decompositions of continuity.
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Submitted October 20, 2016. Published December 25, 2016.
Communicated by Farrukh Mukhamedov.

Definition 1.3. *A subset A of an ideal topological space (X, τ, I) is said to be pre-I-open* [4] *(respect., semi-I-open* [11]*, α-I-open* [11]*, β-I-open* [11]*, t-I-set*[11]*, strong β-I-open* [12]*, b-I-open* [17]*, semi*-I-open* [10]*, β^*-I-open* [6] *) if $A \subset Int(Cl^*(A))$ (resp., $A \subset Cl^*(Int(A))$, $A \subset Int(Cl^*(Int(A)))$, $A \subset Cl(Int(Cl^*(A)))$, $Int(A) = Int(Cl^*(A))$, $A \subset Cl^*(Int(Cl^*(A)))$, $A \subset Cl^*(Int(A)) \cup Int(Cl^*(A))$, $A \subset Cl(Int^*(A))$, $A \subset Cl(Int^*(Cl(A)))$).*

Definition 1.4. *The complement of a pre-I-open (resp., semi-I-open, α-I-open, β-I-open, strong β-I-open, b-I-open, semi*-I-open) set is said to be pre-I-closed* [4] *(resp., semi-I-closed* [11]*, α-I-closed* [11]*, β-I-closed* [11]*, strong β-I-closed* [10]*, b-I-closed* [7]*, semi*-I-closed* [10] *).*

In Theorem 25 of [8], it is shown that the t-I-sets and semi*-I-closed sets coincide.

The paper is organized as follows. In Section 2, we investigate some further properties and characterizations of strong β-I-open sets due to [12] in any ideal topological space and then obtain some decompositions of open sets in I-extremally disconnected spaces. In Section 3, we define a new set called an AK_I-set in an ideal topological space by using the strong β-I-closed sets. In Section 4, we define AK_I^*-sets and obtain a decomposition of an open set by using AK_I^*-sets. In Section 5, we give decompositions of continuity.

2. Further Properties of Strong β-I-open Sets

In this section, first we investigate some further properties and characterizations of strong β-I-open sets due to [12] in any ideal topological spaces. Then, we obtain some decompositions of open sets in I-extremally disconnected spaces.

We start with recalling the following diagram. It is a combined form of diagrams which are given in [17] and [6].

$$\begin{array}{ccccccc}
\text{open} \longrightarrow & \alpha\text{-}I\text{-open} & \longrightarrow & \text{semi-}I\text{-open} & & & \\
& \downarrow & & \downarrow & & & \\
& \text{pre-}I\text{-open} & \longrightarrow & b\text{-}I\text{-open} & \longrightarrow & \beta\text{-}I\text{-open} & \longrightarrow \beta^*\text{-}I\text{-open}
\end{array}$$

Diagram I

Proposition 2.1. *For a subset A of an ideal topological space (X, τ, I), the following properties hold:*

(1) If A is a b-I-open set, then it is a strong β-I-open set.

(2) If A is a semi-open set, then it is a semi*-I-open set.

Proof. (1) Let A be a b-I-open set in X. Then, we have $A \subset Cl^*(Int(A)) \cup Int(Cl^*(A))$. Therefore, we have $A \subset Cl^*(Int(Cl^*(A)))$ and this shows that A is a strong β-I-open set.

(2) This proof is obvious from $\tau \subset \tau^*$. □

Remark 2.2. *The following diagram is obtained from Diagram I by using Proposition 2.1 and [*[12]*, Diagram]. Moreover, the converses of these implications in Diagram II are not true in general as shown in the following examples and related references.*

$$\begin{array}{ccccccccc}
\text{open} & \longrightarrow & \alpha\text{-}I\text{-open} & \longrightarrow & \text{semi-}I\text{-open} & \longrightarrow & \text{semi-open} & \longrightarrow & \text{semi}^*\text{-}I\text{-open} \\
 & \swarrow & & \swarrow & & & & \searrow & \\
\text{pre-}I\text{-open} & \longrightarrow & b\text{-}I\text{-open} & \longrightarrow & \text{strong } \beta\text{-}I\text{-open} & \longrightarrow & \beta\text{-}I\text{-open} & \longrightarrow & \beta^*\text{-}I\text{-open}
\end{array}$$

Diagram II

Example 2.3. *Let $X = \{a, b, c, d\}$, $\tau = \{\varnothing, X, \{a\}, \{b, c\}, \{a, b, c\}\}$ and $I = \{\varnothing, \{a\}\}$. Then, $A = \{b, d\}$ is a strong β-I-open set but it is not a b-I-open set.*

Example 2.4. *[[8], Example 14]. Let $X = \{a, b, c, d\}$, $\tau = \{\phi, X, \{a\}, \{a, b\}, \{c, d\}, \{a, c, d\}\}$ and $I = \{\phi, \{a\}, \{d\}, \{a, d\}\}$. The set $A = \{c\}$ is a semi*-I-open set but it is not a semi-open set.*

We give some characterizations of strong β-I-open sets in the next two theorems.

Theorem 2.5. *Let (X, τ, I) be an ideal topological space and $A \subset X$. Then, A is a strong β-I-open set if and only if $Cl^*(A) = Cl^*(Int(Cl^*(A)))$.*

Proof. Let A be a strong β-I-open set in X. Then, we have $A \subset Cl^*(Int(Cl^*(A)))$. We obtain

$$Cl^*(A) \subset Cl^*(Cl^*(Int(Cl^*(A)))) = Cl^*(Int(Cl^*(A)))$$

and hence $Cl^*(A) \subset Cl^*(Int(Cl^*(A)))$ by using Cl^* closure operation. Also, it is obvious that $Cl^*(Int(Cl^*(A))) \subset Cl^*(A)$. Thus, $Cl^*(A) = Cl^*(Int(Cl^*(A)))$.

Conversely, let $Cl^*(A) = Cl^*(Int(Cl^*(A)))$. Since $Cl^*$is the closure operator, we have $A \subset Cl^*(A)$ for every subset A of X. Therefore, by using hypothesis, we have $A \subset Cl^*(Int(Cl^*(A)))$. This shows that A is a strong β-I-open set. □

Theorem 2.6. *Let (X, τ, I) be an ideal topological space and $A \subset X$. Then, the following properties are equivalent.*

(1) A is a strong β-I-open set,

(2) There exists a pre-I-open set U in X such that $U \subset Cl^*(A) \subset Cl^*(U)$,

(3) $Cl^*(A) = Cl^*(Int(Cl^*(A)))$.

Proof. (1) $\Longrightarrow$ (2). Let A be a strong β-I-open set in X. Then, we have $A \subset Cl^*(Int(Cl^*(A)))$. If we take $U = Int(Cl^*(A))$, we have $U \subset Int(Cl^*(U))$. Therefore, we obtain U is a pre-I-open set and $U \subset Cl^*(A) \subset Cl^*(U)$.

(2) $\Longrightarrow$ (3). Assume that there exists a pre-I-open set U in X such that $U \subset Cl^*(A) \subset Cl^*(U)$. Then, we have $U \subset Int(Cl^*(U))$ and hence $Cl^*(U) \subset Cl^*(Int(Cl^*(U))) \subset Cl^*(Int(Cl^*(Cl^*(A)))) \subset Cl^*(A) \subset Cl^*(U)$. Therefore, we obtain $Cl^*(U) = Cl^*(Int(Cl^*(A)))$. On the other hand, since $U \subset Cl^*(A) \subset Cl^*(U)$, we have $Cl^*(A) = Cl^*(Int(Cl^*(A)))$.

(3) $\Longrightarrow$ (1). This proof is given in Theorem 2.5. □

To obtain a characterization of strong β-I-open sets, we need the definition of weak regular-I-closed sets. A subset A of (X, τ, I) is called a *weak regular-I-closed* set [20] if $A = Cl^*(Int(A))$.

Theorem 2.7. *Let (X, τ, I) be an ideal topological space and $A \subset X$. Then, A is a strong β-I-open set if and only if $Cl^*(A)$ is a weak regular-I-closed set.*

Proof. Let A be a strong β-I-open set in X. Therefore, $Cl^*(A) = Cl^*(Int(Cl^*(A)))$ by using Theorem 1. This shows that $Cl^*(A)$ is a weak regular-I-closed set.

On the other hand, let $Cl^*(A)$ be a weak regular-I-closed set. Then, we have $Cl^*(A) = Cl^*(Int(Cl^*(A))$. Using Theorem 2.5, we obtain A is a strong β-I-open set. □

Theorem 2.8. *Let (X, τ, I) be an ideal topological space and A, $B \subset X$ such that $A \subset B \subset Cl^*(A)$. If A is a strong β-I-open set, then $Cl^*(A) = Cl^*(B)$ and hence B is a strong β-I-open set.*

Proof. It is obtained that $Cl^*(A) = Cl^*(B)$ by taking the $*$-closure of $A \subset B \subset Cl^*(A)$. On the other hand, suppose that A is a strong β-I-open set in X. Then, we have $A \subset Cl^*(Int(Cl^*(A)))$. By hypothesis, since $A \subset B \subset Cl^*(A)$, we obtain that $B \subset Cl^*(Cl^*(Int(Cl^*(A)))) = Cl^*(Int(Cl^*(A))) = Cl^*(Int(Cl^*(B)))$. Therefore, we have $B \subset Cl^*(Int(Cl^*(B)))$ and this shows that B is a strong β-I-open set. □

Corollary 2.9. *Let (X, τ, I) be an ideal topological space and $A \subset X$. If A is a strong β-I-open set, then $Cl^*(A)$ is a strong β-I-open set.*

Proof. Let A be a strong β-I-open set. Since $A \subset Cl^*(A)$ for every subset A of X, we have that $Cl^*(A)$ is a strong β-I-open set by Theorem 2.8. □

Definition 2.10. *A subset A of an ideal topological space (X, τ, I) is said to be α_*-I-open if $Int^*(A) = Int^*(Cl(Int^*(A)))$.*

Theorem 2.11. *Let (X, τ, I) be an ideal topological space and $A \subset X$. Then, A is a strong β-I-closed set if and only if A is an α_*-I-open set.*

Proof. Let A be a strong β-I-closed set in X. Then, $(X\text{-}A)$ is a strong β-I-open set. Hence, we obtain $Cl^*(X\text{-}A) = Cl^*(Int(Cl^*(X\text{-}A)))$ by using Theorem 2.5. Besides, we have $Cl^*(X\text{-}A) = X\text{-}Int^*(A)$ and $Cl^*(Int(Cl^*(X\text{-}A))) = X\text{-}Int^*(Cl(Int^*(A)))$ and hence $X\text{-}Int^*(A) = X\text{-}Int^*(Cl(Int^*(A)))$. Therefore, we obtain $Int^*(A) = Int^*(Cl(Int^*(A)))$ and this shows that A is an α_*-I-open set.

The other side of the proof is obtained similar to the first side by using the necessary definitions. □

Theorem 2.12. *Let (X, τ, I) be an ideal topological space and $A \subset X$. If A is a semi-I-closed set, then A is both a strong β-I-closed set and a t-I-set.*

Proof. We obtain that every semi-I-closed set is a strong β-I-closed set by taking the complement of Diagram II. On the other hand, it is obvious that every semi-I-closed set is a t-I-set according to [[10], Lemma 7] and [[8], Theorem 25]. □

Definition 2.13. *An ideal topological space (X, τ, I) is said to be strongly β-I-compact* [13] *if for every strong β-I-open cover $\{W_\alpha : \alpha \in \Delta\}$, there exists a finite subset Δ_0 of Δ such that $(X\text{-} \cup \{W_\alpha : \alpha \in \Delta\}) \in I$.*

Theorem 2.14. *Let (X, τ, I) be an ideal topological space and I be a minimal ideal on X, i.e., $I = \{\varnothing\}$. If (X, τ, I) is strongly β-I-compact space, then it is compact space.*

Proof. Since every open set is strong β-I-open, it follows that the space is compact using Definition 2.13. □

To obtain decompositions of open sets in any I-extremally disconnected spaces, we recall the following definitions and lemmas.

Definition 2.15. *A subset A of an ideal topological space (X, τ, I) is said to be weakly I-local closed* [19] *(resp. locally closed* [2] *or FG* [16]*, I-locally-closed* [23]*) if $A = U \cap V$, where U is an open set and V is a $*$-closed (resp. closed, *-perfect) set. In the following lemma, author gave a characterization of weakly I-local closed sets.*

Lemma 2.16. *(*[24]*) A subset A of an ideal topological space (X, τ, I) is weakly I-local closed if and only if there exists an open set U such that $A = U \cap Cl^*(A)$.*

Definition 2.17. *An ideal topological space (X, τ, I) is said to be I-extremally disconnected* [20] *(or $*$-extremally disconnected* [9]*) if $Cl^*(A) \in \tau$ for each $A \in \tau$.*

Lemma 2.18. *(*[9]*) An ideal topological space (X, τ, I) is I-extremally disconnected if and only if $Cl^*(Int(A)) \subset Int(Cl^*(A))$ for every subset A of X.*

Remark 2.19. *Note that Theorem 2.20 is an extension of [*[7]*, Theorem* 8*].*

Theorem 2.20. *Let (X, τ, I) be an I-extremally disconnected space and $A \subset X$. Then, the following properties are equivalent.*

(1) A is an open set,
(2) A is an α-I-open and weakly I-local closed set,
(3) A is a pre-I-open and weakly I-local closed set,
(4) A is a semi-I-open and weakly I-local closed set,
(5) A is a b-I-open and weakly I-local closed set,
(6) A is a strong β-I-open and weakly I-local closed set.

Proof. Since (1) $\Longrightarrow$ (2), (2) $\Longrightarrow$ (3), (3) $\Longrightarrow$ (4) and (4) $\Longrightarrow$ (5) are well known due to [10], we will prove only (5) $\Longrightarrow$ (6) and (6) $\Longrightarrow$ (1).

(5) $\Longrightarrow$ (6) : It is obtained directly from Diagram II.

(6) $\Longrightarrow$ (1) : Assume that A is a strong β-I-open and *weakly I-local closed* set. Since A is a strong β-I-open set, we have $A \subset Cl^*(Int(Cl^*(A)))$. By Lemma 2.16, there exists an open set U such that $A = U \cap Cl^*(A)$. Therefore, we have $A \subset [U \cap Cl^*(Int(Cl^*(A)))]$. Since (X, τ, I) is an I-extremally disconnected space, by using Lemma 2.18, we have $A \subseteq U \cap Int(Cl^*(Cl^*(A)))$. Besides since Cl^* is a Kuratowski closure operator, we obtain $A \subset U \cap Int(Cl^*(Cl^*(A)))] = U \cap Int(Cl^*(A))$ and hence we have $A \subset U \cap Int(Cl^*(A)) = Int(U \cap Cl^*(A))$. Consequently, we have $A \subset Int(U \cap Cl^*(A))$ and hence we have $A \subset Int(A)$ by using Lemma 2.16. This shows that A is an open set. □

Remark 2.21. *Since every open set is a locally closed set and every locally closed set is a weakly I-local closed set, we have the following theorem. One can say that it is extended form of [*[7]*, Theorem* 9*].*

Theorem 2.22. *Let (X, τ, I) be an I-extremally disconnected space and $A \subset X$. Then, the following properties are equivalent.*

(1) A is an open set,
(2) A is an α-I-open and locally closed set,
(3) A is a pre-I-open and locally closed set,
(4) A is a semi-I-open and locally closed set,
(5) A is a b-I-open and locally closed set,
(6) A is a strong β-I-open and locally closed set.

Also, one can obtain the following theorem. Because, every open set is an *I-locally-closed* set and every *I-locally-closed* set is a *weakly I-local closed* set.

Theorem 2.23. *Let (X, τ, I) be an I-extremally disconnected space and $A \subset X$. Then, the following properties are equivalent.*

(1) A is an open set,
(2) A is an α-I-open and I-locally-closed set,
(3) A is a pre-I-open and I-locally-closed set
(4) A is a semi-I-open and I-locally-closed set
(5) A is a b-I-open and I-locally-closed set
(6) A is a strong β-I-open and I-locally-closed set.

3. AK_I-sets

In this section, we define a new set called an AK_I-set in an ideal topological space by using the strong β-I-closed sets. A subset A is called a strong β-I-closed set in an ideal topological space (X, τ, I) if $(X\text{-}A)$ is a strong β-I-open set. Hence, we have the following: A is a strong β-I-closed set if and only if $Int^*(Cl(Int^*(A))) \subset A$. Then, we obtain some properties and characterizations of strong β-I-closed sets to give characterizations of AK_I-sets and B_I^*-sets.

First of all, let us recall some modifications of the notion of locally closed sets.

Definition 3.1. *A subset A of an ideal topological space (X, τ, I) is called*

(1) an η_I-set [8] *if $A = U \cap V$, where U is an open set and V is an α-I-closed set in X;*

(2) a C_I-set [8] *if $A = U \cap V$, where U is an open set and V is a pre-I-closed set in X;*

(3) a B_I^*-set [6] *if $A = U \cap V$, where U is an open set and V is a semi-I-closed set in X;*

(4) a BC_I-set [6] *if $A = U \cap V$, where U is an open set and V is a b-I-closed set in X;*

(5) a B_I-set [11] *if $A = U \cap V$, where U is an open set and V is a t-I-set in X.*

Definition 3.2. *A subset A of an ideal topological space (X, τ, I) is called an AK_I-set if $A = U \cap V$, where U is an open set and V is a strong β-I-closed set in X.*

Remark 3.3. *Every open set and every strong β-I-closed set is an AK_I-set. But, the converse of each of these implications is not true in general as shown by the following examples.*

Example 3.4. *Let (X, τ, I) be the same as in Example 2.3,i.e., $X = \{a, b, c, d\}$, $\tau = \{\phi, X, \{a\}, \{b, c\}, \{a, b, c\}\}$ and $I = \{\phi, \{a\}\}$. Then, $A = \{a, c\}$ is an AK_I-set but it is not an open set.*

Example 3.5. *Let $X = \{a, b, c, d\}$, $\tau = \{\phi, X, \{a\}, \{a, b\}, \{c, d\}, \{a, c, d\}\}$and $I = \{\phi, \{a\}, \{d\}, \{a, d\}\}$, i.e., (X, τ, I) be as same as in Example 2.4. Then, $A = \{a, c, d\}$ is an AK_I-set but it is not a strong β-I-closed set.*

For all of sets defined above, we have the following diagram by Diagram I. Note that none of the converses of these implications is true in general as shown in the following examples and [6] and [8].

$$\begin{array}{ccccccc}
\text{locally closed set} & \longrightarrow & \eta_I\text{-set} & \longrightarrow & B_I^*\text{-set} & \longrightarrow & B_I\text{-set} \\
\uparrow & & \downarrow & & \downarrow & & \\
\text{open set} & & C_I\text{-set} & \longrightarrow & BC_I\text{-set} & \longrightarrow & AK_I\text{-set}
\end{array}$$

Diagram III

Example 3.6. *Let (X, τ, I) be the same as in Example 2.3,i.e., $X = \{a, b, c, d\}$, $\tau = \{\phi, X, \{a\}, \{b, c\}, \{a, b, c\}\}$ and $I = \{\phi, \{a\}\}$. By Example 2.3, $A = \{a, c\}$ is a strong β-I-closed set but it is not a b-I-closed set. Besides, A is not an open set. So, we have A is an AK_I-set but it is not a BC_I-set.*

Example 3.7. *Let (X, τ, I) be the same as in Example 2.4,i.e., $X = \{a, b, c, d\}$, $\tau = \{\phi, X, \{a\}, \{a, b\}, \{c, d\}, \{a, c, d\}\}$ and $I = \{\phi, \{a\}, \{d\}, \{a, d\}\}$. Then, $A = \{a, b, d\}$ is a B_I-set, but it is not a B_I^*-set.*

It is shown in [[8], Example 21] that the set $A = \{d\}$ is an η_I-set which is not locally closed.

One can obtain the following result by using Theorem 2.12. Besides, the B_I-sets and AK_I-sets are independent of each other as shown in Example 3.9.

Corollary 3.8. *Let (X, τ, I) be an ideal topological space and $A \subset X$. If A is an B_I^*-set, then it is an AK_I-set and a B_I-set.*

Example 3.9. *(1) Let (X, τ, I) be the same as in Example 2.18,i.e., $X = \{a, b, c, d\}$, $\tau = \{\phi, X, \{a\}, \{a, b\}, \{c, d\}, \{a, c, d\}\}$ and $I = \{\phi, \{a\}, \{d\}, \{a, d\}\}$. Then, $A = \{b, d\}$ is a B_I-set, but it is not an AK_I-set.*

(2) Let $X = \{a, b, c, d\}$, $\tau = \{\phi, X, \{a\}, \{b, c\}, \{a, b, c\}\}$ and $I = \{\phi, \{a\}\}$, i.e., (X, τ, I) be the same as in Example 2.16. Then, $A = \{a, c\}$ is an AK_I-set, but it is not a B_I-set.

Recall that the next fact is given in [20] and [9]: " Strong β-I-open sets and pre-I-open sets coincide in I-extremally disconnected spaces. One can say that strong β-I-open sets, b-I-open sets and pre-I-open sets coincide in I-extremally disconnected spaces using Diagram II. Therefore, we have the following result using Diagram III.

Corollary 3.10. *Let (X, τ, I) be an I-extremally disconnected space and $A \subset X$. Then, we have the following property: A is an AK_I-set if and only if it is a C_I-set.*

Remark 3.11. *In* [6]*, it was proved that the BC_I-sets and C_I-sets coincide in I-extremally disconnected spaces. We can conclude that Corollary 3.8 is an improvement of the above result.*

Example 3.12. *Let (X, τ, I) be the same as in Example 2.5 in* [20]*, i.e., $X = \{a, b, c\}$, $\tau = \{\varnothing, X, \{a\}, \{b\}, \{a, b\}\}$ and $I = \{\varnothing, \{a\}, \{b\}, \{a, b\}\}$. Then, τ is a family of strong β-I-open sets (resp. b-I-open sets and pre-I-open sets) in (X, τ, I). Also, τ is a family of AK_I-sets (resp. BC_I-sets and C_I-sets) in (X, τ, I).*

To obtain a characterization of AK_I-sets in any ideal topological space, we recall the following definition and lemma.

Definition 3.13. *(*[10]*) The strong β-I-closure of a subset A of an ideal topological space (X, τ, I), denoted by $S\beta$-I-$Cl(A)$, is defined by the intersection of all strong β-I-closed sets of X containing A.*

Lemma 3.14. *(*[10]*) For a subset A of an ideal topological space (X, τ, I), $S\beta$-I-$Cl(A) = A \cup Int^*(Cl(Int^*(A)))$.*

Theorem 3.15. *Let (X, τ, I) be an ideal topological space and $A \subset X$. Then, A is an AK_I-set if and only if $A = U \cap S\beta\text{-}I\text{-}Cl(A)$ for an open set U in X.*

Proof. Let A be an AK_I-set. Then, we have $A = U \cap V$, where U is an open set and V is a strong β-I-closed set. It is obvious that $A \subset V$ and hence $A \subset S\beta\text{-}I\text{-}Cl(A) \subset V$. Therefore, we have $A = A \cap S\beta\text{-}I\text{-}Cl(A) = (U \cap V) \cap S\beta\text{-}I\text{-}Cl(A) = U \cap S\beta\text{-}I\text{-}Cl(A)$.

Now we will prove that $S\beta\text{-}I\text{-}Cl(A)$ is a strong β-I-closed set. Assume that V is any strong β-I-closed set V containing A. So, we have $S\beta\text{-}I\text{-}Cl(A) \subset V$. Therefore, we obtain $Int^*(Cl(Int^*(S\beta\text{-}I\text{-}Cl(A))) \subset Int^*(Cl(Int^*(V)) \subset V$ and hence $Int^*(Cl(Int^*(S\beta\text{-}I\text{-}Cl(A))) \subset \cap V$ for any strong β-I-closed set V containing A. Consequently, we have $Int^*(Cl(Int^*(S\beta\text{-}I\text{-}Cl(A))) \subset S\beta\text{-}I\text{-}Cl(A)$ and this shows that $S\beta\text{-}I\text{-}Cl(A)$ is a strong β-I-closed set. According to Definition 3.1, A is an AK_I-set. □

To obtain a decomposition of a strong β-I-closed set, we need to define a new g-closed set as follows.

Definition 3.16. *A subset A of an ideal topological space (X, τ, I) is said to be generalized strong β-I-closed (briefly $gs\beta$-I-closed) if $S\beta\text{-}I\text{-}Cl(A) \subset U$ whenever $A \subset U$ and U is an open set in X.*

Theorem 3.17. *Let (X, τ, I) be an ideal topological space and $A \subset X$. Then, A is a strong β-I-closed set if and only if it is an AK_I-set and a $gs\beta$-I-closed set in X.*

Proof. The first side is immediately coming from Remark 3.3 and the related definitions.

Let A be an AK_I-set and a $gs\beta$-I-closed set in X. Then, we have $A = U \cap S\beta\text{-}I\text{-}Cl(A)$ for an open set U in X by Theorem 3.15. On the other hand, it is obvious that since A is a $gs\beta$-I-closed set in X,we have $S\beta\text{-}I\text{-}Cl(A) \subset U$. Therefore, we obtain $S\beta\text{-}I\text{-}Cl(A) \subset U \cap S\beta\text{-}I\text{-}Cl(A) = A$ and hence $A = S\beta\text{-}I\text{-}Cl(A)$ by using Definition 3.1. This shows that A is a strong β-I-closed set. □

The next theorem is a consequence of some properties of the AK_I-sets in any ideal topological space.

Theorem 3.18. *Let (X, τ, I) be an ideal topological space and $A \subset X$. If A is an AK_I-set in X, then the following properties hold.*

(1) $[S\beta\text{-}I\text{-}Cl(A)\backslash A]$ is a strong β-I-closed set.

(2) $([X\backslash S\beta\text{-}I\text{-}Cl(A)] \cup A)$ is a strong β-I-open set.

Proof. (1) Let A be an AK_I-set in X. Then , by using Theorem 3.15, we have $A = U \cap S\beta\text{-}I\text{-}Cl(A)$ for an open set U in X. Therefore, we have

$$\begin{aligned}
[S\beta\text{-}I\text{-}Cl(A)\backslash A] &= S\beta\text{-}I\text{-}Cl(A)\backslash [U \cap S\beta\text{-}I\text{-}Cl(A)] \\
&= S\beta\text{-}I\text{-}Cl(A) \cap [X\backslash (U \cap S\beta\text{-}I\text{-}Cl(A))] \\
&= S\beta\text{-}I\text{-}Cl(A) \cap [(X\backslash U) \cup (X\backslash S\beta\text{-}I\text{-}Cl(A))] \\
&= [S\beta\text{-}I\text{-}Cl(A) \cap (X\backslash U)] \cup [S\beta\text{-}I\text{-}Cl(A) \cap (X\backslash S\beta\text{-}I\text{-}Cl(A))] \\
&= [S\beta\text{-}I\text{-}Cl(A) \cap (X\backslash U)] \cup \phi \\
&= [S\beta\text{-}I\text{-}Cl(A) \cap (X\backslash U)].
\end{aligned}$$

This shows that $[S\beta\text{-}I\text{-}Cl(A)\backslash A]$ is a strong β-I-closed set.

(2) Since $[S\beta\text{-}I\text{-}Cl(A)\backslash A]$ is a strong β-I-closed set in X, we have $(X\backslash [S\beta\text{-}I\text{-}Cl(A)\backslash A]) = (X\backslash [S\beta\text{-}I\text{-}Cl(A) \cap (X\backslash A)]) = ([X\backslash S\beta\text{-}I\text{-}Cl(A)] \cup A)$ is a strong β-I-open set in X. □

4. AK_I^*-SETS

In this section, we introduce the notion of AK_I^*-sets and obtain a decomposition of an open set by using AK_I^*-sets.

Definition 4.1. *Let (X, τ, I) be an ideal topological space. A subset A of X is called*

(1) a C_I^*-set [8] *if $A = U \cap V$, where U is an open set and V is a pre-I-regular set in X;*

(2) an AC_I-set [6] *if $A = U \cap V$, where U is an open set and V is a b-I-regular set in X;*

(3) an AB_I-set [18] *if $A = U \cap V$, where U is an open set and V is a semi-I-regular set in X, where V is said to be pre-I-regular* [8] *(resp. b-I-regular* [6]*, semi-I-regular* [13] *) if V is pre-I-open and pre-I-closed (resp. b-I-open and b-I-closed, semi-I-open and a t-I-set).*

Definition 4.2. *Let (X, τ, I) be an ideal topological space. A subset A of X is called*

(1) a semi-I^*-regular set if A is a semi-I-open and semi-I-closed set;

(2) an AB_I^*-set if $A = U \cap V$, where U is an open set and V is a semi-I^*-regular set in X;

(3) a strong β-I-regular set if A is a strong β-I-open and strong β-I-closed set;

(4) an AK_I^*-set if $A = U \cap V$, where U is an open set and V is a strong β-I-regular set in X.

Remark 4.3. *In any ideal topological space, every open set and every strong β-I-regular set is an AK_I^*-set. The converses of these implications are not true in general as shown by the following examples.*

Example 4.4. *Let (X, τ, I) be the same as in Example 2.3, i.e., $X = \{a, b, c, d\}$, $\tau = \{\phi, X, \{a\}, \{b, c\}, \{a, b, c\}\}$ and $I = \{\phi, \{a\}\}$. Then, $A = \{b, d\}$ is a strong β-I-regular set and hence an AK_I^*-set but it is not an open set.*

Example 4.5. *Let $X = \{a, b, c, d\}$, $\tau = \{\phi, X, \{a\}, \{a, b\}, \{c, d\}, \{a, c, d\}\}$ and $I = \{\phi, \{a\}, \{d\}, \{a, d\}\}$. Then, $A = \{a, c, d\}$ is an AK_I^*-set but it is not a strong β-I-regular set.*

Theorem 4.6. *In any ideal topological space, each AK_I^*-set is strong β-I-open.*

Proof. Let (X, τ, I) be an ideal topological space and A be an AK_I^*-set in X. Then, we have $A = U \cap V$ such that U is an open set and V is a strong β-I-regular set. By using Definition 4.1(3), V is a strong β-I-open set. According to [[12], Proposition 3], we have $A = U \cap V$ is a strong β-I-open set. □

Remark 4.7. *The converse of Theorem 4.6 need not be true in general as shown by the following example.*

Example 4.8. *Let (X, τ, I) be the same as in Example 2.4,i.e., $X = \{a, b, c, d\}$, $\tau = \{\phi, X, \{a\}, \{a, b\}, \{c, d\}, \{a, c, d\}\}$ and $I = \{\phi, \{a\}, \{d\}, \{a, d\}\}$. Then, $A = \{a, b, c\}$ is a strong β-I-open set, but it is not an AK_I^*-set.*

Remark 4.9. *It is obvious that every strong β-I-regular set is a strong β-I-open set. By using Remark 4.3 and Theorem 4.6, we obtain that the notion of AK_I^*-sets is weaker than the notions of both strong β-I-regular sets and open sets and also is*

stronger than the notion of strong β-I-open sets. Therefore, we have the following diagram.

$$\begin{array}{ccccc} & & \text{strong } \beta\text{-}I\text{-open set} & & \\ & & \uparrow & & \\ \text{strong } \beta\text{-}I\text{-regular set} & \longrightarrow & \mathrm{AK}_I^*\text{-set} & \longleftarrow & \text{open set} \end{array}$$

Diagram IV

We have the following diagram by using Diagrams II and III with Definitions 3.16 and 4.1.

Recall that a subset A of (X, τ, I) is called an $\eta\varsigma$-set [9] if $A = U \cap V$, where U is an open set and V is a clopen set in X.

$$\begin{tikzcd}
\text{open set} \arrow[r] & \eta\varsigma\text{-set} \arrow[r] \arrow[dr] \arrow[dd] & \mathrm{AB}_I^*\text{-set} \arrow[r] \arrow[dr] & \mathrm{AB}_I\text{-set} \arrow[dr] & \\
& & \eta_I\text{-set} \arrow[r] \arrow[d] & B_I^*\text{-set} \arrow[r] \arrow[d] & B_I\text{-set} \\
& & \mathrm{C}_I\text{-set} \arrow[r] & \mathrm{BC}_I\text{-set} \arrow[r] & \mathrm{AK}_I\text{-set} \\
& \mathrm{C}_I^*\text{-set} \arrow[ur] \arrow[r] & \mathrm{AC}_I\text{-set} \arrow[ur] \arrow[r] & \mathrm{AK}_I^*\text{-set} \arrow[ur] &
\end{tikzcd}$$

Diagram V

Next we present decompositions of open sets in any ideal topological space.

Theorem 4.10. *Let (X, τ, I) be an ideal topological space. Then, for a subset A of X the following properties are equivalent:*

(1) A is an open set,

(2) A is an α-I-open set and an AK_I^*-set,

(3) A is an α-I-open set and an AK_I-set.

Proof. (1) $\Longrightarrow$ (2). This proof is obtained from Diagram I and Remark 4.3.

(2) $\Longrightarrow$ (3). Since every strong β-I-regular set is a strong β-I-closed set, the proof is obvious.

(3) $\Longrightarrow$ (1). Let A be an α-I-open set and an AK_I-set. Then, we have $A = U \cap V$, where U is an open set and V is a strong β-I-closed set in X. Since V is strong β-I-closed, by Diagram II, V is a β_I^*-closed set and hence $Int(Cl^*(Int(V))) \subset V$. Therefore, we obtain $Int(V) = Int(Cl^*(Int(V)))$. Since A is an α-I-open set and $A \subset V$, $A \subset Int(Cl^*(Int(A))) \subset Int(Cl^*(Int(V))) = Int(V)$. Therefore, we have $A \subset U \cap Int(V) = Int(U \cap V) = Int(A)$. This shows that A is open. $\square$

We state that Theorem 4.10 is an improvement of Theorem 2.11 in [9] using Diagram V.

5. Decompositions of continuity

In this section, we obtain some decompositions of continuous functions. First we recall the definitions of some generalizations of continuous functions.

Definition 5.1. *A function $f : (X, \tau, I) \longrightarrow (Y, \varphi)$ is said to be*

(1) α-I-continuous [11] *if $f^{-1}(V)$ is an α-I-open set in (X, τ, I) for every open set V in (Y, φ),*

(2) pre-I-continuous [4] *if $f^{-1}(V)$ is a pre-I-open set in (X, τ, I) for every open set V in (Y, φ),*

(3) semi-I-continuous [11] *if $f^{-1}(V)$ is a semi-I-open set in (X, τ, I) for every open set V in (Y, φ),*

(4) b-I-continuous [17] *if $f^{-1}(V)$ is a b-I-open set in (X, τ, I) for every open set V in (Y, φ),*

(5) strongly β-I-continuous [13] *if $f^{-1}(V)$ is a strong β-I-open set in (X, τ, I) for every open set V in (Y, φ),*

(6) $W_I LC$-continuous [19] *if $f^{-1}(V)$ is a weakly I-local closed set in (X, τ, I) for every open set V in (Y, φ),*

(7) LC-continuous [16] *if $f^{-1}(V)$ is a localy closed set in (X, τ, I) for every open set V in (Y, φ).*

Remark 5.2. *The following two theorems are improvements of [*[7]*, Theorem* 26*] and [*[7]*, Theorem* 27*], respectively.*

Theorem 5.3. *Let $f : (X, \tau, I) \longrightarrow (Y, \varphi)$ be a function such that (X, τ, I) is an I-extremally disconnected space. Then, the following properties are equivalent:*
(1) f is continuous,
(2) f is α-I-continuous and $W_I LC$-continuous,
(3) f is pre-I-continuous and $W_I LC$-continuous,
(4) f is semi-I-continuous and $W_I LC$-continuous,
(5) f is b-I-continuous and $W_I LC$-continuous,
(6) f is strongly β-I-continuous and $W_I LC$-continuous.

Proof. It follows from Theorem 2.20. □

Theorem 5.4. *Let $f : (X, \tau, I) \longrightarrow (Y, \varphi)$ be a function such that (X, τ, I) is an I-extremally disconnected space. Then, the following properties are equivalent:*
(1) f is continuous,
(2) f is α-I-continuous and LC-continuous,
(3) f is pre-I-continuous and LC-continuous,
(4) f is semi-I-continuous and LC-continuous,
(5) f is b-I-continuous and LC-continuous,
(6) f is strongly β-I-continuous and LC-continuous.

Proof. It follows from Theorem 2.22. □

Now, to obtain some decompositions of continuity we introduce the notions of *AK_I-continuity* and *AK_I^*-continuity*.

Definition 5.5. *A function $f : (X, \tau, I) \longrightarrow (Y, \varphi)$ is said to be AK_I-continuous (resp. AK_I^*-continuous) if $f^{-1}(V)$ is an AK_I-set (resp. AK_I^*-set) in (X, τ, I) for every open set V in (Y, φ).*

By using Theorem 4.10, we have the following decompositions of continuity.

Theorem 5.6. *The following properties are equivalent for a function $f : (X, \tau, I) \longrightarrow (Y, \varphi)$:*
(1) f is continuous,
(2) f is α-I-continuous and AK_I^*-continuous,
(3) f is α-I-continuous and AK_I-continuous.

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