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# ON THE SINGULARITIES OF THE D-TUBULAR SURFACES

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Abstract. In this article, we study the singularities of the tubular surfaces. The geometric conditions for the tubular surfaces to have generic singularities for a front, i.e. cuspidal edges and swallowtails, are investigated.

## 1. Introduction

The theory of curves and surfaces in $\mathbb{R}^3$ stills an attractive area of research. It has many applications in several areas such as engineering, geometric modeling and computer vision. The differential geometry of curves and surfaces is an attractive area for many mathematicians specially geometers. There are many articles concerning this area. The full fruit aspect of the theory of curves and surface is the singularity theory. The singularity theory of curves and surfaces has many interesting applications in the reality such as computer vision and medical imaging. A surface in Euclidean 3-space is considered as the image of a map from $\mathbb{R}^2$ to $\mathbb{R}^3$. The classification of simple singularities of a map from $\mathbb{R}^2$ to $\mathbb{R}^3$ have been investigated in detail by Mond [11]. After Mond's classification, the singularity theory of surfaces has become an interesting area for many geometers. As a result of this, several articles appeared such as [1, 2, 5, 6, 8, 10].

In the present article, we study the singularity of the directional tubular surface. In fact, the geometric conditions for the tubular surface to have the generic singularities of front are pointed out in this paper. These conditions depend on the quazi curvatures.

This paper is divided into four main parts. The first part is this introduction, which gives a brief idea about this article. The second part is dealing with the basic concepts of the theory of the classical differential geometry of curves in Euclidean 3-space. Also, in the second part, the concepts of the quazi-frame and quazi-equations are highlighted. The third part is dedicated to the main part of this paper. In the third part, we review the related basic terminologies of the singularity theory, that are needed in this paper. The main results of this paper, which deal with the classifications of the singularity of the tubular surface are presented in third part of this paper. In the fourth part, we present two examples which illustrate some of our main results in the third part.

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## 2. Preliminaries

In this section, we review the basic concepts of the theory of curves in $\mathbb{R}^3$. Precisely, we highlight Serret- Frenet frame associated to a regular space curve. For more information in the theory of curves and surfaces in Euclidean 3-space we refer the reader to [4, 13]. Recall that a regular space curve in $\mathbb{R}^3$ is a map $\gamma : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}^3$ such that the first derivative $\gamma'$ does not vanish at any $t \in I$. The Serret- Frenet frame is given by $\{T_\gamma, N_\gamma, B_\gamma\}$, where $T_\gamma$, $N_\gamma$, and $B_\gamma$ are the unit tangent vector, the unit principal normal vector and the unit binormal vector of $\gamma$ respectively, where, $T_\gamma = \dfrac{\gamma'}{\|\gamma'\|}$, $N_\gamma = \dfrac{T'_\gamma}{\|T'_\gamma\|}$ and $B_\gamma = T_\gamma \wedge N_\gamma$. The curvature and the torsion of a regular space curve are given by $\kappa_\gamma = \dfrac{\|\gamma' \wedge \gamma''\|}{\|\gamma'\|^3}$ and $\tau_\gamma = \dfrac{\det(\gamma', \gamma'', \gamma''')}{\|\gamma' \wedge \gamma''\|^2}$ respectively. If $\gamma$ is a unit speed curve, i.e. $\|\gamma'\| = 1$, the Serret-Frenet equations are given by

$$\begin{cases} T'_\gamma = \kappa_\gamma N_\gamma \\ N'_\gamma = -\kappa_\gamma T_\gamma + \tau_\gamma B_\gamma \\ B'_\gamma = -\tau_\gamma N_\gamma. \end{cases}$$

The Serret- Frenet frame is well defined provided the curvature does not vanish. This means that this frame does not work a long the hole curve if there is a point where the curvature vanishes. In [3], R.L. Bishop pointed out that a frame a long a curve can be defined in more than one way. In [12], Mustafa and others defined a Q-frame as the following.
Let $\gamma$ be a regular space curve, the Q-frame a long $\gamma$ is given by $\{T_\gamma, \mathcal{N}_Q, \mathcal{B}_Q, \mathcal{K}\}$, where $\mathcal{K}$ is the projection vector, $\mathcal{N}_Q = \dfrac{T_\gamma \wedge \mathcal{K}}{\|T_\gamma \wedge \mathcal{K}\|}$, and $\mathcal{B}_Q = T_\gamma \wedge \mathcal{N}_Q$. $\mathcal{N}_Q$ and $\mathcal{B}_Q$ are called the quasi-principal normal and quasi-binormal respectively. The vector $\mathcal{K}$ can be chosen such that $T_\gamma$ and $\mathcal{K}$ are not parallel and in this case the Q-frame is well defined along the hole curve. In this paper we chose $\mathcal{K} = (0, 1, 0)$. If $\theta$ is the angle between $N_\gamma$ and $\mathcal{N}_Q$, then the quazi-equations are given by

$$\begin{cases} T'_\gamma = \kappa_1 \mathcal{N}_Q + \kappa_2 \mathcal{B}_Q \\ \mathcal{N}'_Q = -\kappa_1 T_\gamma + \kappa_3 \mathcal{B}_Q \\ \mathcal{B}'_Q = -\kappa_2 T_\gamma - \kappa_3 \mathcal{N}_Q \end{cases}$$

where, prime denotes the derivative with respect to the arc-length of $\gamma$, and ( $\kappa_1 = \kappa_\gamma \cos\theta$, $\kappa_2 = -\kappa_\gamma \sin\theta$ and $\kappa_3 = \tau_\gamma + \theta'$) are the quasi-curvatures.
The directional tubular surface associated to quasi-frame is given by the following definition.

**Definition 2.1.** *Let $\gamma$ be a regular space curve. The directional tubular surface (D-tubular surface) associated to quasi-frame at a distance $d$ from $\gamma$ is*

$$M(s, \phi) = \gamma(s) + d(\cos(\phi)\mathcal{N}_Q(s) + \sin(\phi)\mathcal{B}_Q(s)). \tag{2.1}$$

## 3. Singularities of the D-tubular surfaces

In this section we study the singularities of the D-tubular surface and give the conditions for this surface to have cuspidal edges and swallowtail singularities. First of all, we review some basic concepts of singularity theory which are needed in this section. For more detail we refer the reader to [5, 11, 15].

**Definition 3.1.** *Let $\mathbb{F} : \mathcal{M}^m \longrightarrow \mathcal{N}^n$ be a smooth map between two smooth manifolds. $\mathbb{F}$ is singular at $p$ if and only if the rank of $d\mathbb{F}_p$ ,the differential of $\mathbb{F}$, at $p$ is less than min $(m, n)$.*

**Definition 3.2.** *Let $\mathbb{F} : \mathcal{M}^m \longrightarrow \mathcal{N}^n$ be a smooth map between two smooth manifolds. $\mathbb{F}$ is parametrized by a co-rank $\chi$ singularity at $p$ if and only if $min(m, n) - rank(d\mathbb{F}_p) = \chi$.*

**Definition 3.3.** *Two map-germs $\mathbb{F}_i : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ $(i = 1, 2)$ are $\mathcal{A}$-equivalent if there exist germs of $C^\infty$-diffeomorphisms $\vartheta$ and $\varphi$ such that $\varphi \circ \mathbb{F}_1 = \mathbb{F}_2 \circ \vartheta$ holds, where $\vartheta : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ and $\varphi : (\mathbb{R}^p, 0) \to (\mathbb{R}^p, 0)$.*

The main object of this paper is to study the singularity of D-tubular surface. Precisely, we give the conditions for this surface to have cuspidal edge, swallowtail, cuspidal butterfly, cuspidal lips and cuspidal beaks singularities with normal forms $(x, y^2, y^3)$, $(x, 3y^4+xy^2, 4y^3+2xy)$, $(4x^5+x^2y, 5x^4+2xy, y)$, $(x^3+xy^2, y, 3x^4+2x^2y^2)$ and $(x^3 - xy^2, y, 3x^4 - 2x^2y^2)$ respectively in terms of quasi-curvatures. To do so, we use the criteria for wave front to have such singularities.

**Definition 3.4.** *a map $\mathbb{F} : \mathcal{U} \longrightarrow \mathbb{R}^3$ is called a front (wave front) if there is a unit vector field $\mathbb{V}$ a long $\mathbb{F}$ such that $\mathbb{L} := (\mathbb{F}, \mathbb{V}) : \mathcal{U} \longrightarrow T_1\mathbb{R}^3$ is a Legendrian immersion.*

The vector field $\mathbb{V}$ is called the normal of the front. The crucial function associated to a front is the density function which is given by $\lambda(x, y) = \det(\mathbb{F}_x, \mathbb{F}_y, \mathbb{V})$, where $(x, y)$ is a local coordinate system of $\mathcal{U}$, $\mathbb{F}_x = \dfrac{\partial \mathbb{F}}{\partial y}$ and $\mathbb{F}_y = \dfrac{\partial \mathbb{F}}{\partial y}$. If $p$ is a singular point of $\mathbb{F}$, then $p$ is a non-degenerate singular point if and only if $d\lambda(p) \neq 0$. For a non-degenerate singular point there exists a regular curve $\mathcal{Y}$ near $p$ such that its image under $\mathbb{F}$ coincides with the singular set of $\mathbb{F}$. Also, there is a non-vanishing vector field $\eta$ along $\mathcal{Y}$. This vector field is called null vector field. For more detail regarding this issue we refer the reader to [5, 6, 9, 10].

**Criteria 3.5.** *Let $\mathbb{F} : \mathcal{U} \longleftarrow \mathbb{R}^3$ be a front and $p$ be a non-degenerate singular point of $\mathbb{F}$. Then,*

(1) *$\mathbb{F}$ is diffeomorphic to cuspidal edge at $\mathbb{F}(p)$ if and only if $\eta\lambda \neq 0$*
(2) *$\mathbb{F}$ is diffeomorphic to swallowtail at $\mathbb{F}(p)$ if and only if $\eta\lambda = 0$ and $\eta^2\lambda \neq 0$ at $(0, 0)$.*
(3) *$\mathbb{F}$ is diffeomorphic to cuspidal butterfly at $\mathbb{F}(p)$ if and only if $\eta\lambda = \eta^2\lambda = 0$ and $\eta^3\lambda \neq 0$ at $(0, 0)$.*

The following criteria is about the cuspidal lips and cuspidal beaks singularities at degenerate singular point of a front (see cf.[5, 9, 10]).

**Criteria 3.6.** *Let $\mathbb{F} : \mathcal{U} \longleftarrow \mathbb{R}^3$ be a front and $p$ be a degenerate singular point of $\mathbb{F}$. Then,*

(1) *$\mathbb{F}$ is diffeomorphic to cuspidal lips at $\mathbb{F}(p)$ if and only if the rank$(d\mathbb{F}) = 1$ and $\det(Hess(\lambda(p))) > 0$.*
(2) *$\mathbb{F}$ is diffeomorphic to cuspidal beaks at $\mathbb{F}(p)$ if and only if the rank$(d\mathbb{F}) = 1$ and $\det(Hess(\lambda(p))) < 0$ and $\eta^2\lambda(p) \neq 0$, $Hess(\lambda(p))$ is the Hessian matrix of $\lambda$ at $p$.*

Now it is easy to prove the following lemma.

**Lemma 3.7.** *Let $\gamma$ be a regular space curve parametrized by its arc-length $s$. The D-tubular surface $M$, as in definition 2.1, associated to quasi-frame is a front (wave front).*

**Lemma 3.8.** *Let $M$ be the D-surface as defined in Definition 2.1. $M$ is parametrized by a co-rank one singularity at $M(s_0, \phi_0)$ if and only if*

$$d[\kappa_1 \cos(\phi) + \kappa_2 \sin(\phi)] = 1 \tag{3.1}$$

*at $(s_0, \phi_0)$.*

*Proof.* It has been proved in [12], that the tubular surface is singular if and only if equation (3.1) holds. Our task here is to show that the type of singularity is of co-rank one. Now if equation (3.1) holds, then

$$M_s = d\kappa_3[\cos(\phi)\mathcal{B}_Q - \sin(\phi)\mathcal{N}_Q],$$
$$M_\phi = [\cos(\phi)\mathcal{B}_Q - \sin(\phi)\mathcal{N}_Q].$$

Therefore, the rank of the differential of $M$ is equal to 1, which means that the singularity is of a co-rank one.

□

**Proposition 3.9.** *Let $M$ be the D-surface as defined in Definition 2.1. If $p = M(s_0, \phi_0)$ is a singular point for $M$, then $p$ is non-degenerate singular point if and only if $\kappa_1' \cos(\phi) + \kappa_2' \sin(\phi) \neq 0$ or $\kappa_2 \cos(\phi) - \kappa_1 \sin(\phi) \neq 0$ at $(s_0, \phi_0)$.*

*Proof.* Let $M$ be the D-tubular surface given by parametrization as in equation (2.1). Then, it is singular if and only if equation (3.1) holds. Now assume that $M$ is singular, we define the density function as

$$\lambda(s, \phi) = d[\kappa_1(s) \cos(\phi) + \kappa_2(s) \sin(\phi)] - 1. \tag{3.2}$$

It is clear that $\lambda_s = d[\kappa_1' \cos(\phi) + \kappa_2' \sin(\phi)]$ and $\lambda_\phi = d[\kappa_2 \cos(\phi) - \kappa_1 \sin(\phi)]$. Thus it is clear that $d\lambda \neq 0$ if and only if $\lambda_s \neq 0$ or $\lambda_\phi \neq 0$, which complete the proof of the proposition. □

**Corollary 3.10.** *Assume as in Proposition 3.9. If $p = M(s_0, \phi_0)$ is a singular point, then it is degenerate singular point if and only if $\kappa_1' \cos(\phi) + \kappa_2' \sin(\phi) = 0$ and $\kappa_2 \cos(\phi) - \kappa_1 \sin(\phi) = 0$ at $(s_0, \phi_0)$.*

*Proof.* The proof of this corollary comes directly from the negation of Proposition 3.9. □

Now we state the first main theorem for this paper.

**Theorem 3.11.** *Let $M$ be the D-surface as defined in Definition 2.1. If $M(s_0, \phi_0)$ is a non-degenerate singular point for $M$. Then*

(1) *$M$ is diffeomorphic to cuspidal edge or has cuspidal edge singularity at $M(s_0, \phi_0)$ if and only if at $(s_0, \phi_0)$*

$$(\kappa_1' - \kappa_2\kappa_3) \cos(\phi) + (\kappa_2' + \kappa_1\kappa_3) \sin(\phi) \neq 0.$$

(2) *$M$ is diffeomorphic to swallowtail or has swallowtail singularity at $M(s_0, \phi_0)$ if and only if at $(s_0, \phi_0)$*

$$\begin{cases} (\kappa_1' - \kappa_2\kappa_3)' \cos(\phi) + (\kappa_2' + \kappa_1\kappa_3)' \sin(\phi) = 0 \quad \text{and} \\ (\kappa_1' - \kappa_2\kappa_3)' \cos(\phi) + (\kappa_2' + \kappa_1\kappa_3)' \sin(\phi) \neq \kappa_3[(\kappa_2' + \kappa_1\kappa_3) \cos(\phi) - (\kappa_1' - \kappa_2\kappa_3) \sin(\phi)]. \end{cases}$$

(3) *$M$ is diffeomorphic to cuspidal butterfly at $M(s_0,\phi_0)$ if and only if at $(s_0,\phi_0)$*

$$\begin{cases}(\kappa_1'-\kappa_2\kappa_3)'\cos(\phi)+(\kappa_2'+\kappa_1\kappa_3)'\sin(\phi)=0,\\ (\kappa_1'-\kappa_2\kappa_3)'\cos(\phi)+(\kappa_2'+\kappa_1\kappa_3)'\sin(\phi)=\kappa_3[(\kappa_2'+\kappa_1\kappa_3)\cos(\phi)-(\kappa_1'-\kappa_2\kappa_3)\sin(\phi)] \quad \textit{and}\\ (\kappa_1'-\kappa_2\kappa_3)''\cos(\phi)+(\kappa_2'+\kappa_1\kappa_3)''\sin(\phi)+\kappa_3'[(\kappa_1'-\kappa_2\kappa_3)\sin(\phi)-(\kappa_2'+\kappa_1\kappa_3)\cos(\phi)]\\ \neq 2\kappa_3[(\kappa_2'+\kappa_1\kappa_3)'\cos(\phi)-(\kappa_1'-\kappa_2\kappa_3)'\sin(\phi)].\end{cases}$$

*Proof.* It is clear that $M$ is singular if and only if $\lambda=0$. The vector filed $\eta$ is defined as

$$\eta=\frac{\partial}{\partial s}-\kappa_3\frac{\partial}{\partial\phi}.$$

Therefore,

$$\eta\lambda=d[(\kappa_1'-\kappa_2\kappa_3)\cos(\phi)+(\kappa_2'+\kappa_1\kappa_3)\sin(\phi)]. \tag{3.3}$$

$$\begin{aligned}\eta^2\lambda=&d[(\kappa_1'-\kappa_2\kappa_3)'\cos(\phi)+(\kappa_2'+\kappa_1\kappa_3)'\sin(\phi)]\\&+\kappa_3d[(\kappa_1'-\kappa_2\kappa_3)\sin(\phi)-(\kappa_2'+\kappa_1\kappa_3)\cos(\phi)].\end{aligned} \tag{3.4}$$

$$\begin{aligned}\eta^3\lambda=&d[(\kappa_1'-\kappa_2\kappa_3)''\cos(\phi)+(\kappa_2'+\kappa_1\kappa_3)''\sin(\phi)]\\&+\kappa_3'd[(\kappa_1'-\kappa_2\kappa_3)\sin(\phi)-(\kappa_2'+\kappa_1\kappa_3)\cos(\phi)]\\&+\kappa_3d[(\kappa_1'-\kappa_2\kappa_3)'\sin(\phi)-(\kappa_2'+\kappa_1\kappa_3)'\cos(\phi)]\\&-\kappa_3d[(\kappa_2'+\kappa_1\kappa_3)'\cos(\phi)-(\kappa_1'-\kappa_2\kappa_2)'\sin(\phi)]\\&-\kappa_3^2d[(\kappa_1'-\kappa_2\kappa_3)\cos(\phi)+(\kappa_2'+\kappa_1\kappa_3)\sin(\phi)].\end{aligned} \tag{3.5}$$

Therefore, using Criteria 3.5, the proof is completed. □

Now we give the conditions for the directional tubular surface to have cuspidal lips and cuspidal beaks at degenerate singular points

**Theorem 3.12.** *Let $M$ be the D-surface as defined in Definition 2.1. If $M$ has a degenerate singularity at $M(s_0,\phi_0)$. Then*

(1) *$M$ is diffeomorphic to cuspidal lips at $M(s_0,\phi_0)$ if and only if*

$$\kappa_1''\cos(\phi)+\kappa_2''\sin(\phi)+d(\kappa_2'\cos(\phi)-\kappa_1'\sin(\phi))^2<0.$$

(2) *$M$ is diffeomorphic to cuspidal beaks at $M(s_0,\phi_0)$ if and only if the following conditions hold*
   (*i*) $\kappa_1''\cos(\phi)+\kappa_2''\sin(\phi)+d(\kappa_2'\cos(\phi)-\kappa_1'\sin(\phi))^2>0$
   (*ii*) $(\kappa_1'-\kappa_2\kappa_3)'\cos(\phi)+(\kappa_2'+\kappa_1\kappa_3)'\sin(\phi)\neq\kappa_3[(\kappa_2'+\kappa_1\kappa_3)\cos(\phi)-(\kappa_1'-\kappa_2\kappa_3)\sin(\phi)$.

*Proof.* Let $M(s_0,\phi_0)$ be a degenerate singular point for the tubular surface $M$. The Hessian matrix of $\lambda$ is given by

$$Hess(\lambda)=\begin{pmatrix}\lambda_{ss}&\lambda_{s\phi}\\ \lambda_{s\phi}&\lambda_{\phi\phi}\end{pmatrix}=\begin{pmatrix}d[\kappa_1''\cos(\phi)+\kappa_2''\sin(\phi)]&d[\kappa_2'\cos(\phi)-\kappa_1'\sin(\phi)]\\ d[\kappa_2'\cos(\phi)-\kappa_1'\sin(\phi)]&-d[\kappa_1\cos(\phi)+\kappa_2\sin(\phi)]\end{pmatrix}.$$

At a singular point $\lambda_{\phi\phi}=-1$. Therefore, the determinant of the Hessian matrix of $\lambda$ at a singular point is given by

$$\det(Hess(\lambda))=-d[\kappa_1''\cos(\phi)+\kappa_2''\sin(\phi)]-d^2[\kappa_2'\cos(\phi)-\kappa_1'\sin(\phi)]^2.$$

Therefore, using Criteria 3.6 the results hold. □

## 4. Examples

In this section, we provide two examples of the D-tubular surface to illustrate the result (2), in Theorem 3.11 and result (1) in Theorem 3.12. Also, we draw the visualization of the D-tubular surfaces in both the examples.

## Example 1

Now we give an example for the D-tubular surface when it has swallowtail singularity.
Let $\gamma(t) = (t, t^2, t^3)$, we choose the projection vector to be $\mathcal{K} = (0, 1, 0)$. The Quazi-normal and Quazi-binormal are given by

$$\begin{cases} \mathcal{N}_Q = \left(\dfrac{-3t^2}{\sqrt{1+9t^4}}, 0, \dfrac{1}{\sqrt{1+9t^4}}\right), \\ \\ \mathcal{B}_Q = \left(\frac{2t}{\sqrt{(1+9t^4)(1+4t^2+9t^4)}}, \frac{-(1+9t^4)}{\sqrt{(1+9t^4)(1+4t^2+9t^4)}}, \frac{6t^3}{\sqrt{(1+9t^4)(1+4t^2+9t^4)}}\right). \end{cases}$$

Now if the distance $d = 2$, then the associated D-tubular surface is given by $M(t, \phi) = (M_1(t, \phi), M_2(t, \phi), M_3(t, \phi))$, where

$$M_1(t, \phi) = \frac{t\sqrt{(1+9t^4)(1+4t^2+9t^4)} - 6t\sqrt{1+4t^2+9t^4}\cos(\phi) + 4\sin(\phi)}{\sqrt{(1+9t^4)(1+4t^2+9t^4)}};$$

$$M_2(t, \phi) = \frac{t^2\sqrt{1+4t^2+9t^4} - 2\sin(\phi)\sqrt{1+9t^4}}{\sqrt{1+4t^2+9t)}};$$

$$M_3(t, \phi) = \frac{t^3\sqrt{(1+9t^4)(1+4t^2+9t^4)} + 2\cos(\phi)\sqrt{1+4t^2+9t^4} + 12t^3\sin(\phi)}{\sqrt{(1+9t^4)(1+4t^2+9t^4)}}.$$

The local picture of $M$ is shown in Figure 1

## Example 2

Let $\gamma(t) = (t, \sinh(t), \cosh(t))$, the Frenet frame of $\gamma$ is given by

$$\begin{cases} T_\gamma(t) = \left(\dfrac{\text{sech(t)}}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}, \dfrac{\tanh(t)}{\sqrt{2}}\right), \\ \\ N_\gamma = (-\tanh(t), 0, \text{sech(t)}), \\ \\ B_\gamma = \left(\dfrac{\text{sech(t)}}{\sqrt{2}}, \dfrac{-1}{\sqrt{2}}, \dfrac{\tanh(t)}{\sqrt{2}}\right). \end{cases}$$

Calculation shows that the Q-frame in the direction $\mathcal{K} = (0, 1, 0)$ coincides with Frenet-frame. Therefore, $\{T_\gamma, N_\gamma, B_\gamma\} = \{T_\gamma, \mathcal{N}_Q, \mathcal{B}_Q\}$. Now if $d = 2$, then the associated D-tubular surface is given by
$M(t, \phi) = (M_1(s, \phi), M_2(s, \phi), M_3(s, \phi))$ where,

$$M_1(s, \phi) = \frac{t\cosh(t) - 2\sinh(t)\cos(\phi) + \sqrt{2}\sin(\phi)}{\cosh(t)};$$

$$M_2(s, \phi) = \sinh(t) - \sqrt{2}\sin(\phi);$$

$$M_3(s, \phi) = \frac{\cosh^2(t) + 2\cos(\phi) + \sqrt{2}\sinh(t)\sin(\phi)}{\cosh(t)}.$$

Calculation shows that $M$ has cuspidal lips singularity at $M(0, 0) = (0, 0, 3)$. The local picture of $M$ in a neighborhood of $(0, 0, 3)$ is shown in Figure 2.

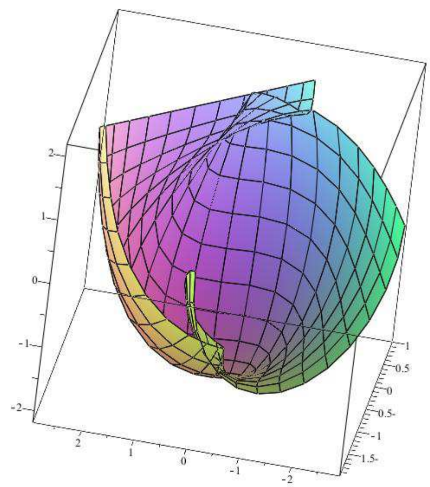

FIGURE 1. Figure of example 1

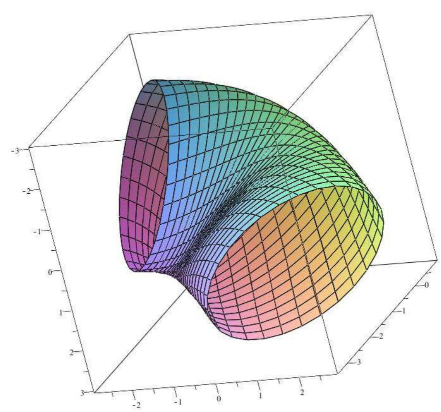

FIGURE 2. Figure of example 2

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