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A NEW TYPE CONTINUITY FOR REAL FUNCTIONS

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ABSTRACT. A real valued function f defined on a subset of $\mathbb{R}$ is δ^2-ward continuous if $\lim_{n\to\infty} \Delta^3 f(x_n) = 0$ whenever $\lim_{n\to\infty} \Delta^3 x_n = 0$, where $\Delta^3 z_n = z_{n+3} - 3z_{n+2} + 3z_{n+1} - z_n$ for each positive integer n, $\mathbb{R}$ denotes the set of real numbers, and a subset E of $\mathbb{R}$ is δ^2-ward compact if any sequence of points in E has a δ^2-quasi Cauchy subsequence where a sequence (x_n) is δ^2-quasi Cauchy if $\lim_{n\to\infty} \Delta^3 z_n$=0. It turns out that the uniform limit process preserves this kind of continuity, and the set of δ^2-ward continuous functions is a closed subset of the set of continuous functions.

1. INTRODUCTION

The concept of continuity and any concept involving continuity play a very important role not only in pure mathematics but also in other branches of sciences involving mathematics especially in computer science, information theory, economics, and biological science.

Throughout this paper, $\mathbb{N}$ and $\mathbb{R}$ will denote the set of all positive integers and the set of real numbers, respectively. We will use letters $\mathbf{x}, \mathbf{y}, \mathbf{z}, ...$ for sequences $\mathbf{x} = (x_n), \mathbf{y} = (y_n), \mathbf{z} = (z_n)$, ... of terms in $\mathbb{R}$. c, Δ, and Δ^2 will denote the set of all convergent sequences, the set of all quasi Cauchy sequences, the set of all δ-quasi Cauchy sequences of points in $\mathbb{R}$ where a sequence $\mathbf{x} = (x_n)$ is called quasi Cauchy in [8], and is called forward convergent to 0 in [11] if $\lim_{n\to\infty} \Delta x_n = 0$; and is called δ-quasi Cauchy if $\lim_{n\to\infty} \Delta^2 x_n = 0$ in [12], where $\Delta z_n = z_{n+1} - z_n$ and $\Delta^2 x_n = x_{n+2} - 2x_{n+1} + x_n$ for each $n \in \mathbb{N}$.

Fast ([25]) introduced the definition of statistical convergence. A sequence (x_k) is statistically convergent to real number ℓ if $\lim_{n\to\infty} \frac{1}{n}|\{k \leq n : |x_k - \ell| \geq \epsilon\}| = 0$ for every $\epsilon > 0$ (see also [33] [26], [41], [44] and [40]). The notion of N_θ convergence was introduced by Freedman, Sember, and M. Raphael ([43]) in the sense that a sequence (x_k) is N_θ convergent to an $\ell \in \mathbb{R}$ if $\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} |x_k - \ell| = 0$, where $I_r = (k_{r-1}, k_r]$, and $k_0 \neq 0$, $h_r : k_r - k_{r-1} \to \infty$ as $r \to \infty$ and $\theta = (k_r)$ is an increasing sequence of positive integers. Later on, the idea of lacunary statistically convergence was given in [27] based on the idea of N_θ-convergence in the sense that a sequence (x_k) is called lacunary statistically convergent to an element ℓ of $\mathbb{R}$ if $\lim_{r\to\infty} \frac{1}{h_r}|\{k \in I_r : |x_k - l| \geq \epsilon\}| = 0$ for every $\epsilon > 0$ (see also [28]).

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Following the idea given in a 1946 American Mathematical Monthly problem [6], a number of authors Posner [31], Iwinski [29], Srinivasan [35], Antoni [3], Antoni and Salat [4], Spigel and Krupnik [34] have studied A-continuity defined by a regular summability matrix A. Some authors, Ozturk [30], Savas and Das [32], Borsik and Salat [5] have studied A-continuity for methods of almost convergence or for related methods.

Recently, Connor and Grosse-Erdman [7] have given sequential definitions of continuity for real functions calling G-continuity instead of A-continuity and their results cover the earlier works related to A-continuity where a method of sequential convergence, or briefly a method, is a linear function G defined on a linear subspace of s, denoted by c_G, into $\mathbb{R}$. A sequence $\mathbf{x} = (x_n)$ is said to be G-convergent to ℓ if $\mathbf{x} \in c_G$ and $G(\mathbf{x}) = \ell$. In particular, lim denotes the limit function $\lim \mathbf{x} = \lim_{n\to\infty} x_n$ on the linear space c, $st - \lim$ denotes the statistical limit function $st - \lim \mathbf{x} = st - \lim_{n\to\infty} x_n$ on the linear space $st(\mathbb{R})$ and $st_\theta - \lim$ denotes the lacunary statistical limit function $st_\theta - \lim \mathbf{x} = st_\theta - \lim_{n\to\infty} x_n$ on the linear space $st_\theta(\mathbb{R})$. A real function f is called G-continuous at a point u provided that whenever a sequence $\mathbf{x} = (x_n)$ of terms in the domain of f is G-convergent to u, then the sequence $f(\mathbf{x}) = (f(x_n))$ is G-convergent to $f(u)$. A method G is called regular if every convergent sequence $\mathbf{x} = (x_n)$ is G-convergent with $G(\mathbf{x}) = \lim \mathbf{x}$. A method is called subsequential if whenever $\mathbf{x}$ is G-convergent with $G(\mathbf{x}) = \ell$, then there is a subsequence (x_{n_k}) of $\mathbf{x}$ with $lim_k\, x_{n_k} = \ell$.

A concept of ward continuity, and a concept of ward compactness have been introduced in [11] in the sense that a real function is called ward continuous if it preserves quasi Cauchy sequences, i.e. $lim_{n\to\infty} \Delta f(x_n) = 0$ whenever $lim_{n\to\infty} \Delta x_n = 0$, and a subset E of $\mathbb{R}$ is called ward compact if any sequence of points in E has quasi Cauchy subsequence, i.e. whenever $\mathbf{x} = (x_n)$ is a sequence of points in E there is a subsequence $\mathbf{z} = (z_k) = (x_{n_k})$ of $\mathbf{x}$ with $lim_{k\to\infty} \Delta z_k = 0$. The concept of δ-ward continuity, and the concept of δ-ward compactness have been introduced in [12] in the sense that a real function is called δ-ward continuous if $lim_{n\to\infty} \Delta^2 f(x_n) = 0$ whenever $lim_{n\to\infty} \Delta^2 x_n = 0$, and a subset E of $\mathbb{R}$ is called δ-ward compact if whenever $\mathbf{x} = (x_n)$ is a sequence of points in E there is a subsequence $\mathbf{z} = (z_k) = (x_{n_k})$ of $\mathbf{x}$ with $lim_{k\to\infty} \Delta^2 z_k = 0$.

A real function f defined on $\mathbb{R}$ is continuous if and only if it preserves Cauchy sequences, i.e., $(f(x_n))$ is a Cauchy sequence whenever (x_n) is. Using the idea of continuity of a real function in terms of sequences in the sense that a function preserves a certain kind of sequences in the above manner, many kinds of continuities were introduced and investigated, not all but some of them we recall in the following: slowly oscillating continuity ([10], [53]), quasi-slowly oscillating continuity ([14], [24]), ward continuity ([11]), δ-ward continuity ([12]), statistical ward continuity ([38]), λ-statistical ward continuity ([22]), ρ-statistical ward continuity ([13]), lacunary statistical ward continuity ([18], strongly lacunary ward continuity ([36]) and Abel ward continuity ([17]) which enabled some authors to obtain conditions on the domain of a function to be uniformly continuous (see [53, Theorem 6],[8, Theorem 1 and Theorem 2],[24, Theorem 2.3], [22, Theorem 5].

The purpose of this paper is to introduce a concept of δ^2-ward continuity of a real function and a concept of δ^2-ward compactness of a subset of $\mathbb{R}$ which cannot be given by means of any G and prove interesting theorems.

2. δ^2-ward continuity

Now in this section, we first introduce a definition of a δ^2-quasi Cauchy sequence in the following.

Definition 2.1. A sequence $\mathbf{x} = (x_k)$ is called δ^2-ward convergent to a number ℓ if $lim_{k\to\infty}\Delta^3 x_k = \ell$ where $\Delta^3 x_k = x_{k+3} - 3x_{k+2} + 3x_{k+1} - x_k$.

For the special case $\ell = 0$, we prefer calling $\mathbf{x}$ is δ^2-quasi Cauchy instead of calling $\mathbf{x}$ is δ^2-ward convergent to 0. We note that any quasi Cauchy sequence is also δ^2-quasi Cauchy, but the converse is not always true as it can be seen by considering the sequence $(x_n) = (n)$. We also note that any δ-quasi Cauchy sequence is also δ^2-quasi Cauchy, but the converse is not always true as it can be seen by considering the sequence $(x_n) = (n^2)$.

Now we give the definition of δ^2-ward compactness of a subset of $\mathbb{R}$.

Definition 2.2. A subset E of $\mathbb{R}$ is called δ^2-ward compact if any sequence of points in E has a δ^2-quasi Cauchy subsequence, i.e. whenever $\mathbf{x} = (x_n)$ is a sequence of points in E there is a subsequence $\mathbf{z} = (z_k) = (x_{n_k})$ of $\mathbf{x}$ with $lim_{k\to\infty}\Delta^3 z_k = 0$.

We see that this definition of δ^2-ward compactness can not be obtained by any G-sequential compactness, i.e. by any summability matrix A, even by the summability matrix $A = (a_{nk})$ defined by $a_{nk} = -1$ if $k = n$, $a_{kn} = 3$ if $k = n+1$, $a_{kn} = -3$ if $k = n+2$, and $a_{kn} = 1$ if $k = n+3$

$$(*) \qquad G(x) = \lim Ax = \lim_{k\to\infty} \sum_{n=1}^{\infty} a_{kn} x_n = \Delta^3 x_k$$

(see [9] for the definition of G-sequential compactness). In this case, despite that G-sequential compact subsets of $\mathbb{R}$ should include the singleton set $\{0\}$, δ^2-ward compact subsets of $\mathbb{R}$ do not have to include the singleton $\{0\}$.

Firstly, we note that any finite subset of $\mathbb{R}$ is δ^2-ward compact, the union of two δ^2-ward compact subsets of $\mathbb{R}$ is δ^2-ward compact and the intersection of any δ^2-ward compact subsets of $\mathbb{R}$ is δ^2-ward compact. Any subset of a δ^2-ward compact set is δ^2-ward compact and any bounded subset of $\mathbb{R}$ is δ^2-ward compact. We note that any slowly oscillating compact subset of $\mathbb{R}$ is δ^2-ward compact (see [10] for the definition of slowly oscillating compactness), and any quasi-slowly oscillating compact subset of $\mathbb{R}$ is δ^2-ward compact where a subset E of $\mathbb{R}$ is called quasi-slowly oscillating compact (see also [24]) if whenever $\mathbf{x} = (x_n)$ is a sequence of points in E, there is a quasi-slowly oscillating subsequence $\mathbf{y} = (x_{n_k})$ of $\mathbf{x} = (x_n)$. Any compact subset of $\mathbb{R}$ is also δ^2-ward compact and the converse is not always true. For example, any open interval is δ^2-ward compact, which is not compact. On the other hand, the set $\mathbb{N}$ is not δ^2-ward compact.

Theorem 2.1. A subset of $\mathbb{R}$ is bounded if and only if it is δ^2-ward compact.

Proof. Let E be any bounded subset of $\mathbb{R}$ and (x_n) be any sequence of points in E. (x_n) is also a sequence of points in $\overline{E}$ where $\overline{E}$ denotes the closure of E. As $\overline{E}$ is sequentially compact there is a quasi Cauchy subsequence (x_{n_k}) of (x_n). This subsequence is δ^2-quasi Cauchy since any quasi Cauchy sequence is δ^2-quasi Cauchy. To prove the converse, suppose that E is unbounded. If it is unbounded above, then choose positive real numbers x_1, x_2 and x_3 in E. Since E is unbounded above, we can find an x_4 in E such that $1 + x_1 + 3x_3 - 3x_2 < x_4$. Then we can find an x_5 in E such that $2 + x_2 + 3x_4 - 3x_3 < x_5$. We can inductively construct a sequence (x_k) of points in E such that $k + x_k + 3x_{k+2} - 3x_{k+1} < x_{k+3}$ for each $k \in \mathbb{N}$, and the sequences $(x_{k+3} - 3x_{k+2})$, $(3x_{k+1} - x_k)$ are strictly increasing. Then $k < x_{k+3} - 3x_{k+2} + 3x_{k+1} - x_k$ for each $k \in \mathbb{N}$. Hence it is easy to see that the sequence (x_k) has no δ^2-quasi Cauchy subsequence, which completes the proof. □

The concept of δ-ward continuity suggests us to give a new type continuity, namely, δ^2-ward continuity, analogous to ward continuity and δ-ward continuity.

Definition 2.3. A function f is called δ^2-ward continuous on E if it preserves δ^2-quasi Cauchy sequences, i.e. the sequence $f(\mathbf{x}) = (f(x_n))$ is δ^2-quasi Cauchy whenever $\mathbf{x} = (x_n)$ is a sequence of terms in E which is a δ^2-quasi Cauchy sequence.

We note that this definition of continuity can not be obtained by any G-continuity, i.e. by any summability matrix A, even by the summability matrix $A = (a_{nk})$ defined by $(*)$ however for this special summability matrix A if A-continuity of f at the point 0 implies δ^2-ward continuity of f, then $f(0) = 0$; and if δ^2-forward continuity of f implies A-continuity of f at the point 0, then $f(0) = 0$ (see [9] and [37]). We also note that the sum of two δ^2-ward continuous functions is δ^2-forward continuous and the composite of two δ^2-ward continuous functions is δ^2-ward continuous but the product of two δ^2-ward continuous functions need not be δ^2-ward continuous as it can be seen by considering product of the δ^2-ward continuous function $f(x) = x$ with itself three times, so function $g(x) = x^3$ is not δ^2-ward continuous. Now we give the following theorem.

Theorem 2.2. If f is δ^2-ward continuous on a subset E of $\mathbb{R}$, then it is δ-ward continuous on E.

Proof. Let (x_n) be any sequence with $\lim_{k\to\infty} \Delta^2 x_k = 0$. Then the sequence

$$(x_1, x_1, x_2, x_2, \cdots, x_n, x_n, \cdots)$$

is also δ^2-quasi Cauchy hence, by the hypothesis, the sequence

$$(f(x_1), f(x_1), f(x_2), f(x_2), \cdots, f(x_n), f(x_n), \cdots)$$

is δ^2-quasi Cauchy. It follows from this that

$$(f(x_1), f(x_2), \cdots, f(x_n), \cdots)$$

is δ-quasi Cauchy, from which follows the δ-ward continuity of f. □

Theorem 2.3. If f is δ^2-ward continuous on a subset E of $\mathbb{R}$, then it is ward continuous on E.

Proof. Although the proof follows from the preceding theorem and [12, Theorem 1], we give a direct proof for completeness. Let (x_n) be any sequence with $\lim_{k\to\infty} \Delta x_k = 0$. Then the sequence

$$(x_1, x_1, x_1, x_2, x_2, x_2, \cdots, x_n, x_n, x_n \cdots)$$

is δ-quasi Cauchy hence, by the hypothesis, the sequence

$$(f(x_1), f(x_1), f(x_1), f(x_2), f(x_2), f(x_2), \cdots, f(x_n), f(x_n), f(x_n), \cdots)$$

is δ^2-δ-quasi Cauchy. It follows from this that

$$(f(x_1), f(x_1), f(x_2), f(x_2), \cdots, f(x_n), f(x_n), \cdots)$$

is δ-quasi Cauchy, from which follows the ward continuity of f. □

The converse of this theorem is not always true for the ward continuous function $f(x) = sinx$ is an example which is not δ^2-ward continuous. This example also explains that not all uniformly continuous functions are δ^2-ward continuous.

Theorem 2.4. If f is δ^2-ward continuous on a subset E of $\mathbb{R}$, then it is continuous on E in the ordinary sense.

Proof. Although the proof follows from the preceding theorem and [11, Theorem 1], we give a direct proof for completeness. Let (x_n) be a convergent sequence with $\lim_{k\to\infty} x_k = \ell$. Then the sequence

$$(x_1, \ell, \ell, x_2, \ell, \ell, \cdots, x_n, \ell, \ell \cdots)$$

is δ^2-quasi Cauchy hence, by the hypothesis, the sequence

$$(f(x_1), f(\ell), f(\ell), f(x_2), f(\ell), f(\ell), \cdots, f(x_n), f\ell), f(\ell), \cdots)$$

is δ^2-δ-quasi Cauchy. It follows from this that

$$(f(x_1), f(x_2), \cdots, f(x_n), \cdots)$$

is convergent to ℓ. This completes the proof. □

The converse is not always true for the function $f(x) = x^3$ is an example since the sequence $x_n = (n)$ is δ^2-quasi Cauchy while $(f(n)) = (n^3)$ is not δ^2-quasi Cauchy. Now we have the following result.

Corollary 2.5. If f is δ^2-ward continuous, then it is statistically continuous.

We state much more general case in the following.

Corollary 2.6. If f is δ^2-ward continuous, then it is G-continuous for any regular subsequential method G.

The preceding corollary ensures that δ^2-ward continuity implies not only ordinary continuity, and statistical continuity, but also either of the following continuities; λ-statistical continuity ([45], [22]), ρ-statistical continuity ([13]), lacunary statistical continuity ([18]), strongly lacunary continuity ([36]) and I-sequential continuity for any non trivial admissible ideal I of $\mathbb{N}$ ([19], [16], [1]).

Theorem 2.7. δ^2-ward continuous image of any δ^1-ward compact subset of $\mathbb{R}$ is δ^2-ward compact.

Proof. Let f be a δ^2-ward continuous function and E be a δ^2-ward compact subset of $\mathbb{R}$. Take any sequence $y = (y_n)$ of terms in $f(E)$. Write $y_n = f(x_n)$ where $x_n \in E$ for each $n \in \mathbb{N}$. δ^2-ward compactness of E implies that there is a subsequence $\mathbf{z} = (z_k) = (x_{n_k})$ of $\mathbf{x}$ with $\lim_{k\to\infty} \Delta^3 z_k = 0$. Since f is δ^2-ward continuous, $(t_k) = f(\mathbf{z}) = (f(z_k))$ is δ^2-quasi Cauchy. Thus (t_k) is a subsequence of the sequence $f(\mathbf{x})$ with $\lim_{k\to\infty} \Delta^3 t_k = 0$. This completes the proof of the theorem. □

Corollary 2.8. δ^2-ward continuous image of any bounded subset of $\mathbb{R}$ is bounded.

The proof of this theorem follows from the preceding theorem.

It is well-known that any continuous function on a compact subset E of $\mathbb{R}$ is uniformly continuous on E. It is also true for a regular subsequential method G that any δ^2-ward continuous function on a G-sequentially compact subset E of $\mathbb{R}$ is uniformly continuous on E (see [9]). For δ^2-ward continuous functions we have the following.

Theorem 2.9. If f is δ^2-ward continuous on a δ^2-ward compact subset E of $\mathbb{R}$, then it is uniformly continuous on E.

Proof. The proof of this theorem follows from Theorem 2.3, and [11, Theorem 7], so is omitted. □

Corollary 2.10. If f is δ^2-ward continuous on a bounded subset E of $\mathbb{R}$, then it is uniformly continuous on E.

Proof. The proof of this theorem follows from Theorem 2.3, and [11, Theorem 7], so is omitted. □

It is known that the uniform limit of a sequence of continuous functions is continuous. This is also true in case of δ^2-ward continuity.

Theorem 2.11. If (f_n) is a sequence of δ^2-ward continuous functions defined on a subset E of $\mathbb{R}$ and (f_n) is uniformly convergent to a function f, then f is δ^2-forward continuous on E.

Proof. Let $\mathbf{x} = (x_n)$ be any δ^2-quasi Cauchy sequence of points in E and $\epsilon > 0$. Then there exists a positive integer $\mathbb{N}$ such that $|f_n(x) - f(x)| < \frac{\epsilon}{9}$ for all $x \in E$ whenever $n \geq \mathbb{N}$. As f_N is δ^2-ward continuous, there exists a positive integer N_1, depending on ϵ and greater than N such that

$$|f_N(x_{n+3}) - 3f_N(x_{n+2}) + 3f_N(x_{n+1}) - f_N(x_n)| < \frac{\epsilon}{9}$$

for $n \geq N_1$. Now for $n \geq N_1$ we have

$$\begin{aligned} |f(x_{n+3}) - 3f(x_{n+2}) + 3f(x_{n+1}) - 3fx_n| &= |f(x_{n+3}) - f_N(x_{n+3})| + \\ 3|f_N(x_{n+2}) - f(x_{n+2})| + 3|f_N(x_{n+1}) - f(x_{n+1})| &+ |f_N(x_n) - f(x_n)| \\ +|f_N(x_{n+3}) - 3f_N(x_{n+2}) &+ 3f_N(x_{n+1}) - f_N(x_n)| \\ &\leq \frac{\epsilon}{9} + \frac{3\epsilon}{9} + \frac{3\epsilon}{9} + \frac{2\epsilon}{9} < \epsilon. \end{aligned}$$

This completes the proof of the theorem. □

Theorem 2.12. The set of all δ^2-ward continuous functions on E is a closed subset of the set of all continuous functions on E, i.e. $\overline{TFC(E)} = TFC(E)$ where $TFC(E)$ is the set of all continuous functions on E, $\overline{TFC(E)}$ denotes the set of all cluster points of $TFC(E)$ and $E$is a bounded subset of $\mathbb{R}$.

Proof. Let us denote the set of all δ^2-ward continuous functions on E by $TFC(E)$ and f be any element in the closure of $TFC(E)$. Then there exists a sequence of points in $TFC(E)$ such that $lim_{k\to\infty} f_k = f$. To show that f is δ^2-forward continuous take any δ^2-quasi Cauchy sequence (x_n). Let $\epsilon > 0$. Since (f_k) converges to f, there exists an N such that for all $x \in E$ and for all $n \in \mathbb{N}$, $|f(x) - f_n(x)| < \frac{\epsilon}{9}$. As f_N is δ^2-ward continuous, there is an N_1, greater than N, such that for all $n \geq N_1$,

$$|f_N(x_{n+3}) - 3f_N(x_{n+2}) + 3f_N(x_{n+1}) - f_N(x_n)| < \frac{\epsilon}{9}.$$

Hence for all $n \geq N_1$,

$$|f(x_{n+3}) - 3f(x_{n+2}) + 3f(x_{n+1}) - f(x_n)| \leq \frac{\epsilon}{9} + \frac{3\epsilon}{9} + \frac{3\epsilon}{9} + \frac{2\epsilon}{9} < \epsilon.$$

This completes the proof of the theorem. □

Corollary 2.13. The set of all δ^2-ward continuous functions on a subset E of $\mathbb{R}$ is a complete subspace of the space of all continuous functions on E.

Proof. The proof follows from the preceding theorem. □

3. Conclusion

In this paper, the concept of a δ^2-quasi-Cauchy sequence is introduced and investigated. In this investigation, we have obtained theorems related to δ^2-ward continuity, and some other kinds of continuities. One may expect this investigation to be a useful tool in the field of analysis in modeling various problems occurring in many areas of science, dynamical systems, computer science, information theory, and biological science.

For a further study, we suggest to investigate δ^2-quasi-Cauchy sequences of fuzzy points or soft points (see [15], for the definitions and related concepts in fuzzy setting, and see [2] related concepts in soft setting). We also suggest to investigate δ^2-quasi-Cauchy double sequences (see for example [47], [50], [46], [51], [49], and [20] for the definitions and related concepts in the double sequences case). For another further study, we suggest to investigate δ^2-quasi-Cauchy sequences in a abstract metric space (see [23], [48], [21], and [52]). Yet another further study, our suggestion is to investigate the theory in 2-normed spaces (see [39], [42] for the related concepts).

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