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BEST PROXIMITY POINTS IN COMPLETE METRIC SPACES WITH (P)-PROPERTY VIA C-CLASS FUNCTIONS

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Abstract. In this paper, using the concept of a C-class function, three types of generalized almost $(f, \psi, \varphi, \theta)$-contractions are defined. By the (P)-property, existence and uniqueness of some best proximity points are stated and proved in this new setting. The results presented herein are a natural continuation of those of Shatanawi and Pitea [Filomat 29(1), 2015, 63-74].

1. Introduction

In this paper, we consider a problem of global optimization in the context of a complete metric space with (P)-property. More accurately, it is the problem of finding the minimum distance between two subsets of a metric space. For this purpose, we utilize our generalized almost contractive non-self-maps. In fact, non-self-maps have been utilized for the said purposes under a category of problems which has been termed proximity point problems. This category of problems has been developed so far in various settings: reflexive Banach spaces in [2] by Al-Thafai and Shahzad, partially ordered metric spaces in [10] by Choudhury *et al.*, uniformly convex Banach spaces in [12] by Eldered and Veeramani, hyperconvex metric spaces and in Hilbert spaces in [16] by Kirk *et al.*, Menger probabilistic metric spaces in [14] by Jamali and Vaezpour. The notion of proximal pointwise contraction and results regarding the existence of a best proximity point on a pair of weakly compact convex subset of a Banach space are obtained by Eldered and Veeramani in [11]. In [15], the notion of cyclic orbital Meir-Keeler contraction is studied by Karpagam and Agrawal, being given sufficient conditions for the existence of best proximity points of such a map. In [22], Sankar Raj and Veeramani use a convergence theorem in order to prove the existence of a best proximity point, without the use of Zorn's lemma. In [24], Vetro refers to the class of p-cyclic φ-contractions, wider than that of p-cyclic contraction mappings and presents convergence and existence results of best proximity points for mappings from this class are stated and proved. In [23], Sankar Raj stated a fixed point theorem for weakly contractive nonselfmappings based on the notion of (P)-property. For some interesting examples of pairs having the (P)-property, we address the reader to: Akbar and Gabeleh [1], Sankar Raj [23], Samet [21].

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We are going to use generalizations of almost contractions: Altun and Acar [4], Miandaragh *et al.* [17, 18], Olatinwo and Postolache [19], Păcurar [20]. We introduce three types of generalized almost $(f, \psi, \varphi, \theta)$-contractions, with the help of a C-class function. Also, we utilize our notions to state and prove some best proximity point theorems. Our results extend and improve the above existing results in literature.

2. Preliminaries

To introduce our new results, it is fundamental to recall the definition of a best proximity point of a nonselfmapping T and the notion of (weak) (P)-property.

Let A and B be nonempty subsets of a metric space. In the sequel, the following two sets associated to A and B will be of paramount importance.

$$A_0 = \{a \in A : d(a, b) = d(A, B), \text{ for some } b \in B\},$$
$$B_0 = \{b \in B : d(a, b) = d(A, B), \text{ for some } a \in A\}.$$

As usual,

$$d(A, B) := \inf\{d(a, b) : a \in A, b \in B\}.$$

Definition 2.1 ([21])**.** Let A and B be two nonempty subsets of a metric space (X, d). An element $u \in A$ is said to be a best proximity point of the nonselfmapping $T\colon A \to B$ if it satisfies the condition

$$d(u, Tu) = d(A, B).$$

A tool of significant importance in the development of the best proximity point results is the notion of pair endowed with the (weak) (P)-property.

Definition 2.2 ([33])**.** Let (A, B) be a pair of nonempty subsets of a metric space (X, d) with $A_0 \neq \emptyset$. Then, the pair (A, B) is said to have the weak (P)-property if, for each x_1, $x_2 \in A$, and y_1, $y_2 \in B$, the following implication holds

$$\begin{pmatrix} d(x_1, y_1) = d(A, B) \\ d(x_2, y_2) = d(A, B) \end{pmatrix} \Rightarrow \quad d(x_1, x_2) \leq d(y_1, y_2).$$

If we replace relation $d(x_1, x_2) \leq d(y_1, y_2)$ by $d(x_1, x_2) = d(y_1, y_2)$ we obtain a less general notion, that of a pair endowed with the (P)-property, see [23].

In [25], Shatanawi and Pitea studied a best proximity point result with regard to an almost contraction for a pair of sets endowed with the weak (P)-property. Before we present their main result, we recall the following

Definition 2.3 ([8])**.** A map $\varphi\colon [0, +\infty) \to [0, +\infty)$ is called a c-comparison function if it satisfies:

(1) φ is a monotone increasing,
(2) $\sum_{n=0}^{+\infty} \varphi^n(t)$ converges for all $t \geq 0$.

By replacing the second condition by $\lim_{n\to+\infty} \varphi^n(t) = 0$, $\forall t \in [0, +\infty)$, we get the notion of comparison function, more general than the one of c-comparison function. It is known that if φ is a comparison function, then $\varphi(t) < t$ for all $t > 0$ and $\varphi(0) = 0$.

Interesting properties of either c-comparison functions or comparison functions can be found in [13] and [30]. Also, contractive conditions with control functions are studied in: Aydi *et al.* [6], Chandok and Postolache [9], Shatanawi *et al.* [26, 27, 28, 29], Sintunavarat and Kumam [31], Sistani and Kazemipour [32].

In the following, denote $[0,+\infty)\times[0,+\infty)\times[0,+\infty)\times[0,+\infty)$ by $[0,+\infty)^4$.

Let Θ be the set of all continuous functions $\theta\colon [0,+\infty)^4 \to [0,+\infty)$ such that

$$\theta(0,t,s,u)=0 \text{ for all } t,s,u\in[0,+\infty)$$

and

$$\theta(t,s,0,u)=0 \text{ for all } t,s,u\in[0,+\infty).$$

Example 2.1 ([21])**.** Define $\theta_1,\theta_2,\theta_3\colon [0,+\infty)^4 \to [0,+\infty)$ by the formulas

$$\theta_1(t,s,u,v)=\tau\inf\{t,s,u,v\},\quad \tau>0,$$

$$\theta_2(t,s,u,v)=\tau\ln(1+tsuv),\quad \tau>0,$$

and

$$\theta_3(t,s,u,v)=\tau tsuv,\quad \tau>0.$$

Then $\theta_1,\theta_2,\theta_3\in\Theta$.

In [25] the following type of contraction was used.

Definition 2.4 ([25])**.** Let φ be a comparison function, and $\theta\in\Theta$. Mapping $T\colon A\to B$ is called a generalized almost (φ,θ)-contraction if, for each x, $y\in A$,

$$\begin{aligned} d(Tx,Ty) \quad \le \quad & \varphi(d(x,y))+\theta(d(y,Tx)-d(A,B),d(x,Ty)-d(A,B),\\ & d(x,Tx)-d(A,B),d(y,Ty)-d(A,B)). \end{aligned}$$

By using this type of contraction function, the following theorem is proved.

Theorem 2.1. *Consider A and B two closed subsets of a complete metric space (X,d) for which A_0 is nonempty. Let $T\colon A\to B$ be a mapping which satisfies the following conditions:*

1) *T is a generalized almost (φ,θ)-contraction;*
2) *$TA_0\subseteq B_0$;*
3) *The pair (A,B) has the weak (P)-property.*

Then, there exists a unique best proximity point of T, $x^*\in A$.

Let Ψ be the set of all continuous functions $\psi\colon \mathbb{R}_+\to\mathbb{R}_+$ satisfying the following conditions:

(ψ_1) ψ is continuous and strictly increasing;
(ψ_2) $\psi(t)=0$ if and only of $t=0$.

Let Φ_u be the set of all continuous functions $\varphi\colon \mathbb{R}_+\to\mathbb{R}_+$ satisfying the following conditions:

(ψ_1) φ is continuous .
(ψ_2) $\varphi(t)>0$ if $t>0$ and $\varphi(0)\ge 0$.

Let Φ be the set of all lower continuous functions $\phi\colon \mathbb{R}_+\to\mathbb{R}_+$, such $\phi(t)=0$ if and only if $t=0$ and $\phi(t)<t$ for all $t>0$.

In 2014, A.H. Ansari [5] introduced the concept of C-class functions. By using this concept we can generalize many fixed point theorems in literature.

Definition 2.5 ([5])**.** Let $f\colon \mathbb{R}_+^2\to\mathbb{R}$ be a continuous mapping. f is called *a C-class function* if it satisfies the following conditions:

$(C_1):$ $f(s,t)\le s$, for all $(s,t)\in\mathbb{R}_+^2$.
$(C_2):$ $f(s,t)=s$ implies that $s=0$, or $t=0$, for all $(s,t)\in\mathbb{R}_+^2$.

Note that if f is a C-class function, then $f(0,0)=0$.

We denote the set of all the C-class functions by $\mathcal{C}$; see [3].

Example 2.2 ([5])**.** The following functions $f\colon [0,\infty)^2 \to \mathbb{R}$ are elements of $\mathcal{C}$:

(1) $f(s,t) = s - t$, $f(s,t) = s \Rightarrow t = 0$;
(2) $f(s,t) = ms$, $0<m<1$, $f(s,t) = s \Rightarrow s = 0$;
(3) $f(s,t) = \frac{s}{(1+t)^r}$; $r \in (0,\infty)$, $f(s,t) = s \Rightarrow s = 0$ or $t = 0$;
(4) $f(s,t) = s\beta(s)$, where $\beta\colon [0,\infty) \to [0,1)$, is continuous;
(5) $f(s,t) = s - \frac{t}{k+t}$;
(6) $f(s,t) = s - \varphi(s)$, where $\varphi\colon [0,\infty) \to [0,\infty)$ is a continuous function such that $\varphi(t) = 0 \Leftrightarrow t = 0$;
(7) $f(s,t) = sh(s,t)$, where $h\colon [0,\infty) \times [0,\infty) \to [0,\infty)$ is a continuous function such that $h(t,s) < 1$ for all $t, s > 0$;
(8) $f(s,t) = \sqrt[n]{\ln(1+s^n)}$;
(9) $f(s,t) = \phi(s)$, where $\phi\colon [0,\infty) \to [0,\infty)$ is a upper semicontinuous function such that $\phi(0) = 0$, and $\phi(t) < t$ for $t > 0$,
(10) $f(s,t) = \frac{s}{(1+s)^r}$; $r \in (0,\infty)$;
(11) $f(s,t) = \vartheta(s)$; $\vartheta\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is a generalized Mizoguchi-Takahashi type function so that $F(s,t) = s$ compels $s = 0$.

Lemma 2.2 ([7])**.** *Suppose (X,d) is a metric space. Let $\{x_n\}$ be a sequence in X such that $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$. If $\{x_n\}$ is not a Cauchy sequence, then there exist an $\varepsilon > 0$ and two sequences of positive integers $\{m_k\}$ and $\{n_k\}$ with $m_k > n_k > k$ such that $d(x_{m_k}, x_{n_k}) \geq \varepsilon$, $d(x_{m_k-1}, x_{n_k}) < \varepsilon$ and*

(i) $\lim_{k\to\infty} d(x_{m_k-1}, x_{n_k+1}) = \varepsilon$;
(ii) $\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = \varepsilon$;
(iii) $\lim_{k\to\infty} d(x_{m_k-1}, x_{n_k}) = \varepsilon$.

We note also that $\lim_{k\to\infty} d(x_{m_k+1}, x_{n_k+1}) = \varepsilon$ and $\lim_{k\to\infty} d(x_{m_k}, x_{n_k-1}) = \varepsilon$.

Let Φ_u denote the class of the functions $\varphi\colon [0,\infty) \to [0,\infty)$ which satisfy the following conditions:

(a) φ is continuous;
(b) $\varphi(t) > 0, t > 0$ and $\varphi(0) \geq 0$.

Let Ψ be the set of all continuous functions $\psi\colon \mathbb{R}_+ \to \mathbb{R}_+$ satisfying the following conditions:

(ψ_1) ψ is continuous and strictly increasing.
(ψ_2) $\psi(t) = 0$ if and only of $t = 0$.

3. Main Results

Our first aim in the paper is to introduce and prove best proximity point theorems for a more general case. For this instance, we introduce the notion of a generalized almost (f,ψ,φ,θ)-contraction of the first type, as follows

Definition 3.1. Let $\psi \in \Psi$, $\varphi \in \Phi_u$, $f \in C$, and $\theta \in \Theta$. The mapping $T\colon A \to B$ is called a generalized almost (f,ψ,φ,θ)-contraction of the first type if, for each x, $y \in A$,

$$\begin{aligned}\psi(d(Tx,Ty)) \quad \leq \quad & f\big(\psi(d(x,y)), \varphi(d(x,y))\big) + \\ & \theta\big(d(y,Tx) - d(A,B), d(x,Ty) - d(A,B), \\ & \quad d(x,Tx) - d(A,B), d(y,Ty) - d(A,B)\big). \qquad (3.1)\end{aligned}$$

Our first result is

Theorem 3.1. *Consider A and B two closed subsets of a complete metric space (X,d) for which A_0 is nonempty. Let $T\colon A\to B$ be a mapping which satisfies the following conditions:*

1) *T is a generalized almost (f,ψ,φ,θ)-contraction of the first type;*
2) *$TA_0\subseteq B_0$;*
3) *the pair (A,B) has the weak (P)-property.*

Then, there exists a unique best proximity point of T, $x^*\in A$.

Proof. Consider $x_0\in A_0$. Since $TA_0\subseteq B_0$, then $Tx_0\in B_0$, and there is $x_1\in A_0$ such that $d(x_1,Tx_0)=d(A,B)$. By continuing this procedure, we obtain a sequence $\{x_n\}\subseteq A_0$,

$$d(x_{n+1},Tx_n)=d(A,B),\quad \forall n\in\mathbb{N}\cup\{0\}.$$

If there is $n\in\mathbb{N}\cup\{0\}$, for which $d(x_{n+1},x_n)=0$, it follows

$$d(A,B)\le d(x_n,Tx_n)\le d(x_n,x_{n+1})+d(x_{n+1},Tx_n)=d(x_{n+1},Tx_n)=d(A,B),$$

hence $d(A,B)=d(x_n,Tx_n)$, so x_n is a best proximity point of T.

Without loss of generality, in the following we may assume that $d(x_n,x_{n+1})>0$, for each $n\in\mathbb{N}\cup\{0\}$.

(A,B) satisfies the weak (P)-property, so $d(x_n,x_{n+1})\le d(Tx_{n-1},Tx_n)$, $n\in\mathbb{N}$.

Using the almost (F,ψ,φ,θ)-contraction property of T, we have

$$\begin{aligned}
&\psi(d(x_n,x_{n+1}))\le\psi(d(Tx_{n-1},Tx_n))\\
\le\;& f\Big(\psi(d(x_{n-1},x_n)),\varphi(d(x_{n-1},x_n))\Big)+\\
&\theta(d(x_n,Tx_{n-1})-d(A,B),d(x_{n-1},Tx_n)-d(A,B),\\
&\qquad\qquad d(x_{n-1},Tx_{n-1})-d(A,B),d(x_n,Tx_n)-d(A,B))\\
=\;& f\Big(\psi(d(x_{n-1},x_n)),\varphi(d(x_{n-1},x_n))\Big)+\\
&\theta(0,d(x_{n-1},Tx_n)-d(A,B),\\
&\qquad\qquad d(x_{n-1},Tx_{n-1})-d(A,B),d(x_n,Tx_n)-d(A,B))\\
=\;& f\Big(\psi(d(x_{n-1},x_n)),\varphi(d(x_{n-1},x_n))\Big),\quad n\in\mathbb{N}.
\end{aligned}$$

So,

$$\begin{aligned}
\psi(d(x_n,x_{n+1}))\;&\le\; f\Big(\psi(d(x_{n-1},x_n)),\varphi(d(x_{n-1},x_n))\Big)\\
&\le\;\psi(d(x_{n-1},x_n)).
\end{aligned}\tag{3.2}$$

Using the monotony of ψ, we get

$$d(x_n,x_{n+1})\;\le\; d(x_{n-1},x_n),\quad n\in\mathbb{N},$$

hence $d(x_n,x_{n+1})\to r\ge 0$, when $n\to\infty$. Letting $n\to+\infty$ in (3.2) we obtain

$$\psi(r)\le f\Big(\psi(r),\varphi(r)\Big)$$

which implies $\psi(r)=0$ or $\varphi(r)=0$. Thus, $r=0$. Therefore,

$$\lim_{n\to\infty} d(x_n,x_{n+1})=0.$$

Taking into account the inequalities

$$d(A,B)\le d(x_n,Tx_n)\le d(x_n,x_{n+1})+d(x_{n+1},Tx_n),$$

and letting $n \to +\infty$, we obtain

$$\lim_{n\to+\infty} d(x_n, Tx_n) = d(A, B).$$

Now, we will show that $\{x_n\}$ is a Cauchy sequence. Suppose, to the contrary, that $\{x_n\}$ is not a Cauchy sequence.

By Lemma 2.2, there exists $\delta > 0$ for which we can find two subsequences $\{x_{n_k}\}$, and $\{x_{m_k}\}$ of $\{x_n\}$, with $n_k > m_k > k$ such that

$$\lim_{k\to\infty} d(x_{n_k}, x_{m_k}) = \lim_{k\to\infty} d(x_{n_k-1}, x_{m_k-1}) = \delta.$$

Setting $x = x_{m_k-1}$ and $y = x_{n_k-1}$ in (3.1), we obtain

$$\begin{aligned} &\psi(d(x_{m_k}, x_{n_k})) \le \psi(d(Tx_{m_k-1}, Tx_{n_k-1})) \\ \le\ & f\Big(\psi(d(x_{m_k-1}, x_{n_k-1})), \varphi(d(x_{m_k-1}, x_{n_k-1}))\Big) + \\ &\theta(d(x_{n_k-1}, Tx_{m_k-1}) - d(A,B), d(x_{m_k-1}, Tx_{n_k-1}) - d(A,B), \\ &\qquad d(x_{n_k-1}, Tx_{n_k-1}) - d(A,B), d(x_{m_k-1}, Tx_{m_k-1}) - d(A,B)). \end{aligned}$$

We have

$$\psi(\delta) \le f\Big(\psi(\delta), \varphi(\delta)\Big),$$

which implies $\psi(\delta) = 0$ or $\varphi(\delta) = 0$. Thus, $\delta = 0$. So we got that (x_n) is a Cauchy sequence in A, which is a closed subset of (X, d), a complete metric space. Therefore, there exists $x^* \in A$ such that $\lim_{n\to+\infty} x_n = x^*$.

Using the triangle inequality, it follows

$$d(x^*, Tx^*) \le d(x^*, x_n) + d(x_n, Tx_n) + d(Tx^*, Tx_n). \tag{3.3}$$

Letting $n \to +\infty$ in the inequality

$$\begin{aligned} \psi(d(Tx^*, Tx_n)) \quad \le \quad & f\Big(\psi(d(x^*, x_n)), \varphi(d(x^*, x_n))\Big) + \\ & \theta(d(x_n, Tx^*) - d(A,B), d(x^*, Tx_n) - d(A,B), \\ & \qquad d(x_n, Tx_n) - d(A,B), d(x^*, Tx^*) - d(A,B)), \end{aligned}$$

it follows $\lim_{n\to+\infty} d(Tx_n, Tx^*) = 0$. Taking $n \to +\infty$ in relation (3.3), we obtain that $d(x^*, Tx^*) = d(A, B)$, so x^* is a best proximity point of T.

We shall focus now on the uniqueness of the best proximity point of T. Suppose there are $x^* \neq y^*$ two best proximity points of T. We obtain

$$\begin{aligned} \psi(d(x^*, y^*)) \quad &\le \quad \psi(d(Tx^*, Ty^*)) \\ &\le \quad f\Big(\psi(d(x^*, y^*)), \varphi(d(x^*, y^*))\Big) + \\ &\qquad \theta(d(y^*, Tx^*) - d(A,B), d(x^*, Ty^*) - d(A,B), \\ &\qquad\qquad d(x^*, Tx^*) - d(A,B), d(y^*, Ty^*) - d(A,B)) \\ &= \quad f\Big(\psi(d(x^*, y^*)), \varphi(d(x^*, y^*))\Big) + \\ &\qquad \theta(d(y^*, Tx^*) - d(A,B), d(x^*, Ty^*) - d(A,B), \\ &\qquad\qquad 0, d(y^*, Ty^*) - d(A,B)) \\ &\le \quad f\Big(\psi(d(x^*, y^*)), \varphi(d(x^*, y^*))\Big), \end{aligned}$$

which implies $\psi(d(x^*, y^*)) = 0$ or $\varphi(d(x^*, y^*)) = 0$. Thus, $d(x^*, y^*) = 0$, which is impossible, since $x^* \neq y^*$. The uniqueness part has been proved now. $\square$

Let us take the particular case of $f(s,t) = kt$, where $k \in (0,1)$, and

$$\theta\colon [0,+\infty)^4 \to [0,+\infty), \quad \theta(t_1,t_2,t,3,t_4) = L\min\{t_1,t_2,t_3,t_4\},$$

for some $L \geq 0$. We obtain the following corollary.

Corollary 3.2. *Let A and B be two closed subsets of a complete metric space (X,d) for which A_0 is nonempty. Let $T\colon A \to B$ be a mapping which satisfies the following conditions:*
1) $TA_0 \subseteq B_0$;
2) *the pair (A,B) has the weak (P)-property.*
Suppose there exist $k \in (0,1)$ and $L \geq 0$ such that

$$\begin{aligned} d(Tx,Ty) \quad &\leq \quad kd(x,y) + L\min\{d(y,Tx) - d(A,B), d(x,Ty) - d(A,B), \\ &\qquad d(x,Tx) - d(A,B), d(y,Ty) - d(A,B)\} \end{aligned}$$

holds for all $x,y \in A$. Then, there exists a unique best proximity point of T, $x^* \in A$.

By considering $A = B$ in Theorem 3.1, we get the next corollary.

Corollary 3.3. *Let A be a closed subset of a complete metric space (X,d). Let $T\colon A \to A$ be a mapping such that*

$$\psi(d(Tx,Ty)) \leq f\Big(\psi(d(x,y)), \varphi(d(x,y))\Big) + \theta(d(y,Tx), d(x,Ty), d(x,Tx), d(y,Ty))$$

holds for all $x, y \in A$, where $\psi \in \Psi, \varphi \in \Phi_u, F \in C$. Then T has a unique fixed point $u \in A$; that is $Tu = u$.

Definition 3.2. Let $\psi \in \Psi$, $\varphi \in \Phi_u$, $f \in C$, and $\theta \in \Theta$. A mapping $T\colon A \to B$ is called a generalized almost (f,ψ,φ,θ)-contraction of the second type if, for each x, $y \in A$,

$$\begin{aligned} \psi(d(Tx,Ty)) \quad &\leq \quad f\Big(\psi\Big(\frac{d(x,Tx)+d(y,Ty)}{2} - d(A,B)\Big), \varphi\Big(\frac{d(x,Tx)+d(y,Ty)}{2} \\ &\qquad -d(A,B)\Big)\Big) + \theta\Big(d(y,Tx) - d(A,B), d(x,Ty) - d(A,B), \\ &\qquad d(x,Tx) - d(A,B), d(y,Ty) - d(A,B)\Big). \end{aligned}$$

Our second result is

Theorem 3.4. *Consider A and B two closed subsets of a complete metric space (X,d) for which A_0 is nonempty. Let $T\colon A \to B$ be a mapping which satisfies the following conditions:*
1) *T is a generalized almost (f,ψ,φ,θ)-contraction of the second type;*
2) $TA_0 \subseteq B_0$;
3) *the pair (A,B) has the weak (P)-property.*
Then, there exists a unique best proximity point of T, $x^* \in A$.

Proof. As in Theorem 3.1, we obtain a sequence $\{x_n\} \subseteq A_0$,

$$d(x_{n+1}, Tx_n) = d(A,B), \quad \forall n \in \mathbb{N} \cup \{0\},$$

and without loss of generality we may assume that $d(x_n, x_{n+1}) > 0$, for each $n \in \mathbb{N} \cup \{0\}$.

(A,B) satisfies the weak (P)-property, so $d(x_n, x_{n+1}) \leq d(Tx_{n-1}, Tx_n)$, $n \in \mathbb{N}$.

Using the almost (F,ψ,φ,θ)-contraction of the second type property of T, we have

$$
\begin{aligned}
& \psi(d(x_n,x_{n+1})) \le \psi(d(Tx_{n-1},Tx_n)) \\
\le\ & f\Big(\psi\Big(\frac{d(x_{n-1},Tx_{n-1})+d(x_n,Tx_n)}{2}-d(A,B)\Big), \\
& \varphi\Big(\frac{d(x_{n-1},Tx_{n-1})+d(x_n,Tx_n)}{2}-d(A,B)\Big)\Big) \\
& +\theta(d(x_n,Tx_{n-1})-d(A,B), d(x_{n-1},Tx_n)-d(A,B), \\
& \qquad d(x_{n-1},Tx_{n-1})-d(A,B), d(x_n,Tx_n)-d(A,B)) \\
=\ & f\Big(\psi\Big(\frac{d(x_{n-1},Tx_{n-1})+d(x_n,Tx_n)}{2}-d(A,B)\Big), \\
& \varphi\Big(\frac{d(x_{n-1},Tx_{n-1})+d(x_n,Tx_n)}{2}-d(A,B)\Big)\Big) \\
& +\theta(0, d(x_{n-1},Tx_n)-d(A,B), \\
& \qquad d(x_{n-1},Tx_{n-1})-d(A,B), d(x_n,Tx_n)-d(A,B))
\end{aligned}
$$

$$
\begin{aligned}
=\ & f\Big(\psi\Big(\frac{d(x_{n-1},Tx_{n-1})+d(x_n,Tx_n)}{2}-d(A,B)\Big), \\
& \varphi\Big(\frac{d(x_{n-1},Tx_{n-1})+d(x_n,Tx_n)}{2}-d(A,B)\Big)\Big) \qquad (3.4) \\
\le\ & \psi\Big(\frac{d(x_{n-1},x_n)+d(x_n,Tx_{n-1})+d(x_n,x_{n+1})+d(x_{n+1},Tx_n)}{2}-d(A,B)\Big) \\
=\ & \psi\Big(\frac{d(x_{n-1},x_n)+d(x_n,x_{n+1})}{2}\Big) \quad n\in\mathbb{N}\cup\{0\}.
\end{aligned}
$$

Having in mind the monotone of ψ, it immediately follows

$$d(x_n,x_{n+1}) \le d(x_{n-1},x_n), \quad n\in\mathbb{N}\cup\{0\},$$

hence $d(x_n,x_{n+1})\to r\ge 0$.

From (3.4), by taking lim sup and denoting

$$\limsup_{n\to\infty}\Big(\frac{d(x_{n-1},Tx_{n-1})+d(x_n,Tx_n)}{2}-d(A,B)\Big)=l,$$

we obtain

$$\psi(r)\le f\big(\psi(l),\varphi(l)\big)\le\psi(l), \tag{3.5}$$

which implies

$$
\begin{aligned}
r\ =\ & \lim_{n\to\infty} d(x_n,x_{n+1})\le l=\limsup_{n\to\infty}\Big(\frac{d(x_{n-1},Tx_{n-1})+d(x_n,Tx_n)}{2}-d(A,B)\Big) \\
\le\ & \limsup_{n\to\infty}\Big(\frac{d(x_{n-1},x_n)+d(x_n,x_{n+1})}{2}\Big)=r,
\end{aligned}
$$

hence $r=l$.

Using inequality (3.5) and the second property of f, which is a C-class function, we get $\psi(r)\le f\big(\psi(r),\varphi(r)\big)$. It follows that $\psi(r)=0$, or $\varphi(r)=0$, so $r=0$.

$$\lim_{n\to\infty} d(x_n,x_{n+1})=0.$$

Taking into account the inequalities

$$d(A,B)\le d(x_n,Tx_n)\le d(x_n,x_{n+1})+d(x_{n+1},Tx_n),$$

and letting $n \to +\infty$, we obtain

$$\lim_{n\to+\infty} d(x_n, Tx_n) = d(A,B). \tag{3.6}$$

Now, we will show that $\{x_n\}$ is a Cauchy sequence.

Taking advantage from the contraction condition, we get

$$\begin{aligned}
\psi(d(x_{m+1}, x_{n+1})) &\leq \psi(d(Tx_m, Tx_n)) \\
&\leq f\Big(\psi\Big(\frac{d(x_m, Tx_m) + d(x_n, Tx_n)}{2} - d(A,B)\Big), \\
&\quad \varphi\Big(\frac{d(x_m, Tx_m) + d(x_n, Tx_n)}{2} - d(A,B)\Big) \\
&\quad +\theta(d(x_n, Tx_m) - d(A,B), d(x_m, Tx_n) - d(A,B), \\
&\qquad d(x_m, Tx_m) - d(A,B), d(x_n, Tx_n) - d(A,B)).
\end{aligned}$$

Using relation (3.6), and taking the limit, it follows $\lim_{m,n\to\infty} d(x_m, x_n) = 0$, hence $\{x_n\}$ is a Cauchy sequence, and from the completness we get $\{x_n\}$ is convergent to x^*.

Using the triangle inequality, it follows

$$d(x^*, Tx^*) \leq d(x^*, x_n) + d(x_n, Tx_n) + d(Tx^*, Tx_n). \tag{3.7}$$

Letting $n \to +\infty$ in the inequality

$$\begin{aligned}
\psi(d(Tx^*, Tx_n)) &\leq f\Big(\psi\Big(\frac{d(x^*, Tx^*) + d(x_n, Tx_n)}{2} - d(A,B)\Big), \\
&\quad \varphi\Big(\frac{d(x^*, Tx^*) + d(x_n, Tx_n)}{2} - d(A,B)\Big)\Big) \\
&\quad +\theta(d(x_n, Tx^*) - d(A,B), d(x^*, Tx_n) - d(A,B), \\
&\qquad d(x_n, Tx_n) - d(A,B), d(x^*, Tx^*) - d(A,B)),
\end{aligned}$$

it follows $\limsup_{n\to+\infty} d(Tx_n, Tx^*) \leq \dfrac{d(x^*, Tx^*) - d(A,B)}{2}$. Taking $n \to +\infty$ in relation (3.7), it follows that $d(x^*, Tx^*) \leq d(A,B)$, that is $d(x^*, Tx^*) = d(A,B)$, so x^* is a best proximity point of T.

We shall focus now on the uniqueness of the best proximity point of T.

Suppose there are $x^* \neq y^*$ two best proximity points of T. We obtain

$$\begin{aligned}
\psi(d(x^*, y^*)) &\leq \psi(d(Tx^*, Ty^*)) \\
&\leq f\Big(\psi\Big(\frac{d(x^*, Tx^*) + d(y^*, Ty^*)}{2} - d(A,B)\Big), \\
&\quad \varphi\Big(\frac{d(x^*, Tx^*) + d(y^*, Ty^*)}{2} - d(A,B)\Big)\Big) + \\
&\quad \theta(d(y^*, Tx^*) - d(A,B), d(x^*, Ty^*) - d(A,B), \\
&\qquad d(x^*, Tx^*) - d(A,B), d(y^*, Ty^*) - d(A,B)) \\
&= 0
\end{aligned}$$

which implies $d(x^*, y^*) = 0$ which is impossible, since $x^* \neq y^*$. The uniqueness part has been proved now. $\square$

We will introduce now our third type of generalized almost $(f, \psi, \varphi, \theta)$, as follows.

Definition 3.3. Let $\psi \in \Psi$, $\varphi \in \Phi_u$, $f \in \mathcal{C}$, and $\theta \in \Theta$. A mapping $T\colon A \to B$ is called a generalized almost (f,ψ,φ,θ)-contraction of the third type if, for each x, $y \in A$,

$$\begin{aligned}\psi(d(Tx,Ty)) \quad \leq \quad & f\Big(\psi\Big(\frac{d(x,Ty)+d(y,Tx)}{2} - d(A,B)\Big), \varphi\Big(\frac{d(x,Ty)+d(y,Tx)}{2} \\ & -d(A,B)\Big)\Big) + \theta\Big(d(y,Tx)-d(A,B), d(x,Ty)-d(A,B), \\ & d(x,Tx)-d(A,B), d(y,Ty)-d(A,B)\Big).\end{aligned}$$

By using this definition, the following result can be obtained.

Theorem 3.5. *Consider A and B two closed subsets of a complete metric space (X,d) for which A_0 is nonempty. Let $T\colon A \to B$ be a mapping which satisfies the following conditions:*
1) *T is a generalized almost (f,ψ,φ,θ)-contraction of the third type;*
2) *$TA_0 \subseteq B_0$;*
3) *the pair (A,B) has the weak (P)-property.*

Then, there exists a unique best proximity point of T, $x^* \in A$.

Proof. As in Theorem 3.1, we obtain a sequence $\{x_n\} \subseteq A_0$,

$$d(x_{n+1}, Tx_n) = d(A,B), \quad \forall n \in \mathbb{N} \cup \{0\},$$

and without loss of generality we may assume that $d(x_n, x_{n+1}) > 0$, for each $n \in \mathbb{N} \cup \{0\}$.

(A,B) satisfies the weak (P)-property, so $d(x_n, x_{n+1}) \leq d(Tx_{n-1}, Tx_n)$, $n \in \mathbb{N}$.

Using the almost (f,ψ,φ,θ)-contraction of the third type property of T, we have

$$\begin{aligned}
& \psi(d(x_n,x_{n+1})) \leq \psi(d(Tx_{n-1},Tx_n)) \\
\leq \quad & f\Big(\psi\Big(\frac{d(x_{n-1},Tx_n)+d(x_n,Tx_{n-1})}{2} - d(A,B)\Big), \\
& \varphi\Big(\frac{d(x_{n-1},Tx_n)+d(x_n,Tx_{n-1})}{2} - d(A,B)\Big)\Big) \\
& +\theta(d(x_n,Tx_{n-1})-d(A,B), d(x_{n-1},Tx_n)-d(A,B), \\
& \qquad d(x_{n-1},Tx_{n-1})-d(A,B), d(x_n,Tx_n)-d(A,B)) \\
\leq \quad & f\Big(\psi\Big(\frac{d(x_{n-1},Tx_n)+d(x_n,Tx_{n-1})}{2} - d(A,B)\Big), \\
& \varphi\Big(\frac{d(x_{n-1},Tx_n)+d(x_n,Tx_{n-1})}{2} - d(A,B)\Big)\Big) \\
& +\theta(0, d(x_{n-1},Tx_n)-d(A,B), \\
& \qquad d(x_{n-1},Tx_{n-1})-d(A,B), d(x_n,Tx_n)-d(A,B)) \\
= \quad & f\Big(\psi\Big(\frac{d(x_{n-1},Tx_n)+d(x_n,Tx_{n-1})}{2} - d(A,B)\Big), \\
& \varphi\Big(\frac{d(x_{n-1},Tx_n)+d(x_n,Tx_{n-1})}{2} - d(A,B)\Big)\Big) \qquad (3.8) \\
\leq \quad & \psi\Big(\frac{d(x_{n-1},x_n)+d(x_n,Tx_{n-1})+d(Tx_{n-1},Tx_n)-d(A,B)}{2}\Big) \\
= \quad & \psi\Big(\frac{d(x_{n-1},x_n)+d(Tx_{n-1},Tx_n)}{2}\Big) \quad n \in \mathbb{N} \cup \{0\}.
\end{aligned}$$

Having in mind the monotone of ψ, it follows immediately

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \frac{d(x_{n-1}, x_n) + d(Tx_{n-1}, Tx_n)}{2}, \quad n \in \mathbb{N}, \\ d(Tx_{n-1}, Tx_n) &\leq \frac{d(x_{n-1}, x_n) + d(Tx_{n-1}, Tx_n)}{2}, \quad n \in \mathbb{N}. \end{aligned} \tag{3.9}$$

It follows that $\{d(x_n, x_{n+1})\}$ is a decreasing sequence with positive terms, therefore $d(x_n, x_{n+1}) \to r \geq 0$.

From (3.8) and (3.9), by taking lim sup, and then lim inf, we obtain that there exists $\lim_{n\to\infty} d(Tx_{n-1}, Tx_n) = r$.

Using inequality (3.8) and the second property of f, which is a C-class function, we get $\psi(r) \leq f\Big(\psi(r), \varphi(r)\Big)$. It follows that $\psi(r) = 0$, or $\varphi(r) = 0$, so $r = 0$.

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0.$$

Taking into account the inequalities

$$d(A, B) \leq d(x_n, Tx_n) \leq d(x_n, x_{n+1}) + d(x_{n+1}, Tx_n),$$

and letting $n \to +\infty$, we obtain

$$\lim_{n\to+\infty} d(x_n, Tx_n) = d(A, B).$$

Now, we will show that $\{x_n\}$ is a Cauchy sequence.If $\{x_n\}$ is not a Cauchy sequence, according to Lemma 2.2, there exist an $\varepsilon > 0$ and two sequences of positive integers $\{m_k\}$ and $\{n_k\}$ with $m_k > n_k > k$ such that $d(x_{m_k}, x_{n_k}) \geq \varepsilon$, $d(x_{m_k-1}, x_{n_k}) < \varepsilon$, $\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = \varepsilon$. Taking advantage from the contraction condition, we get

$$\begin{aligned} \psi(d(x_{m_k+1}, x_{n_k+1})) &\leq \psi(d(Tx_{m_k}, Tx_{n_k})) \\ &\leq f\Big(\psi\Big(\frac{d(x_{m_k}, Tx_{n_k}) + d(x_{n_k}, Tx_{m_k})}{2} - d(A, B)\Big), \\ &\quad \varphi\Big(\frac{d(x_{m_k}, Tx_{n_k}) + d(x_{n_k}, Tx_{m_k})}{2} - d(A, B)\Big) \\ &\quad +\theta(d(x_{n_k}, Tx_{m_k}) - d(A, B), d(x_{m_k}, Tx_{n_k}) - d(A, B), \\ &\qquad d(x_{m_k}, Tx_{m_k}) - d(A, B), d(x_{n_k}, Tx_{n_k}) - d(A, B)). \end{aligned}$$

Taking lim sup as $m, n \to \infty$, by using the property of the function f and the fact that $d(x_{m_k}, Tx_{n_k}) \leq d(x_{m_k}, x_{n_k}) + d(x_{n_k}, Tx_{n_k})$, it follows $\varepsilon = 0$, which is impossible. Hence $\{x_n\}$ is a Cauchy sequence, and from the completness we get $\{x_n\}$ is convergent to x^*.

Using the triangle inequality, it follows

$$d(x^*, Tx^*) \leq d(x^*, x_n) + d(x_n, Tx_n) + d(Tx^*, Tx_n). \tag{3.10}$$

Letting $n \to +\infty$ in the inequality

$$\begin{aligned} \psi(d(Tx^*, Tx_n)) &\leq f\Big(\psi\Big(\frac{d(x^*, Tx_n) + d(x_n, Tx^*)}{2} - d(A, B)\Big), \\ &\quad \varphi\Big(\frac{d(x^*, Tx_n) + d(x_n, Tx^*)}{2} - d(A, B)\Big)\Big) \\ &\quad +\theta(d(x_n, Tx^*) - d(A, B), d(x^*, Tx_n) - d(A, B), \\ &\qquad d(x_n, Tx_n) - d(A, B), d(x^*, Tx^*) - d(A, B)) \end{aligned}$$

$$\begin{aligned} \leq \quad & \psi\Big(\frac{d(x^*,x_{n+1})+d(x_{n+1},Tx_n)+d(x_n,x^*)+d(x^*,Tx^*)}{2}-d(A,B)\Big) \\ & +\theta(d(x_n,Tx^*)-d(A,B),d(x^*,Tx_n)-d(A,B), \\ & \qquad\qquad d(x_n,Tx_n)-d(A,B),d(x^*,Tx^*)-d(A,B)) \end{aligned}$$

it follows $\limsup_{n\to+\infty} d(Tx_n,Tx^*) \leq \dfrac{d(x^*,Tx^*)-d(A,B)}{2}$. Taking $n\to+\infty$ in relation (3.10), it follows that $d(x^*,Tx^*)\leq d(A,B)$, that is $d(x^*,Tx^*)=d(A,B)$, so x^* is a best proximity point of T.

We shall focus now on the uniqueness of the best proximity point of T.

Suppose there are $x^*\neq y^*$ two best proximity points of T. We obtain

$$\begin{aligned} \psi(d(x^*,y^*)) \quad & \leq \quad \psi(d(Tx^*,Ty^*)) \\ & \leq \quad f\Big(\psi\Big(\frac{d(x^*,Ty^*)+d(y^*,Tx^*)}{2}-d(A,B)\Big), \\ & \qquad \varphi\Big(\frac{d(x^*,Ty^*)+d(y^*,Tx^*)}{2}-d(A,B)\Big)\Big) \\ & \qquad +\theta(d(y^*,Tx^*)-d(A,B),d(x^*,Ty^*)-d(A,B), \\ & \qquad\qquad d(x^*,Tx^*)-d(A,B),d(y^*,Ty^*)-d(A,B)) \\ & \leq \quad f\Big(\psi\Big(\frac{d(x^*,Tx^*)+d(Tx^*,Ty^*)+d(y^*,Tx^*)+d(Tx^*,Ty^*)}{2} \\ & \qquad -d(A,B)\Big),\varphi\Big(\frac{d(x^*,Ty^*)+d(y^*,Tx^*)}{2}-d(A,B)\Big)\Big) \end{aligned}$$

so $\psi(d(Tx^*,Ty^*))=f\Big(\psi(d(Tx^*,Ty^*)),\varphi\Big(\dfrac{d(x^*,Ty^*)+d(y^*,Tx^*)}{2}-d(A,B)\Big)\Big)$, a false statement, since $x^*\neq y^*$. The uniqueness part has been proved now. □

4. Conclusion

We introduced three types of generalized almost (f,ψ,φ,θ)-contractions, by means of C-class functions. Using the concept of (P)-property, we stated existence and uniqueness results of some best proximity points. This research is a natural continuation of those of Shatanawi and Pitea [25] and other researchers from the reference list.

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