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CHARACTERIZATION OF p-BESSEL SEQUENCES IN BANACH SPACES

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Abstract. Let X be a reflexive separable Banach space, and let $p\mathfrak{B}$ the set of p-Bessel sequences in X^* for X. We show that $p\mathfrak{B}$ is a non-commutative unital Banach algebra isometrically isomorphic to $B(X)$. Also, we classify p-Bessel sequences for X in terms of different kind of operators in $B(X)$ and $B(X^*)$, and we give important characterizations of p-frames and q-Riesz sequences. Using an isomorphism between the sets $p\mathfrak{B}$ and $B(X)$ we obtain interesting results for p-frames in Banach spaces. Using operator theory tools, we investigate the geometry of p-Bessel sequences. Also, we show that the set of all q-Riesz bases for X^* is a topological group.

1. Introduction

Frames for Hilbert spaces were first introduced by Duffin and Schaeffer in 1952 to study some deep questions in non-harmonic Fourier series. Later, in 1986 Daubechies, Grossmann and Meyer reintroduced frames. Today, frame theory is a central tool in many areas such as function space and signal analysis. For a nice and comprehensive survey on various types of frames, see [6, 14].

Definition 1.1. *A sequence (f_i) in a separable Hilbert space H is called a frame if there exist constants $0 < A \le B < \infty$ such that*

$$A||f||^2 \le \sum_i |\langle f, f_i\rangle|^2 \le B||f||^2.$$

for every $f \in H$.

Definition 1.2. *A sequence $(f_i)_{i\in\mathbb{N}}$ in a separable Hilbert space H is called a Riesz sequence if there exist constants $0 < A \le B < \infty$ such that for any finite subset $J \subset \mathbb{N}$,*

$$A\sum_{k\in J} |c_k|^2 \le \|\sum_{k\in J} c_k f_k\|^2 \le B\sum_{k\in J} |c_k|^2.$$

A natural extension of the frame inequalities to Banach spaces leads to the concept of p-frame. Aldroubi, Sun and Tang [2] introduced the concept of p-frame for a Banach space X. Their main results were about p-frame for $L^p(R^d)$. They obtained series expansions in shift-invariant subspaces of $L^p(R^d)$. Christensen and

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Stoeva [7] further discussed some properties of p-frame for a general Banach space X. For more information about p-frame theory, see [22, 25].
Since the investigation of Bessel sequences (p-Bessel sequences) is essential for studying frames (p-frames), in literature there are many results about Bessel sequences (p-Bessel sequences). For instance, the authors in [16], investigated Bessel sequences in Sobolev space. The authors in [4], studied finite extensions of Bessel sequences in infinite dimensional Hilbert spaces. The authors in [22], studied multipliers for p-Bessel sequences in Banach spaces. In [9] and [3], the authors investigated the set of all Bessel sequences for a separable Hilbert space H as a Banach space and obtained important results. Also, in [10], the authors gave a C^*-algebra structure to the set of all Bessel sequences for H.

Let X be a reflexive separable Banach space. Let $p\mathfrak{B}$ be the set of all p-Bessel sequences in X^* for X. In this paper, we show that $p\mathfrak{B}$ is a non-commutative unital Banach algebra isomorphic to $B(X)$. Also, we show that there is a bijection between $p\mathfrak{B}$ and $B(X^*)$. We classify p-Bessel sequences for X in terms of different kind of operators in $B(X)$ and $B(X^*)$, and we give important characterizations of p-frames and q-Riesz sequences. Using an isomorphism between the sets $p\mathfrak{B}$ and $B(X)$ we obtain interesting results for p-frames in Banach spaces. Using operator theory tools, we investigate the geometry of p-Bessel sequences. Also, we show that the set of all q-Riesz bases for X^* is a topological group.

This paper is organized as follows:
In Section 2, we state some well-known definitions and results. In Section 3, we characterize the p-frames, q-Riesz sequences and q-Riesz bases under the assumption that there exists a p-Riesz basis for X. In Section 4, we give a structure of Banach algebra to $p\mathfrak{B}$ and we show that $p\mathfrak{B}$ is a non-commutative unital Banach algebra isomorphic to $B(X)$. Using the obtained results in Section 3, we classify p-Bessel sequences for X in terms of different kinds of operators in $B(X)$ and $B(X^*)$. Using operator theory tools, we investigate the geometry of p-Bessel sequences. Also, we study some properties of multiplication in $p\mathfrak{B}$ and we show that the set of all q-Riesz bases for X^* is a topological group.

2. Notation and Preliminaries

Throughout this paper X denotes a reflexive and separable Banach space with dual space X^*, $B(X)$ is the algebra of all bounded linear operators on X, and $\mathbb{C}$ denotes the field of complex numbers. For an operator $T \in B(X)$ we write T^* for its adjoint, $ker(T)$ for its kernel, and $R(T)$ for its range. An operator $T : X \to Y$ between normed spaces is said to be *bounded below* if there exists a constant $\delta > 0$ such that for every $x \in X$, $\|Tx\| \geq \delta\|x\|$.

Also we let $p, q > 1$, and $1/p + 1/q = 1$.

Definition 2.1. *A sequence $(f_i)_{i\in\mathbb{N}} \subset X^*$ is called a p-frame for X, if there exist constants $0 < A \leq B < \infty$ such that*

$$A||x|| \leq (\sum_{i\in\mathbb{N}} |f_i(x)|^p)^{\frac{1}{p}} \leq B||x||, \ \forall x \in X.$$

The numbers A,B are called p-frame bounds. We call $(f_i)_{i\in\mathbb{N}}$ a tight p-frame *if $A = B$ and a* Parseval p-frame *for X if $A = B = 1$. If only the right hand side inequality is required, $(f_i)_{i\in\mathbb{N}}$ is called a* p-Bessel sequence *for X.*

Remark 1. [7, Lemma 2.1] *If there exists a p-frame $(f_i)_{i\in\mathbb{N}} \subset X^*$ for X, then X is reflexive.*

If $f = (f_i)_{i\in\mathbb{N}} \subset X^*$ is a p-Bessel sequence for X, then the analysis operator for $(f_i)_{i\in\mathbb{N}}$ is the linear operator $U_f : X \to \ell_p$ with $U_f(x) = (f_i(x))_{i\in\mathbb{N}}$ and the synthesis operator for $(f_i)_{i\in\mathbb{N}}$ is the linear operator,

$$T_f : \ell^q \to X^* \qquad T_f((d_i)_{i\in\mathbb{N}}) = \sum_{i\in\mathbb{N}} d_i f_i.$$

We say $(f_i) \subseteq X^*$ is ℓ^q*-independent* if T_f is injective.

Proposition 2.2. [7] *The sequence $f = (f_i)_{i\in\mathbb{N}} \subset X^*$ is a p-Bessel sequence for X with bound B if and only if T_f is a well-defined (hence bounded) operator from ℓ^q into X^* and $||T_f|| \leq B$. In this case $T_f((d_i)_{i\in\mathbb{N}}) = \sum_{i\in\mathbb{N}} d_i f_i$ converges unconditionally.*

Definition 2.3. *A sequence $(x_i)_{i\in\mathbb{N}} \subset X$ is called a* p-Riesz sequence *for X if there exist two positive constants A and B such that*

$$A(\sum_{i\in\mathbb{J}} |a_i|^p)^{\frac{1}{p}} \leq ||\sum_{i\in\mathbb{J}} a_i x_i|| \leq B(\sum_{i\in\mathbb{J}} |a_i|^p)^{\frac{1}{p}},$$

for any finite subset $J \subset \mathbb{N}$. The numbers A and B are called p-Riesz bounds. If moreover $\overline{span(x_i)_{i\in\mathbb{N}}} = X$, then $\{x_i\}_{i\in\mathbb{N}}$ is a p-Reisz basis.

Considering $X_d = \ell^p$ for [24, Proposition 4.2], we have:

Proposition 2.4. *Let X be a reflexive Banach space, and $(f_i)_{i\in\mathbb{N}}$ be a p-Riesz basis for X, and let $T \in B(X)$. The sequence $(T(f_i))_{i\in\mathbb{N}}$ is a p-Riesz basis for X if and only if T is bijective.*

By the Open Mapping Theorem it is equivalent to T is invertible with bounded inverse.

Lemma 2.5. [7] *Let $(x_i)_{i\in\mathbb{N}} \subset X$ be a p-Riesz basis for X, with bounds A, B. Then there exists a unique q-Riesz basis $(\widetilde{x}_i)_{i\in\mathbb{N}} \subset X^*$ for X^* such that*

$$x = \sum_{i\in\mathbb{N}} \widetilde{x}_i(x) x_i \quad \forall x \in X, \qquad f = \sum_{i\in\mathbb{N}} f(x_i)\widetilde{x}_i \quad \forall f \in X^*.$$

The sequence $(\widetilde{x}_i)_{i\in\mathbb{N}}$ is called the dual of $(x_i)_{i\in\mathbb{N}}$ and has the bounds $\frac{1}{B}, \frac{1}{A}$.

Dual sequences $(x_i)_{i\in I} \subseteq X$ and $(\widetilde{x}_i)_{i\in I} \subseteq X^*$ are *biorthogonal* if $\widetilde{x}_j(x_i) = \delta_{ij}$.

Lemma 2.6. [7, 26] *For any sequence $(f_i)_{i\in\mathbb{N}} \subset X^*$ the following statements are equivalent:*
(1) $(f_i)_{i\in\mathbb{N}}$ *is a p-frame for X with optimal p-frame bounds A, B and is l_q-linearly independent.*
(2) $(f_i)_{i\in\mathbb{N}}$ *is a q-Riesz basis for X^* with optimal q-Riesz basis bounds A, B.*

We will prove some results by introducing particular operators, called multipliers. Bessel multipliers for Hilbert spaces were introduced by Balazs in [5]. Then they generalized to Banach spaces in [22] and to Hilbert C^*-modules in [18].

Lemma 2.7. [22] *Let $f = (f_i)_{i\in\mathbb{N}} \subset X^*$ be a p-Bessel sequence for X with bound B_1, let $h = (h_i)_{i\in\mathbb{N}} \subset X$ be a q-Bessel sequence for X^* with bound B_2. The operator $M_{(h_i)(f_i)} : X \to X$ defined by*

$$M_{(h_i)(f_i)}(x) = T_h T_f^*(x) = \sum_{i\in\mathbb{N}} f_i(x)h_i, \quad \forall x \in X \tag{2.1}$$

is well defined. This sum converges unconditionally and $||M_{(h_i)(f_i)}|| \leq B_1B_2$. The operator $M_{(h_i)(f_i)}$ is called a (p,q)-Bessel multiplier.

Considering $X_d = \ell^p$ for [13, Theorem 3.4], we have:

Theorem 2.8. *Suppose X is a reflexive Banach space. Let $(h_i)_{i\in\mathbb{N}} \subset X$ be a p-Riesz basis for X and $(f_i)_{i\in\mathbb{N}} \subset X^*$ be a p-Bessel sequence for X and let $M_{(h_i)(f_i)}$ be defined as (2.1). Then $M_{(h_i)(f_i)}$ is invertible on X if and only if $(f_i)_{i\in\mathbb{N}}$ is a q-Riesz basis for X^*.*

Theorem 2.9. [12] *Suppose X and Y are Banach spaces. Let $L \in B(X,Y)$, and let $L^* : Y^* \to X^*$ be its adjoint. Then L is surjective if and only if L^* is bounded below, and L is bounded below if and only if L^* is surjective.*

Remark 2. [19] *Suppose X and Y are Banach spaces. Let $L \in B(X,Y)$, and let $L^* : Y^* \to X^*$ be its adjoint. Then L has dense range if and only if L^* is injective.*

Throughout this paper, we need to assume that X is a reflexive Banach space. Also, $p\mathfrak{B}$ denotes the set of all p-Bessel sequences in X^* for X.

3. Characterization of p-Bessel sequences and p-frames in Banach Spaces

First we give a characterization of p-Bessel sequences.

Proposition 3.1. *$(f_i)_{i\in\mathbb{N}} \subset X^*$ is a p-Bessel sequence for X if and only if for every $x \in X$, $\sum_{i\in\mathbb{N}} |f_i(x)|^p < \infty$.*

Proof. The necessity is obvious. We prove the sufficiency. Since for every $x \in X$, $\sum_{i\in\mathbb{N}} |f_i(x)|^p < \infty$ the operator $U : X \to \ell^p$ defined by $U(x) = (f_i(x))_{i\in\mathbb{N}}$ is well-defined and linear. We prove that the graph of U is closed. Let $(x_n) \subseteq X$ such that $x_n \to x \in X$ and $U(x_n) \to y = (y_i) \in \ell^p$. Since for every $i \in \mathbb{N}$,

$$|f_i(x_n) - y_i|^p \leq \sum_j |f_j(x_n) - y_j|^p = \|U(x_n) - y\|^p,$$

then $f_i(x_n) \to y_i$. On the other hand f_i is continuous, so $f_i(x_n) \to f_i(x)$. Hence $y_i = f_i(x)$, for each $i \in \mathbb{N}$ and consequently $y = f(x)$. Therefore by the Closed Graph Theorem, U is a bounded operator. Hence, $(f_i)_{i\in\mathbb{N}}$ is a p-Bessel sequence for X. □

In the following theorem, we give a characterization of p-frames in $p\mathfrak{B}$.

Theorem 3.2. *Let $(f_i)_{i\in\mathbb{N}} \subset X^*$ be a p-Bessel sequence for X, $(h_i^0)_{i\in\mathbb{N}} \subset X$ be a p-Riesz basis for X. Then $(f_i)_{i\in\mathbb{N}}$ is a p-frame for X if and only if there exists a unique injective operator with closed range $T \in B(X)$ such that $f_i = \widetilde{h_i^0}T$, for every $i \in \mathbb{N}$.*

Proof. Let $0 < C_1 \leq C_2$ be p-Riesz bounds for $(h_i^0)_{i\in\mathbb{N}}$.
Suppose that there exists an injective operator with closed range $T \in B(X)$ such that $f_i = \widetilde{h_i^0}T$, for every $i \in \mathbb{N}$.
Since T is an injective operator with closed range on the Banach space X, then T is bounded below. Thus there exists some positive constant A_1 such that for every $x \in X$,

$$A_1\|x\| \leq \|T(x)\|$$

Now for every $x \in X$

$$(\sum_{i\in\mathbb{N}} |f_i(x)|^p)^{\frac{1}{p}} = (\sum_{i\in\mathbb{N}} |\widetilde{h_i^0}T(x)|^p)^{\frac{1}{p}} \geq \frac{1}{C_2}||T(x)|| \geq \frac{A_1}{C_2}\|x\|,$$

and since $(f_i)_{i\in\mathbb{N}}$ is a p-Bessel sequence for X, we have the result.
Conversely, let $(f_i)_{i\in\mathbb{N}}$ be a p-frame for X with bounds $0 < B_1 \leq B_2$. Set $T := M_{(h_i^0)(f_i)}$, by Lemma 2.7, $M_{(h_i^0)(f_i)}$ is a well defined bounded operator, and easily we can see $f_i = \widetilde{h_i^0}(M_{(h_i^0)(f_i)})$ for every $i \in \mathbb{N}$.
Also, for every $x \in X$, we have

$$\begin{aligned}\|M_{(h_i^0)(f_i)}(x)\| &= \|\sum_{i\in\mathbb{N}} f_i(x)h_i^0\| \geq C_1(\sum_{i\in\mathbb{N}} |f_i(x)|^p)^{\frac{1}{p}} \\ &\geq B_1C_1\|x\|.\end{aligned}$$

Therefore $M_{(h_i^0)(f_i)}$ is bounded below, and this implies that $M_{(h_i^0)(f_i)}$ is an injective operator with closed range.
If there exist injective operators T_1, T_2 with closed ranges such that $f_i = \widetilde{h_i^0}T_1 = \widetilde{h_i^0}T_2$, then $M_{(h_i^0)(\widetilde{h_i^0}T_1)} = M_{(h_i^0)(\widetilde{h_i^0}T_2)}$. It follows that $T_1 = T_2$. So the injective operator with closed range which satisfies the condition is unique. □

In the following theorem, we give a characterization of q-Riesz sequences in $p\mathfrak{B}$.

Theorem 3.3. *Let $(f_i)_{i\in\mathbb{N}} \subset X^*$ be a p-Bessel sequence for X and $(h_i^0)_{i\in\mathbb{N}} \subset X$ be a p-Riesz basis for X. Then $(f_i)_{i\in\mathbb{N}}$ is a q-Riesz sequence for X^* if and only if there exists a unique surjective operator $T \in B(X)$ such that $f_i = \widetilde{h_i^0}T$, for every $i \in \mathbb{N}$.*

Proof. Let $(f_i)_{i\in\mathbb{N}}$ be a q-Riesz sequence for X^*. Set $T := M_{(h_i^0)(f_i)}$, by Lemma 2.7, $M_{(h_i^0)(f_i)}$ is a well defined bounded operator, and easily we can see $f_i = \widetilde{h_i^0}(M_{(h_i^0)(f_i)})$ for every $i \in \mathbb{N}$. Since $(f_i)_{i\in\mathbb{N}}$ is a q-Riesz sequence for X^*, then there exist constants $A, B > 0$ such that for every $(d_i)_{i=1}^\infty \in \ell^q$

$$A||(d_i)_{i=1}^\infty||_{\ell^q} \leq ||T_f((d_i)_{i=1}^\infty)|| \leq B||(d_i)_{i=1}^\infty||_{\ell^q}.$$

This implies that T_f is bounded below. Consequently, $T_f^* = U_f$ is surjective, see Theorem 2.9.

As X is a reflexive Banach space, $M_{(h_i^0)(f_i)} = T_{h^0}U_f$. Since $(h_i^0)_{i\in\mathbb{N}}$ is a p-Riesz basis for X, then T_{h^0} is invertible. Hence $M_{(h_i^0)(f_i)}$ is surjective.

Conversely, suppose that there exists a surjective operator $T \in B(X)$ such that $f_i = \widetilde{h_i^0}T$, for every $i \in \mathbb{N}$.
It is obvious that $U_f = U_{\widetilde{h_i^0}}T$. Since $(\widetilde{h_i^0})_{i\in\mathbb{N}}$ is a q-Riesz basis for X^*, $U_{\widetilde{h_i^0}}$ is

invertible. Hence, U_f is surjective. So $T_f = U_f^*$ is an injective operator with closed range. Therefore there exists a constant $A > 0$ such that for every $(d_i)_{i=1}^{\infty} \in \ell^q$

$$A||(d_i)_{i=1}^{\infty}||_{\ell^q} \leq ||T_f((d_i)_{i=1}^{\infty})||.$$

On the other hand, $(f_i)_{i\in\mathbb{N}}$ is a p-Bessel sequence for X, then by Proposition 2.2, the upper bound condition is fulfilled, too. Therefore, $(f_i)_{i\in\mathbb{N}}$ is a q-Riesz sequence for X^*.

If there exist surjective operators T_1, T_2 such that $f_i = \widetilde{h_i^0}T_1 = \widetilde{h_i^0}T_2$, then $M_{(h_i^0)(\widetilde{h_i^0}T_1)} = M_{(h_i^0)(\widetilde{h_i^0}T_2)}$. It follows that $T_1 = T_2$. So the surjective operator which satisfies the condition is unique. □

Corollary 3.4. *Let $(f_i)_{i\in\mathbb{N}} \subset X^*$ be a p-Bessel sequence for X and $(h_i^0)_{i\in\mathbb{N}} \subset X$ be a p-Riesz basis for X. If X is an infinite dimensional Banach space, and $T \in B(X)$ is a compact operator, then $(\widetilde{h_i^0}T)_{i\in\mathbb{N}}$ is never a q-Riesz sequence for X^*.*

Proof. If X is an infinite dimensional Banach space, then the compact operator T is not surjective, see [23]. By the proof of Theorem 3.3, if $(\widetilde{h_i^0}T)_{i\in\mathbb{N}}$ is a q-Riesz sequence for X^*, then $M_{(h_i^0)(\widetilde{h_i^0}T)} = T$ is surjective. Then $(\widetilde{h_i^0}T)_{i\in\mathbb{N}}$ is never a q-Riesz sequence for X^*. □

For results involving frames and finite dimensionality in Hilbert spaces see Chapter 3 in [15].

Remark 3. *Let $(h_i^0)_{i\in\mathbb{N}} \subset X$ be a fixed p-Riesz basis for X. If $(f_i)_{i\in\mathbb{N}} \subset X^*$ is a p-Bessel sequence for X, then*

$$N_{(h_i^0)(f_i)} : X^* \to X^*, \qquad N_{(h_i^0)(f_i)}(g) = \sum_{i\in\mathbb{N}} g(h_i^0) f_i$$

is a well defined linear operator and $N_{(h_i^0)(f_i)} \in B(X^*)$. Also, $N_{(h_i^0)(f_i)}$ is weak-to-weak* continuous. By* [20, Theorem 3.1.11], *$T \in B(Y^*, X^*)$ is weak*-to-weak* continuous if and only if it has the form of $T = S^*$ for some $S \in B(X,Y)$. Since $N_{(h_i^0)(f_i)} = M^*_{(h_i^0)(f_i)}$ and $M_{(h_i^0)(f_i)}$ is a bounded operator, then $N_{(h_i^0)(f_i)}$ is weak*-to-weak* continuous.*

Remark 4. *Let $(h_i^0)_{i\in\mathbb{N}} \subset X$ be a fixed p-Riesz basis for X. If $(f_i)_{i\in\mathbb{N}} \subset X^*$ is a p-Bessel sequence for X, such that $M_{(h_i^0)(f_i)}$ is a compact operator, then $(f_i)_{i\in\mathbb{N}}$ is a relatively compact sequence in X^*.*

Since $M_{(f_i)(h_i^0)}$ is a compact operator, then $M^*_{(h_i^0)(f_i)} = N_{(h_i^0)(f_i)}$ is a compact operator. It is easy to see that $N_{(h_i^0)(f_i)}((\widetilde{h_i^0})_{i\in\mathbb{N}}) = (f_i)_{i\in\mathbb{N}}$. Since $(\widetilde{h_i^0})_{i\in\mathbb{N}}$ is a bounded sequence in X^*, by the compactness of $N_{(h_i^0)(f_i)}$, $(f_i)_{i\in\mathbb{N}}$ is a relatively compact sequence in X^*.

Lemma 3.5. *Let $(f_i)_{i\in\mathbb{N}} \subset X^*$ be a p-Bessel sequence for X and $(h_i^0)_{i\in\mathbb{N}} \subset X$ be a p-Riesz basis for X. Then the following statements are equivalent:*

1. *$(f_i)_{i\in\mathbb{N}}$ is ℓ^q-independent.*
2. *$N_{(h_i^0)(f_i)}$ is injective.*
3. *$M_{(h_i^0)(f_i)} \in B(X)$ has dense range.*

Proof. $(1 \Rightarrow 2)$ Since $(h_i^0)_{i\in\mathbb{N}} \subset X$ is complete in X, we have the result.
$(2 \Rightarrow 1)$ X is a reflexive Banach space, then $N_{(h_i^0)(f_i)} = T_f U_{h^0}$. Since $(h_i^0)_{i\in\mathbb{N}}$ is

a p-Riesz basis for X, then U_{h^0} is invertible. So, T_f is injective. Hence $(f_i)_{i\in\mathbb{N}}$ is ℓ^q-independent.
$(2 \Leftrightarrow 3)$ Since $M^*_{(h_i^0)(f_i)} = N_{(h_i^0)(f_i)}$, by Remark 2, it is obvious. □

4. Banach algebra of p-Bessel sequences

In the following theorem, we show that $p\mathfrak{B}$ is a non-commutative unital Banach algebra.

Theorem 4.1. *Let $p\mathfrak{B}$ be the set of all p-Bessel sequences in X^* for X, and let $(h_i^0)_{i\in\mathbb{N}} \subset X$ be a p-Riesz basis for X. Then the following statements hold:*

1. *There exists a norm $||.||_{p\mathfrak{B}}$ for $p\mathfrak{B}$, such that $(p\mathfrak{B}, ||.||_{p\mathfrak{B}})$ is a Banach space.*
2. *There exist an equivalent norm $|.|_{p\mathfrak{B}}$ and a multiplication $._{p\mathfrak{B}}$ on $p\mathfrak{B}$ with respect to $(h_i^0)_{i\in\mathbb{N}}$ such that $(p\mathfrak{B}, |.|_{p\mathfrak{B}})$ is a non-commutative unital Banach algebra.*

Proof. It is easy to check that $p\mathfrak{B}$ is a vector space. For every $f = (f_i)_{i\in\mathbb{N}}$, set $||(f_i)_{i\in\mathbb{N}}||_{p\mathfrak{B}} = ||T_f^*|| = ||U_f||$, where T_f is the synthesis operator for $f = (f_i)_{i\in\mathbb{N}}$ and U_f is the analysis operator for $f = (f_i)_{i\in\mathbb{N}}$, then by Lemma 2.7, $||.||_{p\mathfrak{B}}$ is a norm. Suppose that $((f_i^{(n)})_{i=1}^\infty)_{n=1}^\infty$ is a Cauchy sequence for $p\mathfrak{B}$. Then, given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that, for every $n, m \geq N$, $x \in X$, and $i \in \mathbb{N}$

$$|f_i^{(n)}(x) - f_i^{(m)}(x)|^p \leq \sum_{i=1}^{\infty} |f_i^{(n)}(x) - f_i^{(m)}(x)|^p \leq \varepsilon ||x||^p.$$

Let $x \in X$ be fixed. So, for every $i \in \mathbb{N}$, $(f_i^{(n)}(x))_{n=1}^\infty$ is a Cauchy sequence in $\mathbb{C}$ thus it has a limit, say g_i^x.
Now, for every $i \in \mathbb{N}$, define $b_i : X \to \mathbb{C}$, by $b_i(x) = g_i^x$. Then b_i is a well-defined linear operator from X to $\mathbb{C}$. For every $M \geq 1$, $m, n \geq N$, and $x \in X$ we have

$$\sum_{i=1}^{M} |f_i^{(n)}(x) - f_i^{(m)}(x)|^p \leq \sum_{i=1}^{\infty} |f_i^{(n)}(x) - f_i^{(m)}(x)|^p \leq \varepsilon ||x||^p.$$

If we let $m \to \infty$ this gives

$$(\sum_{i=1}^{M} |f_i^{(n)}(x) - b_i(x)|^p)^{\frac{1}{p}} = (\sum_{i=1}^{M} |f_i^{(n)}(x) - g_i^x|^p)^{\frac{1}{p}} \leq \sqrt[p]{\varepsilon}||x||, \tag{4.1}$$

for every $M \geq 1$, $n \geq N$, and $x \in X$. By Minkowski inequality,

$$\begin{aligned} (\sum_{i=1}^{M} |b_i(x)|^p)^{\frac{1}{p}} &\leq (\sum_{i=1}^{M} |f_i^{(N)}(x) - b_i(x)|^p)^{\frac{1}{p}} + (\sum_{i=1}^{M} |f_i^{(N)}(x)|^p)^{\frac{1}{p}} \\ &\leq \sqrt[p]{\varepsilon}||x|| + ||(f_i^{(N)})_{i=1}^\infty||_{p\mathfrak{B}}||x||, \end{aligned}$$

Therefore

$$(\sum_{i=1}^{\infty} |b_i(x)|^p)^{\frac{1}{p}} \leq (\sqrt[p]{\varepsilon} + ||(f_i^{(N)})_{i=1}^\infty||_{p\mathfrak{B}})||x||,$$

so $(b_i)_{i=1}^\infty \in p\mathfrak{B}$.
Finally, letting $M \to \infty$ in (4.1), gives

$$(\sum_{i=1}^{\infty} |f_i^{(n)}(x) - b_i(x)|^p)^{\frac{1}{p}} \leq \sqrt[p]{\varepsilon}||x||. \tag{4.2}$$

Therefore $(p\mathfrak{B}, ||.||_{p\mathfrak{B}})$ is a Banach space.
Now, we define a multiplication on the Banach space $p\mathfrak{B}$ with respect to $h^0 = (h_i^0)_{i\in\mathbb{N}}$. Let $0 < A_0 \leq B_0 < \infty$ be p-Riesz basis bounds for $h^0 = (h_i^0)_{i\in\mathbb{N}}$. First, for the p-Bessel sequence $g = (g_i)_{i=1}^{\infty} \subset X^*$, consider the operator $M_{(h_i^0)(g_i)} : X \to X$ defined by $M_{(h_i^0)(g_i)}(x) = \sum_{j=1}^{\infty} g_j(x)h_j^0$. Since $(h_i^0)_{i\in\mathbb{N}} \subset X$ is a q-Bessel sequence for X^*, then by Lemma 2.7, $M_{(g_i)(h_i^0)}$ is well-defined, $M_{(h_i^0)(g_i)} \in B(X)$, and $||M_{(h_i^0)(g_i)}(x)|| \leq B_0||(g_i)_{i=1}^{\infty}||_{p\mathfrak{B}}||x||$.
Define $\cdot_{p\mathfrak{B}} : p\mathfrak{B} \times p\mathfrak{B} \to p\mathfrak{B}$ by $((f_i)_{i=1}^{\infty}, (g_i)_{i=1}^{\infty}) \to (f_i M_{(h_i^0)(g_i)})_{i=1}^{\infty}$. Hence for every $x \in X$, $(f_i M_{(h_i^0)(g_i)}(x))_{i=1}^{\infty} = (\sum_{j=1}^{\infty}(f_i(h_j^0)g_j)(x))_{i=1}^{\infty}$. Note that $\cdot_{p\mathfrak{B}}$ is well-defined, since for every $x \in X$

$$\begin{aligned}(\sum_{i=1}^{\infty}|f_i M_{(h_i^0)(g_i)}(x)|^p)^{\frac{1}{p}} &\leq ||(f_i)_{i=1}^{\infty}||_{p\mathfrak{B}}||M_{(h_i^0)(g_i)}(x)|| \\ &\leq B_0||(f_i)_{i=1}^{\infty}||_{p\mathfrak{B}}||(g_i)_{i=1}^{\infty}||_{p\mathfrak{B}}||x||.\end{aligned}$$

Therefore $(f_i M_{(h_i^0)(g_i)})_{i=1}^{\infty} \in p\mathfrak{B}$. The above relation implies that

$$||((f_i)_{i=1}^{\infty} \cdot_{p\mathfrak{B}} (g_i)_{i=1}^{\infty})||_{p\mathfrak{B}} \leq B_0||(f_i)_i)_{i=1}^{\infty}||_{p\mathfrak{B}}||(g_i)_{i=1}^{\infty}||_{p\mathfrak{B}}.$$

If we consider the equivalent norm $|.|_{p\mathfrak{B}} = B_0||.||_{p\mathfrak{B}}$, then $(p\mathfrak{B}, |.|_{p\mathfrak{B}})$ is a non-commutative unital Banach algebra. Moreover, its unit is $(\widetilde{h_i^0})_{i\in\mathbb{N}}$, where $(\widetilde{h_i^0})_{i\in\mathbb{N}}$ is the dual of $(h_i^0)_{i\in\mathbb{N}}$. □

In the following proposition, we show that $p\mathfrak{B}$ is isometrically isomorphic to $B(X)$.

Proposition 4.2. *Let $h^0 = (h_i^0)_{i\in\mathbb{N}} \subset X$ be a p-Riesz basis for X with bounds $0 < A_0 \leq B_0$ and let $\cdot_{p\mathfrak{B}}$ and $|.|_{p\mathfrak{B}}$ be defined with respect to h^0. Then $(p\mathfrak{B}, |.|_{p\mathfrak{B}})$ is a Banach algebra isometrically isomorphic to $B(X)$.*

Proof. Consider the mapping

$$A_h^0 \; : \; p\mathfrak{B} \to B(X) \tag{4.3}$$

$$(g_i)_{i=1}^{\infty} \to M_{(h_i^0)(g_i)} = \sum_{i=1}^{\infty} g_i(.)h_i^0. \tag{4.4}$$

Clearly, $M_{(h_i^0)(g_i)} \in B(X)$ and A_h^0 is a well-defined bounded linear operator from $p\mathfrak{B}$ to $B(X)$ and $|(g_i)_{i=1}^{\infty}|_{\cdot_{p\mathfrak{B}}} = ||M_{(h_i^0)(g_i)}||$, for every $(g_i)_{i=1}^{\infty} \in p\mathfrak{B}$.
Consider the mapping $B_h^0 : B(X) \to p\mathfrak{B}$ by $B_h^0(T) = (\widetilde{h_i^0}T)_{i=1}^{\infty}$, where $(\widetilde{h_i^0})_{i=1}^{\infty}$ is the dual of $(h_i^0)_{i=1}^{\infty}$. Clearly, $(\widetilde{h_i^0})_{i=1}^{\infty}$ is a p-Bessel sequence for X. For every $x \in X$

$$(\sum_{i=1}^{\infty}|\widetilde{h_i^0}Tx|^p)^{\frac{1}{p}} \leq \frac{1}{A_0}||T(x)|| \leq \frac{1}{A_0}||T||||x||.$$

Therefore $(\widetilde{h_i^0}T)_{i=1}^{\infty} \in P\mathfrak{B}$ and B_h^0 is a well-defined operator. Clearly B_h^0 is the inverse of A_h^0. Moreover, A_h^0 is an algebra isomorphism. □

Remark 5. *Consider the mapping*

$$C_h^0 \; : \; p\mathfrak{B} \to B(X^*) \tag{4.5}$$

$$(f_i)_{i=1}^{\infty} \to N_{(h_i^0)(f_i)}, \tag{4.6}$$

where $N_{(h_i^0)(f_i)}(g)=\sum_{i=1}^{\infty} g(h_i^0)f_i$, *for every* $g\in X^*$. *Clearly,* C_h^0 *is a well-defined linear operator from* $p\mathfrak{B}$ *to* $B(X^*)$. *Let* $(f_i)_{i=1}^{\infty}$ *and* $(g_i)_{i=1}^{\infty}$ *be p-Bessel sequences in* X^* *for* X. *For every* $g\in X^*$ *we have*

$$\begin{aligned} C_h^0((g_i))C_h^0((f_i))(g) &= C_h^0((g_i))(\sum_j g(h_j^0)f_j)\\ &= \sum_i(\sum_j g(h_j^0)f_j)(h_i^0)g_i\\ &= \sum_i\sum_j g(h_j^0)f_j(h_i^0)g_i, \end{aligned}$$

and since $f_i(M_{(h_j^0)(g_j)})=\sum_j f_i(h_j^0)g_j$, *then*

$$C_h^0((f_i(M_{(h_j^0)(g_j)})))=\sum_i g(h_i^0)(\sum_j f_i(h_j^0)g_j)=\sum_i\sum_j g(h_i^0)f_i(h_j^0)g_j.$$

Now for every $x\in X$, *by Holder inequality the corresponding double sequences of numbers are absolutely summable. Therefore*

$$C_h^0((f_i)_{i=1}^{\infty}\cdot_{p\mathfrak{B}}(g_i)_{i=1}^{\infty})=C_h^0((g_i)_{i=1}^{\infty})C_h^0((f_i)_{i=1}^{\infty}).$$

Now define $D_h^0: B(X^*)\to p\mathfrak{B}$ *by* $D_h^0(T)=(\widetilde{Th_i^0})_{i\in\mathbb{N}}$. *Then* D_h^0 *is the inverse of* C_h^0. *So,* C_h^0 *is a bijection between* $p\mathfrak{B}$ *and* $B(X^*)$. □

In the following proposition, by using the isomorphism A_h^0 between $p\mathfrak{B}$ and $B(X)$, we characterize several classes of bounded linear operators on X in terms of their corresponding p-Bessel sequences.

Proposition 4.3. *Let* $(h_i^0)_{i\in\mathbb{N}}\subset X$ *be a fixed p-Riesz basis for* X, *and let* A_h^0 *be defined as* (4.3). *Then the following statements hold:*

(1) *If* $\mathcal{A}$ *is the set of all surjective operators in* $B(X)$, *then*

$$A_{h^0}{}^{-1}(\mathcal{A})=\{(f_i)_{i\in\mathbb{N}}\in p\mathfrak{B}:\ (f_i)_{i\in\mathbb{N}}\ is\ a\ q-Riesz\ sequence\ for\ X^*\}.$$

(2) *If* $\mathcal{A}$ *is the set of all bounded below operators in* $B(X)$, *then*

$$A_{h^0}{}^{-1}(\mathcal{A})=\{(f_i)_{i\in\mathbb{N}}\in p\mathfrak{B}:\ (f_i)_{i\in\mathbb{N}}\ is\ a\ p-frame\ for\ X\}.$$

(3) *If* $\mathcal{A}$ *is the set of all operators with dense range in* $B(X)$, *then*

$$A_{h^0}{}^{-1}(\mathcal{A})=\{(f_i)_{i\in\mathbb{N}}\in p\mathfrak{B}:\ (f_i)_{i\in\mathbb{N}}\ is\ \ell^q-independent\}.$$

(4) *If* $\mathcal{A}$ *is the set of all invertible operators in* $B(X)$, *then*

$$A_{h^0}{}^{-1}(\mathcal{A})=\{(f_i)_{i\in\mathbb{N}}\in p\mathfrak{B}:\ (f_i)_{i\in\mathbb{N}}\ is\ a\ q-Riesz\ basis\ for\ X^*\}.$$

Proof. The proof of (1), (2), (3), and (4) is evident by Theorem 3.3, Theorem 3.2, Lemma 3.5, and Theorem 2.8, respectively. □

In the following proposition, we characterize several classes of bounded linear operators on X^* in terms of their corresponding p-Bessel sequences.

Proposition 4.4. *Let* $(h_i^0)_{i\in\mathbb{N}}\subset X$ *be a p-Riesz basis for* X, *and let* C_h^0 *be defined as* (4.5). *Then the following statements hold:*

(1) *If* $\mathcal{A}$ *is the set of all bounded below operators in* $B(X^*)$, *then*

$$C_{h^0}{}^{-1}(\mathcal{A})=\{(f_i)_{i\in\mathbb{N}}\in P\mathfrak{B}:\ (f_i)_{i\in\mathbb{N}}\ is\ a\ q-Riesz\ sequence\ for\ X^*\}.$$

(2) *If $\mathcal{A}$ is the set of all surjective operators in $B(X^*)$, then*
$${C_{h^0}}^{-1}(\mathcal{A}) = \{(f_i)_{i\in\mathbb{N}} \in P\mathfrak{B} :\ (f_i)_{i\in\mathbb{N}}\ is\ a\ p-frame\ for X\}.$$
(3) *If $\mathcal{A}$ is the set of all injective operators in $B(X^*)$, then*
$${C_{h^0}}^{-1}(\mathcal{A}) = \{(f_i)_{i\in\mathbb{N}} \in p\mathfrak{B} :\ (f_i)_{i\in\mathbb{N}}\ is\ \ell^q - independent\}.$$
(4) *If $\mathcal{A}$ is the set of all invertible operators in $B(X^*)$, then*
$${C_{h^0}}^{-1}(\mathcal{A}) = \{(f_i)_{i\in\mathbb{N}} \in P\mathfrak{B} :\ (f_i)_{i\in\mathbb{N}}\ is\ a\ q-Riesz\ basis\ for\ X^*\}.$$

Proof. Since $M^*_{(h_i^0)(f_i)} = N_{(h_i^0)(f_i)}$, for every $(f_i)_{i\in\mathbb{N}} \in P\mathfrak{B}$, by Proposition 4.3, Theorem 2.9, and Remark 2, the proof is evident. □

Theorem 4.5. *Let $p\mathcal{F}$, $q\mathcal{R}$, and $q\mathcal{R}S$ be the set of all p-frames, q-Riesz bases, and q-Riesz sequences in $p\mathcal{B}$, respectively. Then the sets $p\mathcal{F}$, $q\mathcal{R}$, and $q\mathcal{R}S$ are open subsets of $p\mathfrak{B}$.*

Proof. Let $(h_i^0)_{i\in\mathbb{N}} \subset X$ be a fixed p-Riesz basis for X.
(1) Let $GL(X)$ be the set of all invertible operators in $B(X)$, ε be the set of all surjective operators in $B(X)$, and δ be the set of all bounded below operators in $B(X)$. The sets $GL(X)$, ε and δ are open subsets of $B(X)$, see [1, Lemma 2.4, Theorem 2.10 and Corollary 2.11]. The operator $A_{h^0} : p\mathfrak{B} \to B(X)$ defined by $A_{h^0}((f_i)_{i\in\mathbb{N}}) = M_{(h_i^0)(f_i)}$ is continuous, by Proposition 4.3 , $q\mathcal{R} = {A_{h^0}}^{-1}(GL(X))$, $q\mathcal{R}S = {A_{h^0}}^{-1}(\varepsilon)$ and $p\mathcal{F} = {A_{h^0}}^{-1}(\delta)$. Therefore the sets $p\mathcal{F}$, $q\mathcal{R}$, and $q\mathcal{R}S$ are open subsets of $p\mathfrak{B}$. □

Proposition 4.6. *Let $(h_i^0)_{i\in\mathbb{N}} \subset X$ be a p-Riesz basis for X, and let $\cdot_{p\mathfrak{B}}$ be defined with respect to $(h_i^0)_{i\in\mathbb{N}}$. Let $(f_i)_{i\in\mathbb{N}} \subset X^*$ and $(g_i)_{i\in\mathbb{N}} \subset X^*$ be p-Bessel sequences for X. Then*

(1) *If $(g_i)_{i\in\mathbb{N}}$ is a q-Riesz basis for X^*, then $(g_i)_{i\in\mathbb{N}}\cdot_{p\mathfrak{B}}(f_i)_{i\in\mathbb{N}}$ is a q-Riesz basis for X^* if and only if $(f_i)_{i\in\mathbb{N}}$ is a q-Riesz basis for X^*.*
(2) *If $(g_i)_{i\in\mathbb{N}}$ is a p-frame for X, then $(g_i)_{i\in\mathbb{N}}\cdot_{p\mathfrak{B}}(f_i)_{i\in\mathbb{N}}$ is a p-frame for X if and only if $(f_i)_{i\in\mathbb{N}}$ is a p-frame for X.*

Proof. By the definition $M_{(h_i^0)((g_i)\cdot_{p\mathfrak{B}}(f_i))} = M_{(h_i^0)(g_i)} \circ M_{(h_i^0)(f_i)}$, therefore $N_{(h_i^0)((g_i)\cdot_{pB}(f_i))} = N_{(h_i^0)(f_i)} \circ N_{(h_i^0)(g_i)}$. Easily, we can see that if $N_{(h_i^0)(g_i)}$ is invertible (surjective), then $N_{(h_i^0)((g_i)\cdot_{p\mathfrak{B}}(f_i))}$ is invertible (surjective) if and only if $N_{(h_i^0)(f_i)}$ is invertible (surjective). Now, by Proposition 4.4, the claims are obvious. □

Theorem 4.7. *Let $(h_i^0)_{i\in\mathbb{N}} \subset X$ be a fixed p-Riesz basis for X. Then the following statements hold:*

(1) *$(f_i)_{i\in\mathbb{N}} \in p\mathfrak{B}$ is invertible with respect to $(h_i^0)_{i\in\mathbb{N}}$ if and only if $(f_i)_{i\in\mathbb{N}}$ is a q-Riesz basis for X^*.*
(2) *The set of all q-Riesz bases for X^* is a topological group.*

Proof. (1) Suppose that $(g_i)_{i\in\mathbb{N}} \in p\mathfrak{B}$ is the inverse of $(f_i)_{i\in\mathbb{N}}$. Then $(f_i)_{i\in\mathbb{N}}\cdot_{p\mathfrak{B}}$ $(g_i)_{i\in\mathbb{N}} = (\widetilde{h_i^0})_{i\in\mathbb{N}}$ if and only if $M_{(h_i^0)((f_i)\cdot_{p\mathfrak{B}}(g_i))} = M_{(h_i^0)(\widetilde{h_i^0})} = I$. But $M_{(h_i^0)((f_i)\cdot_{p\mathfrak{B}}(g_i))} = M_{(h_i^0)(f_i)} \circ M_{(h_i^0)(g_i)}$. So $M_{(h_i^0)(f_i)} \circ M_{(h_i^0)(g_i)} = I$. By the same argument $M_{(h_i^0)((g_i)\cdot_{p\mathfrak{B}}(f_i))} =$

$M_{(h_i^0)(g_i)} \circ M_{(h_i^0)(f_i)} = I$. Therefore $(f_i)_{i\in\mathbb{N}}$ is invertible with respect to $(h_i^0)_{i\in\mathbb{N}}$ if and only if $M_{(h_i^0)(f_i)}$ is invertible in $B(X)$. Hence, by Proposition 4.3 we have the result.
(2) Suppose that $GL(X)$ is the set of all invertible operators in $B(X)$. Since $A_h^0 : p\mathfrak{B} \to B(X)$ defined by $A_h^0((f_i)_{i\in\mathbb{N}}) = M_{(h_i^0)(f_i)}$ is an algebra isomorphism and $(GL(X), \|.\|_{OP})$ is a topological group, then $A_{h^0}^{-1}(GL(X)) = q\mathcal{R}$ is a topological group in $p\mathfrak{B}$. □

First we recall the following result.

Theorem 4.8. [21, Theorem 15.9.3] *Let X and Y be Banach spaces, suppose that Y has a Schauder basis and let P_n denote the natural projection associated with the basis. Then for any compact $A \in L(X,Y)$ the finite rank operators $P_nA \to A$.*

Proposition 4.9. *Let $f = (f_i)_{i\in\mathbb{N}} \subset X^*$ be a p-Bessel sequence for X and $F_N = (f_i)_{i>N}$. Let $h^0 = (h_i^0)_{i\in\mathbb{N}} \subset X$ be a p-Riesz basis for X and A_h^0 be defined as* (4.3)*. Then $M_{(h_i^0)(f_i)}$ is a compact operator if and only if* $\lim_{N\to\infty} ||A_{h^0}(F_N)|| = 0$.

Proof. Suppose that $M_{(h_i^0)(f_i)}$ is a compact operator. Since $h^0 = (h_i^0)_{i\in\mathbb{N}}$ is a p-Riesz basis for X, then it is a Schauder basis for X. Suppose that $P_N = P_{span\{h_1,\ldots h_N\}}$. By the above theorem, $\lim_{N\to\infty} ||P_N M_{(h_i^0)(f_i)} - M_{(h_i^0)(f_i)}|| = 0$. But for every $x \in X$,

$$M_{(h_i^0)(f_i)}(x) = \sum_{i\in\mathbb{N}} \widetilde{h_i^0}(M_{(h_i^0)(f_i)}x)h_i^0 = \sum_{i\in\mathbb{N}} (M^*_{(h_i^0)(f_i)}\widetilde{h_i^0})(x)h_i^0.$$

Therefore

$$||P_N M_{(h_i^0)(f_i)}(x) - M_{(h_i^0)(f_i)}(x)|| = ||\sum_{i>N} (M^*_{(h_i^0)(f_i)}\widetilde{h_i^0})(x)h_i^0||.$$

Suppose that C is the lower p-Riesz bound for $h^0 = (h_i^0)_{i\in\mathbb{N}}$, then for every $x \in X$

$$\begin{aligned}(\sum_{i>N} |M^*_{(h_i^0)(f_i)}(\widetilde{h_i^0})(x)|^p)^{\frac{1}{p}} &\leq \frac{1}{C}||(P_N M_{(h_i^0)(f_i)} - M_{(h_i^0)(f_i)})(x)|| \\ &\leq \frac{1}{C}||P_N M_{(h_i^0)(f_i)} - M_{(h_i^0)(f_i)}||||x||.\end{aligned}$$

Thus $(M^*_{(h_i^0)(f_i)}(\widetilde{h_i^0}))_{i>N} = (f_i)_{i>N} \in p\mathfrak{B}$ and

$$\lim_{N\to\infty} ||A_{h^0}(F_N)|| = \lim_{N\to\infty} ||P_N M_{(h_i^0)(f_i)} - M_{(h_i^0)(f_i)}|| = 0.$$

Conversely, let $f = (f_i)_{i\in\mathbb{N}} \in p\mathfrak{B}$ such that $\lim_{N\to\infty} ||A_{h^0}(F_N)|| = 0$. We show that $M_{(h_i^0)(f_i)}$ is a compact operator. For every $x \in X$ we have

$$||(P_N M_{(h_i^0)(f_i)} - M_{(h_i^0)(f_i)})(x)|| = ||A_{h^0}(F_N)(x)|| \leq ||A_{h^0}(F_N)||||x||.$$

Since $\lim_{N\to\infty} ||A_{h^0}(F_N)|| = 0$, then $\lim_{N\to\infty} ||P_N M_{(h_i^0)(f_i)} - M_{(h_i^0)(f_i)}|| = 0$. Therefore $M_{(h_i^0)(f_i)}$ is a compact operator. □

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