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MIXED FORM OF HILBERT INTEGRAL INEQUALITY

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Abstract. A mixed form of Hilbert integral inequality are given with a best constant factor.

1. Introduction

Suppose that $p > 1, \frac{1}{p} + \frac{1}{q} = 1$, f and g are positive functions such that $f \in L_p(\mathbb{R}_+)$ and $g \in L_q(\mathbb{R}_+)$, the famous Hardy-Hilbert's inequality is given as

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy < \frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \|f\|_p \|g\|_q , \tag{1.1}$$

the constant $\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}$ is the best possible [1]. Inequality (1.1) was extended in different ways. Krnic et al. [2] adding some parameters to introducing the following best extension of (1.1)

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda} dxdy < B(1-pA_2, \lambda+pA_2-1) \left\{ \int_0^\infty x^{pqA_1-1} f^p(x)dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty y^{pqA_2-1} g^q(x)dx \right\}^{\frac{1}{q}}, \tag{1.2}$$

where $B(1-pA_2, \lambda+pA_2-1)$ is the best possible constants ($B(x,y)$ is the Beta function), $\lambda > 0$, $A_1 \in (\frac{1-\lambda}{q}, \frac{1}{q})$, $A_2 \in (\frac{1-\lambda}{p}, \frac{1}{p})$ and $pA_2 + qA_1 = 2 - \lambda$. For $0 < p < 1, \frac{1}{p} + \frac{1}{q} = 1, \lambda > 0$, $A_1 \in (\frac{1}{q}, \frac{1-\lambda}{q})$, $A_2 \in (\frac{1-\lambda}{p}, \frac{1}{p})$ and $pA_2 + qA_1 = 2 - \lambda$ the reverse form of (1.2) is also valid.

Refinements of some Hilbert-type inequalities by virtue of various methods were obtained in [3] and [4]. A survey of some recent results concerning Hilbert and Hilbert- type inequalities can be found in [5,6,7,8] .

If $p > 1, f(x) > 0$, and $F(x) = \int_0^x f(t)dt$, then the well-known Hardy inequality [1] is given as

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$$\int_0^\infty \left(\frac{F(x)}{x}\right)^p dx < \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)dx, \tag{1.3}$$

where the constant $\left(\frac{p}{p-1}\right)^p$ is the best possible. A weighted form of (1.3) is giving also by Hardy [1] as

$$\int_0^\infty x^a \left(\frac{F(x)}{x}\right)^p dx < \left(\frac{p}{p-1-a}\right)^p \int_0^\infty x^a f^p(x)dx, \tag{1.4}$$

where $p>1, a<p-1$ or $p<0, a>p-1$ and the constant $\left(\frac{p}{p-1-a}\right)^p$ is the best possible. Inequality (1.3) was discovered by Hardy while he was trying to introduce a simple proof of Hilbert inequality.

For the history and development of (1.3), we recommend interested readers to see the papers [10,11].

Recently in [12] for $f,g>0$, $f,g\in L(0,\infty)$, $F(x)=\int_0^x f(u)du$ and $G(x)=\int_0^x g(u)du$, $\lambda>0$, the following form of (1.1) is obtained

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda}dxdy \le \frac{\lambda^2}{pq}B\left(\frac{\lambda}{q},\frac{\lambda}{p}\right)\left(\int_0^\infty x^{-\lambda-1}F^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^\infty y^{-\lambda-1}G^q(y)dy\right)^{\frac{1}{q}}, \tag{1.5}$$

where the constant $\frac{\lambda^2}{pq}B\left(\frac{\lambda}{q},\frac{\lambda}{p}\right)$ is the best possible. By introducing a new parameter $\gamma\in\left(\frac{-\lambda}{p},\frac{\lambda}{q}\right)$ an extended form of (1.5) was given in [13,14] as

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda}dxdy \le C\left(\int_0^\infty x^{-\lambda-1-p\gamma}F^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^\infty y^{-\lambda-1+q\gamma}G^q(y)dy\right)^{\frac{1}{q}}, \tag{1.6}$$

where $C=\left(\frac{\lambda}{p}+\gamma\right)\left(\frac{\lambda}{q}-\gamma\right)B\left(\frac{\lambda}{p}+\gamma,\frac{\lambda}{q}-\gamma\right)$ is the best possible constant. Also, in [12] a differential form of the Hilbert's inequality was obtained, namely

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda}dxdy \le L\left(\int_0^\infty x^{p(n+1)-\lambda-1}\left(f^{(n)}(x)\right)^p dx\right)^{\frac{1}{p}}\left(\int_0^\infty y^{q(n+1)-\lambda-1}\left(g^{(n)}(y)\right)^q dy\right)^{\frac{1}{q}}, \tag{1.7}$$

where the constant $L=\frac{\Gamma\left(\frac{\lambda}{p}-n\right)\Gamma\left(\frac{\lambda}{q}-n\right)}{\Gamma(\lambda)}$ is the best possible. Here, $\Gamma(u)$ is the gamma function, $\lambda>0$, $n=0,1,\ldots$. Note that if we let $n=0$, $\lambda=1$, and $p=q=2$ we obtain the classical Hilbert's inequality. It should be mentioned here that [14] extended it to a general homogeneous functions. In this paper we introduce some mixed forms of Hilbert's inequality which includes (1.6) and (1.7). The obtained inequalities are with a best constant factor.

2. Preliminaries and Lemmas

Recall that the Gamma function $\Gamma(\theta)$ and the Beta function $B(\mu,\nu)$ are defined respectively by

$$\Gamma(\theta)=\int_0^\infty t^{\theta-1}e^{-t}dt,\quad \theta>0,$$

$$B(\mu,\nu)=\int_0^\infty \frac{t^{\mu-1}}{(t+1)^{\mu+\nu}}dt,\quad \mu,\nu>0.$$

By the definition of the gamma function, we may write

$$\frac{1}{(x+y)^{\lambda}}=\frac{1}{\Gamma(\lambda)}\int_0^\infty t^{\lambda-1}e^{-(x+y)t}dt. \tag{2.1}$$

We will need the following two Lemmas which are given in [9]:

Lemma.2.1[12]. *Let* $r>1$, $\frac{1}{r}+\frac{1}{s}=1$, $\varphi>0, \varphi\in L(0,\infty)$, $\Phi(x)=\int_0^x\varphi(u)du$, *then for* $t>0$ *we have*

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$$\int_0^\infty e^{-tx}\varphi(x)dx\le t^{\frac{1}{r}-\alpha}\Gamma(\alpha s+1)^{\frac{1}{s}}\left\{\int_0^\infty x^{-\alpha r}e^{-tx}\Phi^r(x)dx\right\}^{\frac{1}{r}}. \tag{2.2}$$

Lemma.2.2[12]. *Let* $r>1$, $\frac{1}{r}+\frac{1}{s}=1$, $\varphi>0$ *, the derivatives* $\varphi',\varphi'',...,\varphi^{(n)}$ *exists and positive and* $\varphi^{(n)}\in L(0,\infty)\,(n=0,1,...)\left(\varphi^{(0)}:=\varphi\right)$*, moreover, suppose that* $\varphi(0)=\varphi'(0)=\cdots=\varphi^{(n-1)}=0$*, then for* $t,\alpha>0$ *we have*

Body Math

$$\int_0^\infty e^{-tx}\varphi(x)dx\le t^{-n-\frac{1}{r}-\alpha}\Gamma(\alpha s+1)^{\frac{1}{s}}\left\{\int_0^\infty x^{-\alpha r}e^{-tx}\left(\varphi^{(n)}(x)\right)^r dx\right\}^{\frac{1}{r}}. \tag{2.3}$$

3. Main Results

Theorem 3.1. *Let* $p>1, \frac{1}{p}+\frac{1}{q}=1$, $\lambda>0$, $-\frac{\lambda}{p}<\gamma<\frac{\lambda}{q}-n$. Suppose that f satisfies the conditions of φ *in Lamma 2.1 and* g satisfies the conditions of φ *in Lamma 2.2 .*

If $\int_0^\infty x^{-\lambda-p\gamma-1}F^p(x)dx<\infty$ *and* $\int_0^\infty y^{q(\gamma+n+1)-\lambda-1}(g^{(n)}(y))^q dy<\infty$*, then*

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x+y)^{\lambda}}dxdy\le C\left(\int_0^\infty x^{-\lambda-p\gamma-1}F^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^\infty y^{q(\gamma+n+1)-\lambda-1}(g^{(n)}(y))^q dy\right)^{\frac{1}{q}}, \tag{3.1}$$

where the constant $C=\left(\frac{\lambda}{p}+\gamma\right)\frac{\Gamma(\frac{\lambda}{p}+\gamma)\Gamma(\frac{\lambda}{q}-\gamma-n)}{\Gamma(\lambda)}$ *is the best possible.*

Proof. By using (2.1) and applying Hölder's inequality, we have

$$\begin{aligned} I =: \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda} dxdy &= \frac{1}{\Gamma(\lambda)} \int_0^\infty \int_0^\infty f(x)g(y) \left(\int_0^\infty t^{\lambda-1} e^{-(x+y)t} dt \right) dxdy \\ &= \frac{1}{\Gamma(\lambda)} \int_0^\infty \left(t^{\frac{\lambda-1}{p}+\gamma} \int_0^\infty e^{-xt} f(x)dx \right) \left(t^{\frac{\lambda-1}{q}-\gamma} \int_0^\infty e^{-yt} g(y)dy \right) dt \\ &\leq \frac{1}{\Gamma(\lambda)} \left(\int_0^\infty t^{\lambda-1+p\gamma} \left(\int_0^\infty e^{-xt} f(x)dx \right)^p dt \right)^{\frac{1}{p}} \\ &\quad \times \left(\int_0^\infty t^{\lambda-1-q\gamma} \left(\int_0^\infty e^{-yt} g(y)dy \right)^q dt \right)^{\frac{1}{q}}. \end{aligned} \tag{3.1}$$

By Lemma 2.1 for $r = p, s = q$, we get

$$\left(\int_0^\infty e^{-xt} f(x)dx \right)^p \leq t^{1-p\alpha} \Gamma(q\alpha+1)^{\frac{p}{q}} \int_0^\infty x^{-p\alpha} e^{-tx} F^p(x)dx$$

and by Lemma 2.2 for $r = p, s = q$ we obtain

$$\left(\int_0^\infty e^{-yt} g(y)dy \right)^q \leq t^{-qn-q\beta-q+1} \Gamma(p\beta+1)^{\frac{q}{p}} \int_0^\infty y^{-q\beta} e^{-ty} \left(g^{(n)}(y) \right)^q dy.$$

Substituting these two inequalities in (3.2) we have

$$\begin{aligned} I &\leq \frac{\Gamma(q\alpha+1)^{\frac{1}{q}} \Gamma(p\beta+1)^{\frac{1}{p}}}{\Gamma(\lambda)} \left(\int_0^\infty x^{-p\alpha} F^p(x) \int_0^\infty t^{\lambda+p(\gamma-\alpha)} e^{-tx} dtdx \right)^{\frac{1}{p}} \\ &\quad \times \left(\int_0^\infty y^{-q\beta} (g^{(n)})^q \int_0^\infty t^{\lambda-q(\gamma+\beta+n)-q} e^{-ty} dtdy \right)^{\frac{1}{q}} \\ &= D \left(\int_0^\infty x^{-\lambda-p\gamma-1} F^p(x)dx \right)^{\frac{1}{p}} \left(\int_0^\infty y^{q(\gamma+n+1)-\lambda-1} (g^{(n)}(y))^q dy \right)^{\frac{1}{q}}, \end{aligned}$$

where $D = \frac{\Gamma(q\alpha+1)^{\frac{1}{q}} \Gamma(p\beta+1)^{\frac{1}{p}} \Gamma(\lambda+p\gamma-p\alpha+1)^{\frac{1}{p}} \Gamma(\lambda-q\gamma-q\beta-q(n+1)+1)^{\frac{1}{q}}}{\Gamma(\lambda)}$. Now if we set $\alpha = \frac{\lambda+p\gamma}{pq}$ and $\beta = \frac{\lambda-q(\gamma+n+1)}{pq}$, we find $D = C$. Thus inequality (3.1) is proved. We need to prove that the constant factor C contained in (3.1) is the best possible. For $0 < \varepsilon < \lambda - q\gamma$ we define two functions $f_\varepsilon(x) = \frac{\lambda+p\gamma-\varepsilon}{p} x^{\frac{\lambda+p\gamma-\varepsilon}{p}-1}$ and $g_\varepsilon(x) = \frac{\Gamma(\frac{\lambda-q\gamma-\varepsilon}{q}-n)}{\Gamma(\frac{\lambda-q\gamma-\varepsilon}{q})} x^{\frac{\lambda-q\gamma-\varepsilon}{q}-1}$ for $x \geq 1$ and $f_\varepsilon(x) = g_\varepsilon(x) = 0$ for $x \in (0,1)$. Then, we get $F_\varepsilon(x) = \left(x^{\frac{\lambda+p\gamma-\varepsilon}{p}} - 1 \right)$ for $x \geq 1$, $F_\varepsilon(x) = 0$ for $x \in (0,1)$ and $g_\varepsilon^{(n)}(x) =$

$x^{\frac{\lambda-q\gamma-\varepsilon}{q}-n-1}$ for $x\geq 1$ and $g_{\varepsilon}^{(n)}(x)=0$ for $x\in(0,1)$. Suppose that C is not the best possible, then there exist $0<K<C$ such that

$$\int_1^\infty\int_1^\infty \frac{f_\varepsilon(x)g_\varepsilon(y)}{(x+y)^\lambda}dxdy < K\left(\int_1^\infty x^{-\lambda-p\gamma-1}\left(x^{\frac{\lambda+p\gamma-\varepsilon}{p}}-1\right)^p dx\right)^{\frac{1}{p}}$$

$$\times\left(\int_1^\infty y^{q(\gamma+n+1)-\lambda-1}(y^{\frac{\lambda-q\gamma-\varepsilon}{q}-n-1})^q dy\right)^{\frac{1}{q}}$$

$$< K\left(\int_1^\infty x^{-\lambda-p\gamma-1}x^{\lambda+p\gamma-\varepsilon}dx\right)^{\frac{1}{p}}\left(\int_1^\infty y^{q(\gamma+n+1)-\lambda-1}y^{\lambda-q\gamma-\varepsilon-q(n+1)}dy\right)^{\frac{1}{q}}$$

$$= K\left(\int_1^\infty x^{-\varepsilon-1}dx\right)^{\frac{1}{p}}\left(\int_1^\infty y^{-\varepsilon-1}dy\right)^{\frac{1}{q}} = \frac{K}{\varepsilon}. \tag{3.3}$$

On the other hand, we have $\left(\theta(\varepsilon)=\frac{\lambda+p\gamma-\varepsilon}{p}\frac{\Gamma(\frac{\lambda-q\gamma-\varepsilon}{q}-n)}{\Gamma(\frac{\lambda-q\gamma-\varepsilon}{q})}\right)$

$$\int_1^\infty\int_1^\infty \frac{f_\varepsilon(x)g_\varepsilon(y)}{(x+y)^\lambda}dxdy = \theta(\varepsilon)\int_1^\infty\int_1^\infty \frac{x^{\frac{\lambda+p\gamma-\varepsilon}{p}-1}y^{\frac{\lambda-q\gamma-\varepsilon}{q}-1}}{(x+y)^\lambda}dxdy$$

$$= \theta(\varepsilon)\int_1^\infty x^{-\varepsilon-1}\int_{\frac{1}{x}}^\infty \frac{u^{\frac{\lambda-q\gamma-\varepsilon}{q}-1}}{(u+1)^\lambda}dudx$$

$$= \theta(\varepsilon)\int_1^\infty x^{-\varepsilon-1}\left\{\int_0^\infty \frac{u^{\frac{\lambda-q\gamma-\varepsilon}{q}-1}}{(u+1)^\lambda}du - \int_0^{\frac{1}{x}} \frac{u^{\frac{\lambda-q\gamma-\varepsilon}{q}-1}}{(u+1)^\lambda}du\right\}dx$$

$$= \theta(\varepsilon)\frac{B\left(\frac{\lambda-q\gamma-\varepsilon}{q},\frac{\lambda+p\gamma}{p}+\frac{\varepsilon}{q}\right)}{\varepsilon} - \theta(\varepsilon)\int_1^\infty x^{-\varepsilon-1}\int_0^{\frac{1}{x}} \frac{u^{\frac{\lambda-q\gamma-\varepsilon}{q}-1}}{(u+1)^\lambda}dudx$$

$$> \theta(\varepsilon)\frac{B\left(\frac{\lambda-q\gamma-\varepsilon}{q},\frac{\lambda+p\gamma}{p}+\frac{\varepsilon}{q}\right)}{\varepsilon} - \theta(\varepsilon)\int_1^\infty x^{-\varepsilon-1}\int_0^{\frac{1}{x}} u^{\frac{\lambda-q\gamma-\varepsilon}{q}-1}dudy$$

$$= \theta(\varepsilon)\frac{B\left(\frac{\lambda-q\gamma-\varepsilon}{q},\frac{\lambda+p\gamma}{p}+\frac{\varepsilon}{q}\right)}{\varepsilon} - O(1). \tag{3.4}$$

Clearly, when $\varepsilon \to 0^{+}$from (3.3) and (3.4) we obtain a contradiction. Thus the proof of the theorem is completed.

Remark.

If we put $n = 0$ in (3.1) we get

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda} dxdy \le C_1 \left(\int_0^\infty x^{-\lambda-p\gamma-1} F^p(x)dx \right)^{\frac{1}{p}} \left(\int_0^\infty y^{q(\gamma+1)-\lambda-1} g^q(y)dy \right)^{\frac{1}{q}} \tag{3.5}$$

where $C_1 = \left(\frac{\lambda}{p} + \gamma\right) B(\frac{\lambda}{p} + \gamma, \frac{\lambda}{q} - \gamma)$ is the best possible constant. If we apply the weighted Hardy inequality (1.4) to the first integral on the right of (3.5) and let $\gamma = \frac{p-\lambda-pqA_1}{p} \left(-\frac{\lambda}{p} < \gamma < \frac{\lambda}{q}\right)$ under the condition $pA_2 + qA_1 = 2 - \lambda$ we get inequality (1.2) from the introduction.

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