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IDEAL VERSION OF WEIGHTED LACUNARY STATISTICAL CONVERGENCE OF SEQUENCES OF ORDER α

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Abstract. In this work, we are interested in ideal version of weighted lacunary statistical convergence of sequences of order α and we examine some inclusion relations.

1. Introduction

The idea of statistical convergence was given by Zygmund [32] in the first edition of his monograph published in Warsaw in 1935. The concept of statistical convergence was introduced by Steinhaus [30] and Fast [17] and later reintroduced by Schoenberg [29] independently. Many years later, it has been discussed in the theory of Fourier analysis, ergodic theory and number theory under different names. Further, it was investigated from varied points of view (see, [1]-[3], [5]-[9], [11]-[12], [15]-[16], [21], [26]-[27]).

The order of statistical convergence of a sequence of numbers was given by Gadjiev and Orhan [19]. By this way, a different direction was given to the study of statistical convergence, where the notion of statistical convergence of order α, was introduced by replacing n by n^α in the denominator in the definition of statistical convergence by Çolak [8] and later studied by Bhunia et al. [4]. After then λ-statistical convergence of order α was introduced by Çolak and Bektaş [9]; λ-statistical convergence of order α of sequences of function by Et et al. [14]; lacunary statistical convergence of order α by Şengül and Et [31]; statistical convergence of order α in probability theory by Das et al. [13]; weighted statistical convergence of order α and its applications by Ghosal [20] and many other different fields of mathematics.

The notion of I-convergence was studied at initial stage by Kostyrko et al. [23] as a generalization of statistical convergence which was further studied in topological spaces by Lahiri et al. [25]. Kostyrko et al. [24] gave some basic properties of I-convergence and dealt with extremal I-limit points.

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In this paper, we define a new concept called weighted lacunary I-statistical convergence of order α and examine some inclusion relations.

2. Definitions and Preliminaries

In this section, we present some definitions and notations needed throughout the paper.

The idea of statistical convergence depends on the density of subsets of the set $\mathbb{N}$ of natural numbers. The density of a subset E of $\mathbb{N}$ is defined by $\delta(E) = lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\chi_E(k)$ provided the limit exists, where χ_E is the characteristic function of E. It is clear that any finite subset of $\mathbb{N}$ has zero natural density and $\delta(E^c) = \delta(E)$.

Let α be a real number such that $0 < \alpha \leq 1$. The α-density of a subset E of $\mathbb{N}$ is defined by

$$\delta_\alpha(E) = \lim_n \frac{1}{n^\alpha}\left|\{k \leq n : k \in E\}\right|$$

provided the limit exists, where $\{k \leq n : k \in E\}$ denotes the number of elements of E not exceeding n. It is clear that any finite subset of $\mathbb{N}$ has zero α density and $\delta_\alpha(E^c) = 1 - \delta_\alpha(E)$ does not hold for $0 < \alpha < 1$ in general, the equality holds only if $\alpha = 1$. Note that the α-density of any set reduces to the natural density of the set in case $\alpha = 1$. The sequence $x = (x_k)$ is said to be statistically convergent of order α if there is a real number L such that

$$\lim_{n\to\infty}\frac{1}{n^\alpha}\left|\{k \leq n :\ |x_k - L| \geq \varepsilon\}\right| = 0,$$

for every $\varepsilon > 0$. In this case, we write S^α-$\lim x = L$. The set of all statistically convergent sequences of order α will be denoted by S^α [8]. For $\alpha = 1$, S^α reduces to S (see, in [17]).

Let (p_k) be a sequence of positive real numbers and $P_n = p_1 + p_2 + ... + p_n$ for $n \in \mathbb{N}$. Then the Riesz transformation of $x = (x_k)$ is defined as $t_n = 1/P_n\sum_{k=1}^{n}p_k x_k$. If the transformation sequence (t_n) has a finite limit L then the sequence $x = (x_k)$ is said to be Riesz convergent to L. Let us note that if $P_n \to \infty$ as $n \to \infty$ then Riesz mean is a regular summability method. Throughout the paper, let $P_n \to \infty$ as $n \to \infty$ and $P_0 = p_0 = 0$ [28].

Recall that a number sequence $x = (x_k)$ is said to be weighted statistically convergent to a number L (denoted by S_R-$\lim_k x_k = L$) provided that for every $\varepsilon > 0$

$$\lim_{n\to\infty}\frac{1}{P_n}\left|\{k \leq P_n : p_k\,|x_k - L| \geq \varepsilon\}\right| = 0.$$

S_R denotes the set of all weighted statistically convergent sequences of numbers [28].

A sequence $x = (x_k)$ is statistically summable to L by the weighted mean method determined by the sequence (p_k) or briefly statistically summable (R, p_n) to L if st-$\lim_n t_n(x) = L$. In this case, we write $R(st)$-$\lim x = L$. We denote the set of all sequences which are statistically summable (R, p_n) by $R(st)$ [28].

A lacunary sequence is an increasing integer sequence $\theta = (k_r)$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by θ will be defined by $\mathrm{I}_r = (k_{r-1}, k_r]$ and the ratio $\frac{k_r}{k_{r-1}}$ $(r \neq 1)$ will be defined by q_r [18].

Let $\theta=(k_r)$ be a lacunary sequence and $0<\alpha\leq 1$ be given. The sequence $x=(x_k)$ is said to be S_θ^α-statistically convergent (or lacunary statistically convergent of order α) if there is a real number L such that

$$\lim_{r\to\infty}\frac{1}{h_r^\alpha}\left|\{k\in I_r:\ |x_k-L|\geq\varepsilon\}\right|=0,$$

where $I_r=(k_{r-1},k_r]$ and h_r^α denote the α-th power $\left(h_r^\alpha\right)$ of h_r, that is $h^\alpha=(h_r^\alpha)=(h_1^\alpha,\ h_2^\alpha,\ \dots\ ,\ h_r^\alpha,\ \dots)$. In this case, we write S_θ^α-$\lim x=L$ [31]. If we take $\alpha=1$ then we obtain the definition of S_θ-statistically convergent sequence (see, [18]).

Let $\theta=(k_r)$ be a lacunary sequence, (p_k) be a sequence of positive real numbers such that $H_r:=\sum_{k\in \mathrm{I}_r}p_k,\ P_{k_r}:=\sum\limits_{k\in(0,k_r]}p_k,\ P_{k_{r-1}}:=\sum\limits_{k\in(0,k_{r-1}]}p_k,\ Q_r=\frac{P_{k_r}}{P_{k_{r-1}}}$ $(r\neq 1)$, $P_0=0$, $\mathrm{I}'_r=\left(P_{k_{r-1}},P_{k_r}\right]$. It is easy to see that $H_r=P_{k_r}-P_{k_{r-1}}$. If we take $p_k=1$ for all $k\in\mathbb{N}$ then $H_r,\ P_{k_r},\ P_{k_{r-1}},\ Q_r$ and I'_r reduce to $h_r,\ k_r,\ k_{r-1},\ q_r$ and I_r, respectively. Throughout the paper, we assume that $P_n\to\infty$ as $n\to\infty$ such that $H_r\to\infty$ as $r\to\infty$ and we mean $\lim_{k\to\infty}x_k$ by $\lim_k x_k$ or $\lim x$ for brevity [1].

A sequence $x=(x_k)$ is said to be weighted lacunary statistically convergent to L if for every $\varepsilon>0$,

$$\lim_{r\to\infty}\frac{1}{H_r}\left|\{k\in \mathrm{I}'_r: p_k\left|x_k-L\right|\geq\varepsilon\}\right|=0.$$

In this case, we write $S_{(R,\theta)}$-$\lim x=L$. We denote the set of all weighted lacunary statistically convergent sequences by $S_{(R,\theta)}$ [1].

Definition 2.1. [23] Let $X\neq\emptyset$ and $P\left(X\right)=2^X$ be the family of all subsets of X. Then a family of sets $I\subset 2^X$ is said to be an ideal on X if and only if I satisfies these conditions:

(1) $\emptyset\in \mathrm{I}$,
(2) $A,\ B\in I$ imply $A\cup B\in \mathrm{I}$,
(3) $A\in I,\ B\subset A$ imply $B\in I$.

Definition 2.2. [23] Let $X\neq\emptyset$. A non-empty family of sets $F\subset 2^X$ is said to be a filter on X if and only if

(1) $\emptyset\notin F$,
(2) $A,\ B\in F$ imply $A\cap B\in F$,
(3) $A\in F,\ \ B\supset A$ imply $B\in F$.

An ideal is I is called non-trivial if $I\neq\emptyset$ and $X\notin I$, that is $I\neq 2^X$. A non-trivial ideal $I\subset 2^X$ is called admissible if $\{x\}\in I$ for each $x\in X$.

$I\subset 2^X$ is a non-trivial (proper) ideal of X if and only if

$$F=F\left(I\right)=\{X\backslash A:\ A\in I\}$$

is a filter on X. $F\left(I\right)$ is called the filter associated with the ideal I.

Throughout the paper, I will be considered as a non-trivial admissible ideal and w will be the space of all sequences, unless otherwise stated.

Definition 2.3. [23] Let I denote a non-trivial ideal of subsets of $\mathbb{N}$. A sequence $\left(x_n\right)_{n\in\mathbb{N}}$ of elements of X is said to be I-convergent to $L\in X$ ($L=I$-$\lim_{n\to\infty}x_n$), if and only if for each $\varepsilon > 0$ the set $A(\varepsilon) = \{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\}$ belongs to I. The element L is called I-limit of the sequence $x = (x_n)_{n\in\mathbb{N}}$.

Definition 2.4. [11] A sequence $x = (x_k)$ of numbers is said to be I-statistically convergent or $S(\mathrm{I})$-convergent to L, if for every $\varepsilon > 0$ and $\delta > 0$,

$$\left\{n \in \mathbb{N} : \frac{1}{n} |\{k \leq n : |x_k - L| \geq \varepsilon\}| \geq \delta\right\} \in \mathrm{I}.$$

In this case, we write $x_k \to L\,(S(\mathrm{I}))$ or $S(\mathrm{I})\text{-}\lim_k x_k = L$. Let $S(\mathrm{I})$ denotes the set of all I-statistically convergent sequences of numbers. For $\mathrm{I} = \mathrm{I}_{fin}$ (the ideal of all finite subsets of $\mathbb{N}$), I-statistical convergence coincides with statistical convergence.

Definition 2.5. [10] Let θ be a lacunary sequence and $0 < \alpha \leq 1$. A sequence $x = (x_k)$ is said to be I-lacunary statistically convergent of order α to L (or $S_\theta^\alpha(I)$-convergent to L), if for every $\varepsilon > 0$ and $\delta > 0$,

$$\left\{r \in \mathbb{N} : \frac{1}{{h_r}^\alpha} |\{k \in \mathrm{I}_r : |x_k - L| \geq \varepsilon\}| \geq \delta\right\} \in \mathrm{I}.$$

In this case, we write $S_\theta^\alpha(I)\text{-}\lim x = L$ or $x_k \to L\,(S_\theta^\alpha(I))$. The class of all I-lacunary statistically convergent sequences of order α will be denoted by $S_\theta^\alpha(I)$.

Definition 2.6. [22] A sequence $x = (x_k)$ is said to be weighted lacunary I-statistically convergent (or $S_{(R,\theta)}(I)$-convergent to L), if for every $\varepsilon > 0$ and $\delta > 0$,

$$\left\{r \in \mathbb{N} : \frac{1}{H_r} |\{k \in \mathrm{I}'_r : p_k |x_k - L| \geq \varepsilon\}| \geq \delta\right\} \in \mathrm{I}.$$

In this case, we write $S_{(R,\theta)}(I)\text{-}\lim x = L$ or $x_k \to L\left(S_{(R,\theta)}(I)\right)$. The class of all I-weighted lacunary statistically convergent sequences will be denoted by $S_{(R,\theta)}(I)$.

For $\mathrm{I} = \mathrm{I}_{fin}$, $S_{(R,\theta)}(I)$ convergence coincides with $S_{(R,\theta)}$. If $p_k = 1$ for all $k \in \mathbb{N}$, then $S_{(R,\theta)}(\mathrm{I})$-convergence reduces to $S_\theta(\mathrm{I})$-convergence.

3. Main Results

Definition 3.1. Let $\theta = (k_r)$ be a lacunary sequence and $0 < \alpha \leq 1$ be given. A sequence $x = (x_k)$ is said to be I-weighted lacunary statistically convergent of order α to L ($S_{(R,\theta)}^\alpha(I)$-convergent to L) if for every $\varepsilon > 0$ and $\delta > 0$,

$$\left\{r \in \mathbb{N} : \frac{1}{H_r^\alpha} |\{k \in I'_r : p_k |x_k - L| \geq \varepsilon\}| \geq \delta\right\} \in I.$$

In this case, we write $S_{(R,\theta)}^\alpha(I)\text{-}\lim x = L$ or $x_k \to L\left(S_{(R,\theta)}^\alpha(I)\right)$. The class of all I-weighted lacunary statistically convergent sequence of order α will be denoted by $S_{(R,\theta)}^\alpha(I)$.

Remark. (1) For $I = I_{fin}$, $S_{(R,\theta)}^\alpha(I)$ coincides with $S_{(R,\theta)}^\alpha$, that is:
The sequence $x = (x_k)$ is said to be weighted lacunary statistically convergent of order α to L ($S_{(R,\theta)}^\alpha$-convergent to L) if

$$\lim_{r\to\infty} \frac{1}{H_r^\alpha} |\{k \in I'_r : p_k |x_k - L| \geq \varepsilon\}| = 0.$$

The weighted lacunary statistical convergence of order α is well-defined for $0<\alpha\leq 1$, but it is not well-defined for $\alpha>1$ in general. To show this, let $(p_k)=\left(1+2^{-k}\right)$ for $k\in I_r$ and $(p_k)=0$ for otherwise and $x=(x_k)$ be defined as follows;

$$x_k=\begin{cases}1, & \text{if } k=2r\\ 0, & \text{if } k\neq 2r\end{cases}\qquad r=1,2,3,...$$

Then we have

$$P_{k_r}=\sum_{k\in(0,k_r]}p_k=\sum_{k=1}^{k_r}\left(1+2^{-k}\right)=k_r+1-2^{-k_r}$$

and

$$P_{k_{r-1}}=\sum_{k\in(0,k_{r-1}]}p_k=\sum_{k=1}^{k_{r-1}}\left(1+2^{-k}\right)=k_{r-1}+1-2^{-k_{r-1}},$$

hence

$$H_r=P_{k_r}-P_{k_{r-1}}=(k_r-k_{r-1})-\left[2^{-k_r}-2^{-k_{r-1}}\right]\leq(k_r-k_{r-1})\,.$$

Since for $k\in I_r$, $p_k=1+2^{-k}\geq 1$ then $H_r^{\alpha}\geq h_r^{\alpha}$ for $r\in\mathbb{N}$ and $\alpha>1$. Then both

$$\lim_{r\to\infty}\frac{1}{H_r^{\alpha}}\left|\left\{k\in {I_r}':(1+2^{-k})\left|x_k-1\right|\geq\varepsilon\right\}\right|\leq\lim_{r\to\infty}\frac{k_r-k_{r-1}}{H_r^{\alpha}}=\lim_{r\to\infty}\frac{h_r}{H_r^{\alpha}}\leq\lim_{r\to\infty}\frac{h_r}{h_r^{\alpha}}=0\quad(\alpha>1)$$

and

$$\lim_{r\to\infty}\frac{1}{H_r^{\alpha}}\left|\left\{k\in {I_r}':(1+2^{-k})\left|x_k-0\right|\geq\varepsilon\right\}\right|\leq\lim_{r\to\infty}\frac{k_r-k_{r-1}}{H_r^{\alpha}}=\lim_{r\to\infty}\frac{h_r}{H_r^{\alpha}}\leq\lim_{r\to\infty}\frac{h_r}{h_r^{\alpha}}=0\quad(\alpha>1)\,.$$

So it follows that $x_k\to 1\left(S_{(R,\theta)}^{\alpha}\right)$ and $x_k\to 0\left(S_{(R,\theta)}^{\alpha}\right)$ but this is impossible.

(2) For $I=I_{fin}$ and $\alpha=1$, $S_{(R,\theta)}^{\alpha}(I)$ coincides with $S_{(R,\theta)}$ [1].

(3) If $p_k=1$ for all $k\in\mathbb{N}$, then $S_{(R,\theta)}^{\alpha}(I)$ reduces to $S_{\theta}^{\alpha}(I)$ [10].

Theorem 3.2. Let $0<\alpha\leq 1$ and $x=(x_k)$, $y=(y_k)$ be sequences of real numbers, then the followings hold:

1. If $S_{(R,\theta)}^{\alpha}(I)$-$\lim x_k=x_0$ and $S_{(R,\theta)}^{\alpha}(I)$-$\lim y_k=y_0$ then $S_{(R,\theta)}^{\alpha}(I)$-$\lim(x_k+y_k)=x_0+y_0$.
2. If $S_{(R,\theta)}^{\alpha}(I)$-$\lim x_k=x_0$ and $c\in\mathbb{R}$ then $S_{(R,\theta)}^{\alpha}(I)$-$\lim cx_k=cx_0$.

Proof. (1) Let $S_{(R,\theta)}^{\alpha}(I)$-$\lim x_k=x_0$ and $S_{(R,\theta)}^{\alpha}(I)$-$\lim y_k=y_0$. Hence, for $\varepsilon>0$ and $\delta>0$, there exist A_1, $A_2\in F(I)$ such that

$$A(1)=\left\{r\in\mathbb{N}:\frac{1}{H_r^{\alpha}}\left|\left\{k\in {I'}_r:p_k\left|x_k-x_0\right|\geq\frac{\varepsilon}{2}\right\}\right|<\frac{\delta}{2}\right\}\in F(I)$$

and

$$A(2)=\left\{r\in\mathbb{N}:\frac{1}{H_r^{\alpha}}\left|\left\{k\in {I'}_r:p_k\left|y_k-y_0\right|\geq\frac{\varepsilon}{2}\right\}\right|<\frac{\delta}{2}\right\}\in F(I).$$

From the definition of $F(I)$, since A_1, $A_2\in F(I)$ then $A_1\cap A_2\in F(I)$ and $A_1\cap A_2\neq\varnothing$. From the following inequality

$$\begin{aligned}p_k\left|(x_k+y_k)-(x_0+y_0)\right|&=p_k\left|(x_k-x_0)+(y_k-y_0)\right|\\&\leq p_k\left|x_k-x_0\right|+p_k\left|y_k-y_0\right|,\end{aligned}$$

for $r \in A_1 \cap A_2$ we have

$$\frac{1}{H_r^\alpha}\left|\{k \in I'_r : p_k\left|(x_k+y_k)-(x_0+y_0)\right| \geq \varepsilon\}\right|$$
$$\leq \frac{1}{H_r^\alpha}\left|\left\{k \in I'_r : p_k\left|x_k-x_0\right| \geq \frac{\varepsilon}{2}\right\}\right| + \frac{1}{H_r^\alpha}\left|\left\{k \in I'_r : p_k\left|y_k-y_0\right| \geq \frac{\varepsilon}{2}\right\}\right|$$
$$< \frac{\delta}{2}+\frac{\delta}{2}=\delta.$$

Thus we obtain

$$\left\{r \in \mathbb{N} : \frac{1}{H_r^\alpha}\left|\{k \in I'_r : p_k\left|(x_k+y_k)-(x_0+y_0)\right| \geq \varepsilon\}\right| < \delta\right\} \in F(I)$$

and hence $S_{(R,\theta)}^\alpha(I)\text{-}\lim(x_k+y_k)=x_0+y_0$.

(2) Suppose that $S_{(R,\theta)}^\alpha(I)\text{-}\lim x_k = x_0$ and $c \in \mathbb{R}$. It is clear for the case $c=0$. Suppose that $c \neq 0$, since $S_{(R,\theta)}^\alpha(I)\text{-}\lim x_k = x_0$, then there exists $A(1) \in F(I)$ such that

$$A(1)=\left\{r \in : \frac{1}{H_r^\alpha}\left|\left\{k \in I'_r : p_k\left|x_k-x_0\right| \geq \frac{\varepsilon}{|c|}\right\}\right| < \frac{\delta}{2}\right\} \in F(I).$$

Then for every $r \in A(1)$ we have

$$\frac{1}{H_r^\alpha}\left|\{k \in I'_r : p_k\left|cx_k-cx_0\right| \geq \varepsilon\}\right| = \frac{1}{H_r^\alpha}\left|\left\{k \in I'_r : p_k\left|x_k-x_0\right| \geq \frac{\varepsilon}{|c|}\right\}\right| < \frac{\delta}{2}$$

and hence the result is obtained.

□

Theorem 3.3. If $0<\alpha\leq\beta\leq 1$ then $S_{(R,\theta)}^\alpha(I) \subset S_{(R,\theta)}^\beta(I)$.

Proof. The proof follows from the inequality

$$\frac{1}{H_r^\beta}\left|\{k \in I'_r : p_k\left|x_k-L\right| \geq \varepsilon\}\right| \leq \frac{1}{H_r^\alpha}\left|\{k \in I'_r : p_k\left|x_k-L\right| \geq \varepsilon\}\right|.$$

Thus for any $\delta>0$, we have

$$\left\{r \in \mathbb{N} : \frac{1}{H_r^\beta}\left|\{k \in I'_r : p_k\left|x_k-L\right| \geq \varepsilon\}\right| \geq \delta\right\}$$
$$\subseteq \left\{r \in \mathbb{N} : \frac{1}{H_r^\alpha}\left|\{k \in I'_r : p_k\left|x_k-L\right| \geq \varepsilon\}\right| \geq \delta\right\} \in I.$$

Since $x_k \to L\left(S_{(R,\theta)}^\alpha(I)\right)$, then the right-hand side of the set belongs to I. This completes the proof. □

Corollary 3.4. If a sequence $x=(x_k)$ is $S_{(R,\theta)}^\alpha(I)$-statistically convergent to L, then it is $S_{(R,\theta)}(I)$-statistically convergent to L.

Definition 3.5. (1) A sequence $x=(x_k)$ is said to be $(R,p_r,\theta)^\alpha(I)$-summable to L if $I\text{-}\lim_r W_r(x) \to L$ i.e. for any $\varepsilon>0$, $\{r \in \mathbb{N} : |W_r(x)-L| \geq \varepsilon\} \in I$ where $W_r(x) := \frac{1}{H_r^\alpha}\sum_{k \in I_r} p_k x_k$. In this case, we write $((R,p_r,\theta)^\alpha(I))\text{-}\lim x = L$ or $x_k \to L((R,p_r,\theta)^\alpha(I))$.

(2) A sequence $x=(x_k)$ is said to be $[R,p_r,\theta]^\alpha(I)$-summable (or strongly $(R,p_r,\theta)^\alpha(I)$-summable) to L if for any $\varepsilon>0$,

$$\left\{r \in \mathbb{N} : \frac{1}{H_r^\alpha}\sum_{k \in I_r} p_k\left|x_k-L\right| \geq \varepsilon\right\} \in I$$

In this case, we write $[R,p_r,\theta]^\alpha(I)\text{-}\lim x = L$ or $x_k \to L([R,p_r,\theta]^\alpha(I))$.

For $I=I_{fin}$ and $\alpha=1$, $(R,p_r,\theta)^{\alpha}(I)$-summability becomes (R,p_r,θ)-summability.
If $I=I_{fin}$ and $\alpha=1$, $[R,p_r,\theta]^{\alpha}(I)$-summability becomes $[R,p_r,\theta]$-summability.

Theorem 3.6. Let $I\subset P(\mathbb{N})$ be an admissible ideal, $\theta=(k_r)$ be a lacunary sequence and $I_r'\subset I_r$. Then $x_k\to L\left([R,p_r,\theta]^{\alpha}(I)\right)$ implies $x_k\to L\left(S_{(R,\theta)}^{\alpha}(I)\right)$.

Proof. Suppose $x_k\to L\left([R,p_r,\theta]^{\alpha}(I)\right)$ and $K_r(\varepsilon)$ and ${K_r}^{c}(\varepsilon)$ be defined as follows

$$K_r(\varepsilon):=\{k\in I_r':\,p_k\left|x_k-L\right|\geq\varepsilon\},\,{K_r}^{c}(\varepsilon):=\{k\in {I'}_r:p_k\left|x_k-L\right|<\varepsilon\}\quad(3.1)$$

Then we have

$$\begin{aligned}\tfrac{1}{H_r^{\alpha}}\sum_{k\in I_r}p_k\left|x_k-L\right| &\geq \tfrac{1}{H_r^{\alpha}}\sum_{k\in I_r'}p_k\left|x_k-L\right|\\ &=\tfrac{1}{H_r^{\alpha}}\sum_{\substack{k\in I_r'\\ k\in K_r(\varepsilon)}}p_k\left|x_k-L\right|+\tfrac{1}{H_r^{\alpha}}\sum_{\substack{k\in I_r'\\ k\in {K_r}^{c}(\varepsilon)}}p_k\left|x_k-L\right|\\ &\geq \tfrac{1}{H_r^{\alpha}}\sum_{\substack{k\in I_r'\\ k\in K_r(\varepsilon)}}p_k\left|x_k-L\right|=\varepsilon\tfrac{1}{H_r^{\alpha}}\left|\{k\in I_r':\,p_k\left|x_k-L\right|\geq\varepsilon\}\right|.\end{aligned}$$

So for a given $\delta>0$,

$$\frac{1}{H_r^{\alpha}}\left|\{k\in {I'}_r:p_k\left|x_k-L\right|\geq\varepsilon\}\right|\geq\delta\Rightarrow\frac{1}{H_r^{\alpha}}\sum_{k\in I_r}p_k\left|x_k-L\right|\geq\varepsilon\delta$$

i.e.

$$\left\{r\in\mathbb{N}:\frac{1}{H_r^{\alpha}}\left|\{k\in {I'}_r:p_k\left|x_k-L\right|\geq\varepsilon\}\right|\geq\delta\right\}\subset\left\{r\in\mathbb{N}:\frac{1}{H_r^{\alpha}}\sum_{k\in I_r}p_k\left|x_k-L\right|\geq\varepsilon\delta\right\}.$$

Since $x_k\to L\left([R,p_r,\theta]^{\alpha}(I)\right)$, the set on the right hand side belongs to I and so the result is obtained. □

Theorem 3.7. (1) Let $p_k\left|x_k-L\right|\leq M$ for all $k\in\mathbb{N}$ and $I_r\subset I_r'$. If $x_k\to L\left(S_{(R,\theta)}^{\alpha}(I)\right)$ then $x_k\to L\left([R,p_r,\theta]^{\alpha}(I)\right)$. (2) Let $p_k\left|x_k-L\right|\leq M$ for all $k\in\mathbb{N}$ and $I_r\subset I_r'$. If a sequence $x=(x_k)$ is $S_{(R,\theta)}^{\alpha}(I)$-convergent to L then it is $(R,p_r,\theta)^{\alpha}(I)$-summable to L.

Proof. (1) Suppose that $p_k\left|x_k-L\right|\leq M$ for all $k\in\mathbb{N}$ and $I_r\subset I_r'$. Let $x_k\to L\left(S_{(R,\theta)}^{\alpha}(I)\right)$ and $K_r(\varepsilon)$ and ${K_r}^{c}(\varepsilon)$ be defined as in 3.1. For each $\varepsilon>0$ we have

$$\begin{aligned}\tfrac{1}{H_r^{\alpha}}\sum_{k\in I_r}p_k\left|x_k-L\right| &\leq \tfrac{1}{H_r^{\alpha}}\sum_{k\in {I'}_r}p_k\left|x_k-L\right|\\ &=\tfrac{1}{H_r^{\alpha}}\sum_{\substack{k\in {I'}_r\\ k\in K_r(\varepsilon)}}p_k\left|x_k-L\right|+\tfrac{1}{H_r^{\alpha}}\sum_{\substack{k\in {I'}_r\\ k\in {K_r}^{c}(\varepsilon)}}p_k\left|x_k-L\right|\\ &\leq M\tfrac{1}{H_r^{\alpha}}\left|K_r(\varepsilon)\right|+\varepsilon.\end{aligned}$$

Since $x_k\to L\left(S_{(R,\theta)}^{\alpha}(I)\right)$,

$$A(\varepsilon):=\left\{r\in\mathbb{N}:\frac{1}{H_r^{\alpha}}\left|\{k\in {I'}_r:p_k\left|x_k-L\right|\geq\varepsilon\}\right|\geq\frac{\varepsilon}{M}\right\}\in I.$$

If $r\in A^{c}(\varepsilon)$, then

$$\frac{1}{H_r^{\alpha}}\sum_{k\in I_r}p_k\left|x_k-L\right|<2\varepsilon.$$

Consequently, we obtain

$$\left\{r\in\mathbb{N}:\ \frac{1}{H_r^\alpha}\sum_{k\in I_r}p_k\left|x_k-L\right|\geq 2\varepsilon\right\}\subset A(\varepsilon).$$

It follows that the latter set belongs to I which immediately implies

$$\left\{r\in\mathbb{N}:\ \frac{1}{H_r^\alpha}\sum_{k\in I_r}p_k\left|x_k-L\right|\geq \varepsilon\right\}\in I.$$

This shows that $x_k\to L\left([R,p_r,\theta]^\alpha(I)\right)$.

(2) Assume that $p_k\left|x_k-L\right|\leq M$ for all $k\in\mathbb{N}$ and $I_r\subset I_r'$. Let $K_r(\varepsilon)$ and ${K_r}^c(\varepsilon)$ be defined as in 3.1. Then

$$\begin{aligned}
|W_r(x)-L| &\leq \left|\tfrac{1}{H_r^\alpha}\sum_{k\in {I'}_r}p_kx_k-L\right| = \left|\tfrac{1}{H_r^\alpha}\sum_{k\in {I'}_r}p_k\left(x_k-L\right)\right| \\
&\leq \left|\tfrac{1}{H_r^\alpha}\sum_{\substack{k\in {I'}_r\\ k\in K_r(\varepsilon)}}p_k\left(x_k-L\right)+\tfrac{1}{H_r^\alpha}\sum_{\substack{k\in {I'}_r\\ {K_r}^c(\varepsilon)}}p_k\left(x_k-L\right)\right| \\
&\leq \tfrac{1}{H_r^\alpha}\sum_{\substack{k\in {I'}_r\\ k\in K_r(\varepsilon)}}p_k\left|x_k-L\right|+\tfrac{1}{H_r^\alpha}\sum_{\substack{k\in {I'}_r\\ {K_r}^c(\varepsilon)}}p_k\left|x_k-L\right| \\
&\leq \tfrac{M}{H_r^\alpha}\left|\left\{k\in {I'}_r:p_k\left|x_k-L\right|\geq\varepsilon\right\}\right|+\varepsilon.
\end{aligned}$$

Since $x_k\to L\left(S^\alpha_{(R,\theta)}(I)\right)$, there exists $A(\varepsilon)\in I$ such that

$$A(\varepsilon):=\left\{r\in\mathbb{N}:\frac{1}{H_r^\alpha}\left|\left\{k\in {I'}_r:p_k\left|x_k-L\right|\geq\varepsilon\right\}\right|\geq\frac{\varepsilon}{M}\right\}\in I.$$

If $r\in(A(\varepsilon))^c$ then $|W_r(x)-L|<2\varepsilon$. Hence

$$\{r\in\mathbb{N}:|W_r(x)-L|\geq 2\varepsilon\}\subset A(\varepsilon)$$

and so belongs to I. This shows that I-lim $(R,p_r,\theta)^\alpha(I)=L$, hence $x_k\to L\left((R,p_r,\theta)^\alpha(I)\right)$. □

Theorem 3.8. The following statements are true:

1. If $p_k\leq 1$ for all $k\in\mathbb{N}$ and $x_k\to L\left(S^\alpha_\theta(I)\right)$ then $x_k\to L\left(S^\alpha_{(R,\theta)}(I)\right)$.
2. Let $\frac{H_r}{h_r}$ be upper bounded. If $p_k\geq 1$ for all $k\in\mathbb{N}$ and $x_k\to L\left(S^\alpha_{(R,\theta)}(I)\right)$ then $x_k\to L\left(S^\alpha_\theta(I)\right)$.

Proof. (1) If $p_k\leq 1$ for all $k\in\mathbb{N}$ then $H_r\leq h_r$. Thus $(H_r)^\alpha\leq(h_r)^\alpha$ for all $r\in\mathbb{N}$. So, there exists M_1 constant such that $0<M_1\leq\frac{H_r}{h_r}\leq 1$ and hence $(M_1h_r)^\alpha\leq(H_r)^\alpha$ for all $r\in\mathbb{N}$. Let $x_k\to L\left(S^\alpha_\theta(I)\right)$ then for an arbitrary $\varepsilon>0$ we have

$$\begin{aligned}
\tfrac{1}{H_r^\alpha}\left|\left\{k\in I_r':\ p_k\left|x_k-L\right|\geq\varepsilon\right\}\right| &= \tfrac{1}{H_r^\alpha}\left|\left\{P_{k_{r-1}}<k\leq P_{k_r}:\ p_k\left|x_k-L\right|\geq\varepsilon\right\}\right| \\
&\leq \tfrac{1}{(M_1h_r)^\alpha}\left|\left\{P_{k_{r-1}}\leq k_{r-1}<k\leq P_{k_r}\leq k_r:\ p_k\left|x_k-L\right|\geq\varepsilon\right\}\right| \\
&\leq \tfrac{1}{M_1^\alpha}\tfrac{1}{h_r^\alpha}\left|\left\{k_{r-1}<k\leq k_r\ :\ \left|x_k-L\right|\geq\varepsilon\right\}\right| \\
&= \tfrac{1}{M_1^\alpha}\tfrac{1}{h_r^\alpha}\left|\left\{k\in I_r\ :\ \left|x_k-L\right|\geq\varepsilon\right\}\right|.
\end{aligned}$$

Thus for a given $\delta > 0$,

$$\frac{1}{H_r^\alpha}\left|\{k \in I_r' : p_k |x_k - L| \geq \varepsilon\}\right| \geq \delta \Rightarrow \frac{1}{h_r^\alpha}\left|\{k \in I_r \ : \ |x_k - L| \geq \varepsilon\}\right| \geq M_1^\alpha \delta.$$

Hence

$$\left\{r \in \mathbb{N} : \ \tfrac{1}{H_r^\alpha}\left|\{k \in I_r' : p_k |x_k - L| \geq \varepsilon\}\right| \geq \delta\right\} \subset \left\{r \in \mathbb{N} \ : \ \tfrac{1}{h_r^\alpha}\left|\{k \in I_r \ : \ |x_k - L| \geq \varepsilon\}\right| \geq M_1^\alpha \delta\right\}.$$

Since $x_k \to L\left(S_\theta^\alpha(I)\right)$, the set on the right hand side belongs to I and so it follows that $x_k \to L\left(S_{(R,\theta)}^\alpha(I)\right)$.

(2) If $p_k \geq 1$ for all $k \in \mathbb{N}$, then we have $H_r \geq h_r$ for all $r \in \mathbb{N}$. Let $\frac{H_r}{h_r}$ be upper bounded, so there exists M_2 constant such that $1 \leq \frac{H_r}{h_r} \leq M_2 < \infty \Rightarrow H_r \leq M_2 h_r$ for all $r \in \mathbb{N}$. Assume that $x = (x_k)$ converges to the limit L in $S_{(R,\theta)}^\alpha(I)$, then for an arbitrary $\varepsilon > 0$ we have

$$\begin{aligned}
\tfrac{1}{h_r^\alpha}\left|\{k \in I_r \ : \ |x_k - L| \geq \varepsilon\}\right| &= \tfrac{1}{h_r^\alpha}\left|\{k_{r-1} < k \leq k_r \ : \ |x_k - L| \geq \varepsilon\}\right| \\
&\leq \tfrac{M_2^\alpha}{H_r^\alpha}\left|\left\{k_{r-1} \leq P_{k_{r-1}} < k \leq k_r \leq P_{k_r} \ : \ p_k \ |x_k - L| \geq \varepsilon\right\}\right| \\
&= M_2^\alpha \cdot \tfrac{1}{H_r^\alpha}\left|\left\{P_{k_{r-1}} < k \leq P_{k_r} \ : \ p_k \ |x_k - L| \geq \varepsilon\right\}\right| \\
&= M_2^\alpha \cdot \tfrac{1}{H_r^\alpha}\left|\{k \in \ I_r' : \ p_k \ |x_k - L| \geq \varepsilon\}\right|.
\end{aligned}$$

Thus for a given $\delta > 0$,

$$\frac{1}{h_r^\alpha}\left|\{k \in I_r : |x_k - L| \geq \varepsilon\}\right| \geq \delta \Rightarrow \frac{1}{H_r^\alpha}\left|\{k \in I_r' : p_k |x_k - L| \geq \varepsilon\}\right| \geq \frac{\delta}{M_2^\alpha}.$$

Hence

$$\left\{r \in \mathbb{N} \ : \ \frac{1}{h_r^\alpha}\left|\{k \in I_r \ : \ |x_k - L| \geq \varepsilon\}\right| \geq \delta\right\} \subset \left\{r \in \mathbb{N} : \ \frac{1}{H_r^\alpha}\left|\{k \in I_r' : \ p_k |x_k - L| \geq \varepsilon\}\right| \geq \frac{\delta}{M_2^\alpha}\right\}.$$

Since $x_k \to L\left(S_{(R,\theta)}^\alpha(I)\right)$, the set on the right-hand side belongs to I and so it follows that $x_k \to L\left(S_\theta^\alpha(I)\right)$.

□

Theorem 3.9. If $\liminf\limits_{r\to\infty} \frac{H_r^\alpha}{P_{k_r}} > 0$ and $x_k \to L\left(S_R(I)\right)$ then $x_k \to L\left(S_{(R,\theta)}^\alpha(I)\right)$.

Proof. Suppose that $\liminf\limits_{r\to\infty} \frac{H_r^\alpha}{P_{k_r}} > 0$ then there exists a $\gamma > 0$ such that $\frac{H_r^\alpha}{P_{k_r}} \geq \gamma$ for sufficiently large values of r. Let $x = (x_k) \in S_R(I)$ with $S_R(I)$-$\lim x = L$, then for every $\varepsilon > 0$ and for sufficiently large values of r, we have

$$\begin{aligned}
\tfrac{1}{P_{k_r}}\left|\{k \leq P_{k_r} \ : \ p_k \ |x_k - L| \geq \varepsilon\}\right| &\geq \tfrac{1}{P_{k_r}}\left|\left\{P_{k_{r-1}} < k \leq P_{k_r} \ : \ p_k \ |x_k - L| \geq \varepsilon\right\}\right| \\
&= \tfrac{H_r^\alpha}{P_{k_r}}\left(\tfrac{1}{H_r^\alpha}\left|\left\{P_{k_{r-1}} < k \leq P_{k_r} \ : \ p_k \ |x_k - L| \geq \varepsilon\right\}\right|\right) \\
&\geq \gamma\left(\tfrac{1}{H_r^\alpha}\left|\left\{P_{k_{r-1}} < k \leq P_{k_r} \ : \ p_k \ |x_k - L| \geq \varepsilon\right\}\right|\right) \\
&= \gamma\left(\tfrac{1}{H_r^\alpha}\left|\{k \in I_r' \ : \ p_k \ |x_k - L| \geq \varepsilon\}\right|\right).
\end{aligned}$$

So for a given $\delta > 0$,

$$\frac{1}{H_r^{\alpha}} \left| \{ k \in I'_r \, : \, p_k \, |x_k - L| \geq \varepsilon \} \right| \geq \delta \Rightarrow \frac{1}{P_{k_r}} \left| \{ k \leq P_{k_r} \, : \, p_k \, |x_k - L| \geq \varepsilon \} \right| \geq \gamma\delta.$$

Hence

$$\begin{aligned} &\left\{ r \in \mathbb{N} : \tfrac{1}{H_r^{\alpha}} \left| \{ k \in I'_r \, : \, p_k \, |x_k - L| \geq \varepsilon \} \right| \geq \delta \right\} \\ &\subset \left\{ r \in \mathbb{N} : \tfrac{1}{P_{k_r}} \left| \{ k \leq P_{k_r} \, : \, p_k \, |x_k - L| \geq \varepsilon \} \right| \geq \gamma\delta \right\}. \end{aligned}$$

Since $x_k \to L \left(S_R \left(I \right) \right)$, the set on the right-hand side belongs to I and so it follows that $x_k \to L \left(S^{\alpha}_{(R,\theta)} \left(I \right) \right)$. □

Theorem 3.10. *Let $\theta = (k_r)$ and $\theta' = (s_r)$ be two lacunary sequences such that $I_r \subset J_r$ for all $r \in \mathbb{N}$ such that $I_r = (k_{r-1}, k_r]$ and $J_r = (s_{r-1}, s_r]$ and let α and β be such that $0 < \alpha \leq \beta \leq 1$. Then the followings hold.*

(1) *If $\lim_{r\to\infty} \inf \frac{H_r^{\alpha}}{L_r^{\beta}} > 0$, then $S^{\beta}_{(R,\theta')} \left(I \right) \subseteq S^{\alpha}_{(R,\theta)} \left(I \right)$,*

(2) *If $\lim_{r\to\infty} \frac{L_r}{H_r^{\beta}} = 1$, then $S^{\alpha}_{(R,\theta)} \left(I \right) \subset S^{\beta}_{(R,\theta')} \left(I \right)$.*

Proof. (1) Suppose that $\lim_{r\to\infty} \inf \frac{H_r^{\alpha}}{L_r^{\beta}} > 0$ then there exists a $\gamma > 0$ such that $\frac{H_r^{\alpha}}{L_r^{\beta}} \geq \gamma$ for sufficiently large values of r. Since $I_r \subset J_r$ for all $r \in \mathbb{N}$, then it clearly can be seen that $I'_r \subset J'_r$ (for all $r \in \mathbb{N}$) where $I'_r = \left(P_{k_{r-1}}, P_{k_r} \right]$ and $J'_r = \left(P_{s_{r-1}}, P_{s_r} \right]$ such that $P_{k_r} = \sum_{k\in(0,k_r]} p_k$, $P_{k_{r-1}} = \sum_{k\in(0,k_{r-1}]} p_k$, and $H_r = P_{k_r} - P_{k_{r-1}}$ and $L_r = P_{s_r} - P_{s_{r-1}}$. For given $\varepsilon > 0$, we have

$$\{ k \in J'_r : p_k \, |x_k - L| \geq \varepsilon \} \supseteq \{ k \in I'_r : p_k \, |x_k - L| \geq \varepsilon \}$$

and so

$$\begin{aligned} \tfrac{1}{L_r^{\beta}} \left| \{ k \in J'_r : p_k \, |x_k - L| \geq \varepsilon \} \right| &\geq \tfrac{H_r^{\alpha}}{L_r^{\beta}} \tfrac{1}{H_r^{\alpha}} \left| \{ k \in I'_r : p_k \, |x_k - L| \geq \varepsilon \} \right| \\ &\geq \gamma \tfrac{1}{H_r^{\alpha}} \left| \{ k \in I'_r : p_k \, |x_k - L| \geq \varepsilon \} \right|. \end{aligned}$$

Then we obtain

$$\frac{1}{H_r^{\alpha}} \left| \{ k \in I'_r : p_k \, |x_k - L| \geq \varepsilon \} \right| \geq \frac{\delta}{\gamma} \Rightarrow \frac{1}{L_r^{\beta}} \left| \{ k \in J'_r : p_k \, |x_k - L| \geq \varepsilon \} \right| \geq \delta.$$

Since $(x_k) \in S^{\beta}_{(R,\theta')} \left(I \right)$, then there exists $A(\varepsilon) \in I$ such that

$$A(\varepsilon) := \left\{ r \in \mathbb{N} : \frac{1}{L_r^{\beta}} \left| \{ k \in J'_r : p_k \, |x_k - L| \geq \varepsilon \} \right| \geq \delta \right\} \in I.$$

Hence

$$\left\{ r \in \mathbb{N} : \frac{1}{H_r^{\alpha}} \left| \{ k \in I'_r : p_k \, |x_k - L| \geq \varepsilon \} \right| \geq \frac{\delta}{\gamma} \right\} \subset A(\varepsilon) \in I.$$

Since $A(\varepsilon) \in I$ then the set on the left hand side belongs to I and so it follows that $(x_k) \in S^{\alpha}_{(R,\theta)} \left(I \right)$.

(2) Let $(x_k) \in S^{\alpha}_{(R,\theta)}(I)$ and $\lim\limits_{r\to\infty} \frac{L_r}{H_r^{\beta}} = 1$ be satisfied. Since $I_r \subset J_r$ for all $r \in \mathbb{N}$, then we have $I'_r \subset J'_r$ for all $r \in \mathbb{N}$. For every $\varepsilon > 0$ we may write,

$$\begin{aligned}
\frac{1}{L_r^{\beta}} \left|\left\{k \in J'_r : p_k \left|x_k - L\right| \geq \varepsilon\right\}\right| &= \frac{1}{L_r^{\beta}} \left|\left\{P_{s_{r-1}} < k \leq P_{k_{r-1}} : p_k \left|x_k - L\right| \geq \varepsilon\right\}\right| \\
&+ \frac{1}{L_r^{\beta}} \left|\left\{P_{k_r} < k \leq P_{s_r} : p_k \left|x_k - L\right| \geq \varepsilon\right\}\right| \\
&+ \frac{1}{L_r^{\beta}} \left|\left\{P_{k_{r-1}} < k \leq P_{k_r} : p_k \left|x_k - L\right| \geq \varepsilon\right\}\right| \\
&\leq \frac{P_{k_{r-1}} - P_{s_{r-1}}}{L_r^{\beta}} + \frac{P_{s_r} - P_{k_r}}{L_r^{\beta}} + \frac{1}{L_r^{\beta}} \left|\left\{k \in I'_r : p_k \left|x_k - L\right| \geq \varepsilon\right\}\right| \\
&= \frac{L_r - H_r}{L_r^{\beta}} + \frac{1}{L_r^{\beta}} \left|\left\{k \in I'_r : p_k \left|x_k - L\right| \geq \varepsilon\right\}\right| \\
&\leq \frac{L_r - H_r^{\beta}}{H_r^{\beta}} + \frac{1}{H_r^{\beta}} \left|\left\{k \in I'_r : p_k \left|x_k - L\right| \geq \varepsilon\right\}\right| \\
&\leq \left(\frac{L_r}{H_r^{\beta}} - 1\right) + \frac{1}{H_r^{\alpha}} \left|\left\{k \in I'_r : p_k \left|x_k - L\right| \geq \varepsilon\right\}\right|
\end{aligned}$$

for all $r \in \mathbb{N}$. Then we have

$$\left\{r \in \mathbb{N} : \frac{1}{L_r^{\beta}} \left|\left\{k \in J'_r : p_k \left|x_k - L\right| \geq \varepsilon\right\}\right| \geq \delta\right\} \subset \left\{r \in \mathbb{N} : \frac{1}{H_r^{\alpha}} \left|\left\{k \in I'_r : p_k \left|x_k - L\right| \geq \varepsilon\right\}\right| \geq \delta\right\} \in I.$$

Since $\lim\limits_{r\to\infty} \frac{L_r}{H_r^{\beta}} = 1$ and $x = (x_k) \in S^{\alpha}_{(R,\theta)}(I)$, then $S^{\alpha}_{(R,\theta)}(I) \subseteq S^{\beta}_{(R,\theta')}(I)$. □

Corollary 3.11. *Let $\theta = (k_r)$ and $\theta' = (s_r)$ be two lacunary sequences such that $I_r \subset J_r$ for all $r \in \mathbb{N}$. If $\lim\limits_{r\to\infty} \inf \frac{H_r^{\alpha}}{L_r^{\alpha}} > 0$, then*

(1) *$S^{\alpha}_{(R,\theta')}(I) \subseteq S^{\alpha}_{(R,\theta)}(I)$ for each $\alpha \in (0,1]$*
(2) *$S_{(R,\theta')}(I) \subseteq S^{\alpha}_{(R,\theta)}(I)$ for each $\alpha \in (0,1]$*
(3) *For $\alpha = 1$, $S_{(R,\theta')}(I) \subseteq S_{(R,\theta)}(I)$.*

If $\lim\limits_{r\to\infty} \frac{L_r}{H_r^{\alpha}} = 1$, then

(1) *$S^{\alpha}_{(R,\theta)}(I) \subset S^{\alpha}_{(R,\theta')}(I)$ for each $\alpha \in (0,1]$.*
(2) *$S^{\alpha}_{(R,\theta)}(I) \subset S_{(R,\theta')}(I)$ for each $\alpha \in (0,1]$.*
(3) *For $\alpha = 1$, $S_{(R,\theta)}(I) \subset S_{(R,\theta')}(I)$.*

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