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MATHEMATICAL ANALYSIS AND PHYSICAL INTERPRETATION ON NEW MULTIPLE SOLITONIC SOLUTIONS OF N-COUPLED MODIFIED KDV SYSTEM

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Abstract. In this paper, the n-coupled modified KdV system (nc-mKdV) is considered. A combination of Backlund transformations and simplified Hirota's method is used to construct new multiple-soliton and multiple-singular-soliton solutions of the proposed model. Dispersion relations on the effects of the inhomogeneities of the model "due to the variable coefficients" are derived and interpreted for deterministic of the characteristic-line and velocity of each obtained soliton-waves.

1. Introduction

Complex phenomena in various fields of sciences especially in physics are modeled by nonlinear partial differential equations (PDEs). A progress on better understanding of the realistic of such phenomena urged researchers to seek solutions of these PDEs by constructing and conducting well-posed mathematical methods. Such methods are, the sine-cosine method, rational sine-cosine, tanh method, extended tanh method, sech-tanh method, Exp-function method, the first integral method, the (G'/G)-expansion method and others [38, 39, 40, 37, 2, 3, 4, 5, 30, 6, 7, 8, 9, 29, 31, 10, 11, 27, 28, 26]. The aforementioned methods are classified into two types: The first type, the solution of the nonlinear equation is sought to be a special structure in terms of trigonometric or hyperbolic functions. The second type, the solution of the PDE is sought to be a finite series/polynomial of degree n in terms of a trigonometric/hyperbolic function where the order of this series/polynomial is to be determined by a balance procedure.

A more powerful method that produces multiple soliton solutions of different types and more realistic to the field of nonlinear evolutionary systems is called simplified Hirota's method. This latter method has been established by Hirota and been adopted by Wazwaz and other researches to study a wide nonlinear mathematical models arise in the science of physics [16, 17, 18, 19, 20, 21, 22, 12, 23, 14, 15, 13, 33, 34, 35, 24, 25, 1].

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In this work, we proceed with this trend and we will explore new multiple solitonic solutions to the new n-coupled modified KdV (nc-mkdV) with time variable coefficients that reads as

$$
\begin{aligned}
&0 = (v_1)_t + f(t)(v_1)_{xxx} + g(t)v_1^2(v_1)_x,\\
&0 = (v_2)_t + f(t)(v_2)_{xxx} + g(t)v_2^2(v_2)_x + g(t)(v_1^2 v_2)_x + g(t)(v_2^2 v_1)_x,\\
&0 = (v_3)_t + f(t)(v_3)_{xxx} + g(t)v_3^2(u_3)_x + g(t)((v_1+v_2)^2 v_3)_x\\
&\quad + g(t)((v_1+v_2)v_3^2)_x,\\
&0 = (v_4)_t + f(t)(v_4)_{xxx} + g(t)v_4^2(u_4)_x + g(t)((v_1+v_2+v_3)^2 v_4)_x\\
&\quad + g(t)((v_1+v_2+v_3)v_4^2)_x,\\
&\quad \vdots\\
&0 = (v_n)_t + f(t)(v_n)_{xxx} + g(t)v_n^2(v_n)_x + g(t)((\sum_{k=1}^{n-1} v_k)^2 v_n)_x\\
&\quad + g(t)((\sum_{k=1}^{n-1} v_k)v_n^2)_x,
\end{aligned}
\tag{1.1}
$$

where the amplitude of the relevant wave models $v_1(x,t)$, $v_2(x,t)$.......$v_n(x,t)$, and $f(t), g(t)$ are time-dependent functions. In the case where $f(t) = 1, g(t) = 6$ coupling (1.1) reduces to the classical coupled that solved by Wazwaz [36]. The main contribution of this paper is to derive solitons and singular soliton solutions of the coupling system (1.1) by means of the Backlund transformations and simplified Hirota's method under the constraint condition

$$f(t) = c\ g(t), \tag{1.2}$$

We would like to mention that the results in this paper are new and there are no previous studies of this system discussed under the condition given by (1.2).

This paper is organized as follows: A new Multiple-soliton solutions and Multiple-singular-soliton solutions for the (nc-mkdV) system (1.1) are constructed in Section 2 and 3. The last section is devoted for discussion and concluding remarks regards the effects of the variable coefficients and the collision behavior and propagation properties.

2. Multiple-soliton solutions for the nc-mKdV system

In this section, we apply the simplified bilinear method to construct Multiple-soliton solutions of nc-mKdV system (1.1). If we substitute

$$v_j(x,t) = e^{\theta_j(x,t)}, \quad j = 1,2,3,.....,n$$
$$\text{with} \ \ \theta_j(x,t) = h_j x - \omega_j(t)$$

into the linear terms of Eq. (1.1), we get the dispersion relation as follows

$$\omega_j(t) = \int h_j^3 f(t) dt. \tag{2.1}$$

Thus, we obtain

$$\theta_j(x,t) = h_j x - \int h_j^3 f(t) dt. \tag{2.2}$$

Assuming the multiple-soliton solutions (1.1) are

$$v_j(x,t) = R_j(\tan^{-1}(\frac{a(x,t)}{b(x,t)}))_x, \quad j = 1,2,3,.....,n, \tag{2.3}$$

where $a(x,t)$ and $b(x,t)$ for single-soliton solutions, is given by

$$\begin{aligned} a(x,t) &= e^{\theta_1(x,t)} = e^{h_1 x - \int h_j^3 f(t)dt}, \\ b(x,t) &= 1. \end{aligned} \tag{2.4}$$

Substituting (2.3) and (2.4) into (1.1), then solving for $R_1, R_2, R_3, ..., R_n$ we find two sets distinct none zero solutions given by

$$R_j = \pm(-1)^{j+1}\sqrt{\frac{24f(t)}{g(t)}}, \quad j = 1,2,3,.....,n \tag{2.5}$$

and

$$R_j = \begin{cases} \pm\sqrt{\frac{24f(t)}{g(t)}}, & j = 1 \\ \pm(-1)^{j+1}\sqrt{\frac{96f(t)}{g(t)}}, & j = 2,3,4,.....,n \end{cases} \tag{2.6}$$

To obtain a numerical value for the first and second sets of of the value of R_j, we set the constraint $\frac{f(t)}{g(t)} = c$, where c are arbitrary constants. That means the constraint condition for (1.1) given by $f(t) = c\, g(t)$.

Now, substituting (2.5) into (2.3), we obtain two sets of single-soliton solutions for (1.1)

$$\left.\begin{aligned} v_j(x,t) &= \pm(-1)^{j+1}\sqrt{\tfrac{24f(t)}{g(t)}} \times h_1 \tfrac{e^{\theta_1(x,t)}}{\left(1+e^{2\theta_1(x,t)}\right)}, \\ &= \pm(-1)^{j+1}\sqrt{\tfrac{24f(t)}{g(t)}} \times h_1 \frac{e^{h_1 x - \int h_j^3 f(t)dt}}{1+e^{2h_1 x - 2\int h_j^3 f(t)dt}}, \\ &= \pm(-1)^{j+1}\sqrt{\tfrac{6f(t)}{g(t)}} \times h_1 \text{sech}\,(\theta_1(x,t)),\ j = 1,2,3,.....,n \end{aligned}\right\} \tag{2.7}$$

where

$$\theta_1(x,t) = h_1 x - \int h_j^3 f(t)dt,$$

and

$$v_j(x,t) = \begin{cases} \pm\sqrt{\frac{24f(t)}{g(t)}} \times h_1 \frac{e^{\theta_1(x,t)}}{\left(1+e^{2\theta_1(x,t)}\right)}, \\ = \pm\sqrt{\frac{6f(t)}{g(t)}} \times h_1 \text{sech}\,(\theta_1(x,t)),\ j = 1 \\ \pm(-1)^{j+1}\sqrt{\frac{96f(t)}{g(t)}} \times h_1 \frac{e^{\theta_1(x,t)}}{\left(1+e^{2\theta_1(x,t)}\right)}, \\ = \pm(-1)^{j+1}\sqrt{\frac{24f(t)}{g(t)}} \times h_1 \text{sech}\,(\theta_1(x,t)),\ j = 2,3,4,.....,n. \end{cases} \tag{2.8}$$

To obtain the two-soliton solutions, we let

$$\left.\begin{aligned} a(x,t) &= e^{h_1 x - \int h_1^3 f(t)dt} + e^{h_2 x - \int h_2^3 f(t)dt}, \\ b(x,t) &= 1 - b_{12} e^{(h_1+h_2)x - \int (h_1^3+h_2^3) f(t)dt} \end{aligned}\right\} \tag{2.9}$$

Using (2.9) in (2.3) and substituting the results in (1.1), we obtained the phase shift as follows

$$b_{12} = \frac{h_1^2 - 2h_1h_2 + h_2^2}{h_1^2 + 2h_1h_2 + h_2^2}. \tag{2.10}$$

Substituting (2.10), (2.9), (2.6) and (2.5) into (2.3), we obtain sets of two solitons solutions for (1.1) as

$$\begin{aligned} v_j(x,t) &= \pm(-1)^{j+1}\sqrt{\frac{24f(t)}{g(t)}} \\ &\times \frac{h_1e^{\theta_1(x,t)}\left(1+b_{12}e^{2\theta_2(x,t)}\right) + h_2e^{\theta_2(x,t)}\left(1+b_{12}e^{2\theta_1(x,t)}\right)}{(1-b_{12}e^{\theta_1(x,t)+\theta_2(x,t)})^2 + (e^{\theta_1(x,t)} + e^{\theta_2(X,T)})^2},\ j=1,2,..,n, \end{aligned} \tag{2.11}$$

and

$$\begin{aligned} v_j(x,t) &= \pm\sqrt{\frac{24f(t)}{g(t)}} \\ &\times \frac{h_1e^{\theta_1(x,t)}\left(1+b_{12}e^{2\theta_2(x,t)}\right) + h_2e^{\theta_2(x,t)}\left(1+b_{12}e^{2\theta_1(x,t)}\right)}{(1-b_{12}e^{\theta_1(x,t)+\theta_2(x,t)})^2 + (e^{\theta_1(x,t)} + e^{\theta_2(x,t)})^2},\ j=1, \\ v_j(x,t) &= \pm(-1)^{j+1}\sqrt{\frac{96f(t)}{g(t)}} \\ &\times \frac{h_1e^{\theta_1(x,t)}\left(1+b_{12}e^{2\theta_2(x,t)}\right) + h_2e^{\theta_2(x,t)}\left(1+b_{12}e^{2\theta_1(x,t)}\right)}{(1-b_{12}e^{\theta_1(x,t)+\theta_2(x,t)})^2 + (e^{\theta_1(x,t)} + e^{\theta_2(x,t)})^2},\ j=2,..,n. \end{aligned}$$

To this end, (1.1) does not possess any resonant phenomenon [20] due to the phase shift term b_{12} in (2.10) could not be 0 or ∞ when $|h_1| \neq |h_2|$. In the meanwhile, two-soliton solutions can degenerate into a resonant trial under the conditions

$$b_{12} = 0 \quad \text{or} \quad (b_{12})^{-1} = 0, \quad \text{for} \quad |h_1| \neq |h_2|.$$

The three-soliton solutions is determined by

$$\begin{aligned} a(x,t) &= e^{h_1x - \int h_1^3 f(t)dt} + e^{h_2x - \int h_2^3 f(t)dt} + e^{h_3x - \int h_3^3 f(t)dt} + b_{123}e^{(h_1+h_2+h_3)x - \int (h_1^3+h_2^3+h_3^3)f(t)dt} \\ b(x,t) &= 1 - b_{12}e^{(h_1+h_2)x - \int (h_1^3+h_2^3)f(t)dt} - b_{13}e^{(h_1+h_3)x - \int (h_1^3+h_3^3)f(t)dt} - b_{23}e^{(h_2+h_3)x - \int (h_2^3+h_3^3)f(t)dt}, \end{aligned} \tag{2.12}$$

where

$$b_{ij} = \frac{h_i^2 - 2h_ih_j + h_j^2}{h_i^2 + 2h_ih_j + h_j^2}, 1 \leq i < j \leq 3, \tag{2.13}$$

Proceeding as before we find

$$b_{123} = b_{12}b_{13}b_{23},$$

then the two sets of three-soliton solution for (1.1) can be determined by substituting (2.12), (2.13), (2.5) and (2.6) into (2.3).

As a result, we may say that the system (1.1) is completely integrable and $N-$soliton solutions exists for $N \geq 1$ [16].

3. Multiple-singular-soliton solutions for the nc-mKdV system

In order to obtain the single singular-soliton solutions of (1.1), we propose

$$v_j(x,t) = R_j(\ln \frac{a(x,t)}{b(x,t)})_x, \quad j = 1, 2, 3, ..., n, \tag{3.1}$$

where $a(x,t)$ and $b(x,t)$ are given by

$$\left.\begin{array}{l} a(x,t) = 1 + e^{h_1 x - \int h_1^3 f(t)dt}, \\ b(x,t) = 1 - e^{h_1 x - \int h_1^3 f(t)dt}. \end{array}\right\} \tag{3.2}$$

When we substitute (3.2) into (1.1), and solving for R_j, we found two sets of solutions

$$\left. R_j = \pm(-1)^{j+1}\sqrt{\tfrac{-6f(t)}{g(t)}}, \quad j = 1, 2, 3, ..., n, \right\} \tag{3.3}$$

and

$$R_j = \left\{ \begin{array}{cc} \pm\sqrt{\tfrac{-6f(t)}{g(t)}}, & j = 1 \\ \pm(-1)^{j+1}\sqrt{\tfrac{-24f(t)}{g(t)}}, & j = 2, 3, 4, ..., n \end{array} \right. . \tag{3.4}$$

Similarly as before, we set the constraints $\frac{f(t)}{g(t)} = c$, where c is arbitrary constants to obtain a numerical value of R_j.

Then the single singular-soliton solutions of (1.1)

$$\left.\begin{array}{c} v_j(x,t) = \pm(-1)^{j+1}\sqrt{\tfrac{-6f(t)}{g(t)}} \times \tfrac{2h_1 e^{\theta_1(x,t)}}{\left(1-e^{2\theta_1(x,t)}\right)}, \\ = \pm(-1)^{j+1}\sqrt{\tfrac{-6f(t)}{g(t)}} \times \dfrac{2h_1 e^{h_1 x - \int h_1^3 f(t)dt}}{1-e^{2h_1 x - 2\int h_1^3 f(t)dt}}, \\ = \pm(-1)^{j+1}\sqrt{\tfrac{-6f(t)}{g(t)}} \times h_1 \mathrm{csch}\,(\theta_1(x,t)), \; j = 1, 2, 3, ..., n \end{array}\right\}$$

and

$$v_j(x,t) = \left\{ \begin{array}{c} \pm\sqrt{\tfrac{-6f(t)}{g(t)}} \times \tfrac{2h_1 e^{\theta_1(X,T)}}{\left(1-e^{2\theta_1(X,T)}\right)} \\ = \pm\sqrt{\tfrac{-6f(t)}{g(t)}} \times h_1 \mathrm{csch}\,(\theta_1(x,t)), \; j = 1 \\ \pm(-1)^{j+1}\sqrt{\tfrac{-24f(t)}{g(t)}} \times \tfrac{2h_1 e^{\theta_1(X,T)}}{\left(1-e^{2\theta_1(X,T)}\right)} \\ = \pm(-1)^{j+1}\sqrt{\tfrac{-24f(t)}{g(t)}} \times h_1 \mathrm{csch}\,(\theta_1(x,t)), \; j = 2, 3, 4, ..., n. \end{array} \right.$$

where

$$\theta_1(x,t) = h_1 x - \int h_1^3 f(t)dt.$$

The two-singular-soliton solutions are obtained by setting

$$
\begin{aligned}
a(x,t) &= 1 + e^{h_1 x - \int h_1^3 f(t)dt} + e^{h_2 x - \int h_2^3 f(t)dt} \\
&\quad + b_{12} e^{(h_1+h_2)x - (h_1^3+h_2^3)\int f(t)dt} \\
b(x,t) &= 1 - e^{h_1 x - \int h_1^3 f(t)dt} - e^{h_2 x - \int h_2^3 f(t)dt} \\
&\quad + b_{12} e^{(h_1+h_2)x - (h_1^3+h_2^3)\int f(t)dt}.
\end{aligned} \tag{3.5}
$$

Now, by substituting (3.5) into (3.1), then in (1.1), we find the phase shift b_{12} as

$$
b_{12} = \frac{h_1^2 - 2h_1h_2 + h_2^2}{h_1^2 + 2h_1h_2 + h_2^2}, \tag{3.6}
$$

Substituting (3.6), (3.5) and (3.3) into (3.1), then we can determine two-singular-soliton solutions for (1.1).

For three-singular-soliton solutions we use

$$
\begin{aligned}
a(x,t) &= 1 + e^{h_1 x - \int h_1^3 f(t)dt} + e^{h_2 x - \int h_2^3 f(t)dt} + e^{h_3 x - \int h_3^3 f(t)dt} \\
&\quad + b_{12} e^{(h_1+h_2)x - (h_1^3+h_2^3)\int f(t)dt} + b_{13} e^{(h_1+h_3)x - (h_1^3+h_3^3)\int f(t)dt} \\
&\quad + b_{23} e^{(h_2+h_3)x - (h_2^3+h_3^3)\int f(t)dt} + b_{123} e^{(h_1+h_2+h_3)x - (h_1^3+h_2^3+h_3^3)\int f(t)dt}, \\
b(x,t) &= 1 - e^{h_1 x - \int h_1^3 f(t)dt} - e^{h_2 x - \int h_2^3 f(t)dt} - e^{h_3 x - \int h_3^3 f(t)dt} \\
&\quad + b_{12} e^{(h_1+h_2)x - (h_1^3+h_2^3)\int f(t)dt} + b_{13} e^{(h_1+h_3)x - (h_1^3+h_3^3)\int f(t)dt} \\
&\quad + b_{23} e^{(h_2+h_3)x - (h_2^3+h_3^3)\int f(t)dt} - b_{123} e^{(h_1+h_2+h_3)x - (h_1^3+h_2^3+h_3^3)\int f(t)dt}.
\end{aligned} \tag{3.7}
$$

where

$$
b_{ij} = \frac{h_i^2 - 2h_ih_j + h_j^2}{h_i^2 + 2h_ih_j + h_j^2}, 1 \le i < j \le 3, \tag{3.8}
$$

Proceeding as before we find

$$
b_{123} = b_{12}b_{13}b_{23},
$$

then the two sets of three-singular-soliton solution for (1.1) can be determined by substituting (3.7), (3.8), (3.3) and (3.4) into (3.1).

4. Discussion and concluding remarks

As mentioned earlier, we aim to study the effect of having variable coefficients appear in our nc-mKdV model. Dispersion relations will be investigated in a novel way to retrieve characteristic line and velocity v for each soliton.

In equations (2.7) and (2.8), we obtain two sets of single soliton solutions where the amplitudes *amp* for $v_j(x,t), j = 1, 2, 3, ..., n,$, respectively, are of the form

$$amp = \left| 2h_1 \sqrt{\frac{6f(t)}{g(t)}} \right|, j = 1, 2, 3, ..., n,$$

and

$$amp = \begin{cases} \left| h_1 \sqrt{\frac{6f(t)}{g(t)}} \right|, j = 1, \\ \left| h_1 \sqrt{\frac{24f(t)}{g(t)}} \right|, \ j = 2, 3, 4, ..., n. \end{cases}$$

Considering the characteristic-line method [32, 41], the characteristic wedge of each solitary wave for $u_j(x,t)$ is given by

$$x = \int h_i^2 f(t) dt, \tag{4.1}$$

and the velocity v of each solitary wave for $v_j(x,t), j = 1, 2, 3, ..., n$ is

$$v_x = h_i^2 f(t) \tag{4.2}$$

We may say that soliton amplitude *amp* depends on the variable coefficients $f(t)$ and $g(t)$ while the propagation velocity of the solitary wave depends only on the coefficient function $f(t)$. Also, another finding in (4.2) that the soliton travels along the positive x-axis direction when the following inequality holds

$$f(t) > 0. \tag{4.3}$$

Moreover, the solitonic amplitude increases as well as the ratio $f(t)/g(t)$ does increase. Therefore, the ratio $f(t)/g(t)$ can be regarded as the damping. However, Expression (4.2) indicates that the propagation velocity of the solitary wave is influenced by the coefficient function $f(t)$ only.

In Figure 1, we choose $h_1 = 0.5, h_2 = 0.75, f(t) = \frac{3t}{\sqrt{\pi}}$ and $g(t) = \frac{2t}{\sqrt{\pi}}$, then the characteristic curve (4.1) given by

$$x - \frac{3h_i^2}{2\sqrt{\pi}} t^2 + \eta = 0,$$

and the soliton reveals the parabolic type propagation trajectory with the unalterable amplitude but continuously changeable velocity.

In Figure 2, we choose $h_1 = 1, h_2 = 1.5, f(t) = \frac{9 \sin 3t}{10\Gamma(1.9)}$ and $g(t) = \frac{9 \sin 3t}{20\Gamma(1.9)}$, then the characteristic curve (4.1) given by

$$x + \frac{3h_i^2}{10\Gamma(1.9)} \cos 3t + \eta = 0,$$

and thus, the propagation trajectory of the soliton presents the periodicity oscillation.

Another feature one can extract from the solution given in (2.11), which is, elastic interaction of two solitary waves maintain their original amplitudes and velocities after the collision except for phase shifts and illustrates the overtaking collision of two parallel solitary waves with different velocities, see Figure 3.

Finally, the solutions given in (2.11) present two types of solitons; the elevation and depression solitons, which depend on the sign of h_i. In general, for $h_i > 0$, the elevation soliton is derived for (1.1), while the depression soliton appears in

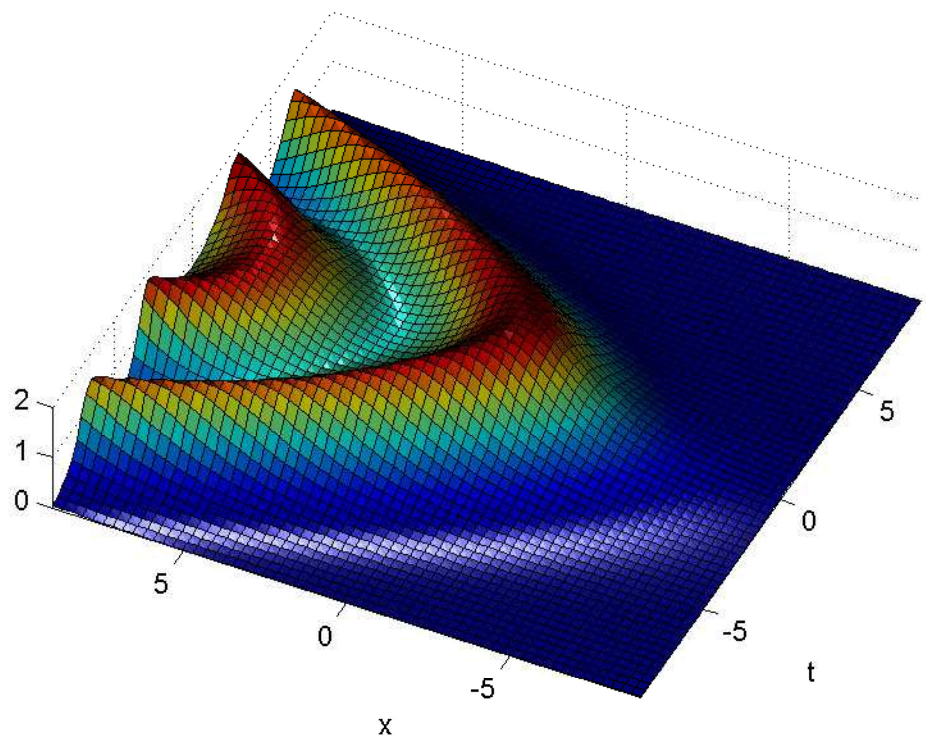

FIGURE 1. The profile figure for solution (2.11) with $f(t) = \frac{3t}{\sqrt{\pi}}$, $g(t) = \frac{2t}{\sqrt{\pi}}$, $h_1 = 0.5$, $h_2 = 0.75$.

the case of $h_i < 0$. To give a clear image on the collision process, we construct a collision between an elevation and a depression, see Figure 4.

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REFERENCES

[1] A.K. Alomari, et. Al, Multiple solutions for fractional differential equations: Analytic approach, Applied Mathematics and Computation, **(219(17)** (2013) 8893-8903.

[2] M. Alquran, K. Al-Khaled, The tanh and sine-cosine methods for higher order equations of Korteweg-de Vries type, Physica Scripta, Volume **84** (2011) 025010.

[3] M. Alquran, Solitons and periodic solutions to nonlinear partial differential equations by the Sine-Cosine method, Applied Mathematics and Information Sciences, **6(1)** (2012) 85-88.

[4] M. Aquran, K. Al-Khaled, H. Ananbeh, New Soliton Solutions for Systems of Nonlinear Evolution Equations by the Rational Sine-Cosine Method, Studies in Mathematical Sciences, **3(1)** (2011) 1-9.

[5] M. Alquran, K. Al-Khaled, Sinc and solitary wave solutions to the generalized Benjamin-Bona-Mahony-Burgers equations, Physica Scripta, **83** (2011) 065010.

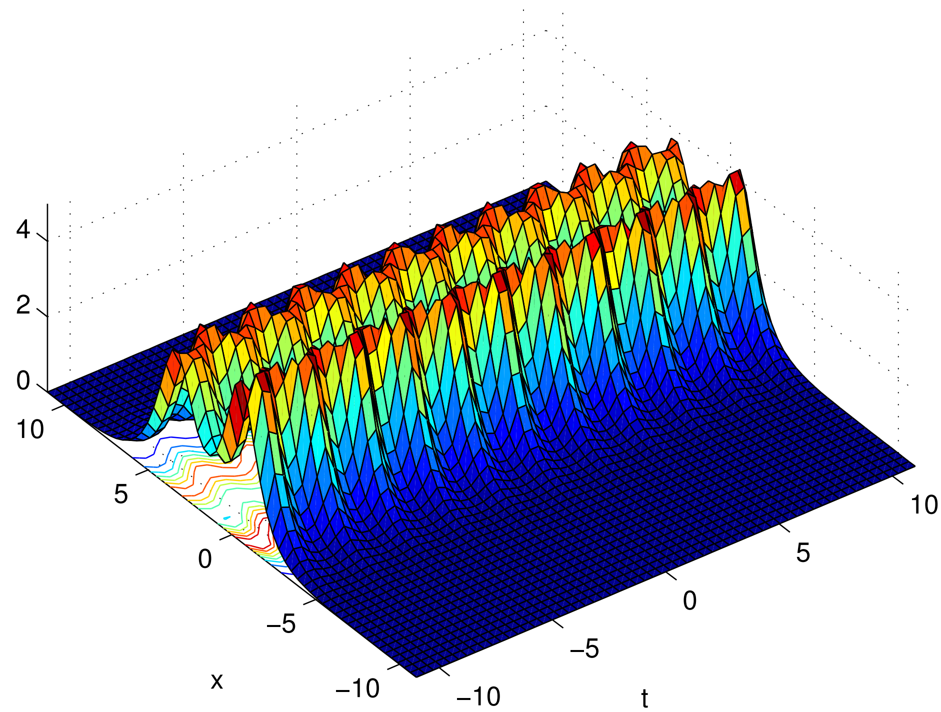

FIGURE 2. The profile figure for solution (2.11) with $f(t) = \frac{9\sin 3t}{10\Gamma(1.9)}$, $g(t) = \frac{9\sin 3t}{20\Gamma(1.9)}$, $h_1 = 1$, $h_2 = 1.5$.

[6] M. Alquran, K. Al-khaled, Mathematical methods for a reliable treatment of the $(2+1)$-dimensional Zoomeron equation, Mathematical Sciences, **6:12** (2012).

[7] M. Alquran, M. Ali, K. Al-Khaled, Solitary wave solutions to shallow water waves arising in fluid dynamics, Nonlinear Studies, **19(4)** (2012) 555-562.

[8] M. Alquran, Bright and dark soliton solutions to the Ostrovsky-Benjamin-Bona-Mahony (OS-BBM) equation, J. Math. Comput. Sci., **2(1)** (2012) 15-22.

[9] M. Alquran, R. Al-Omary, Q. Katatbeh, New explicit solutions for homogeneous KdV equations of third order by trigonometric and hyperbolic function methods, Applications and Applied Mathematics, **7(1)** (2012) 211-225.

[10] M. Alquran, Q. Katatbeh, B. Al-Shrida, Applications of First Integral Method to Some Complex Nonlinear Evolution Systems, Appl. Math. Inf. Sci., **9(2)** (2015) 825-831.

[11] M. Alquran, A. Qawasmeh, Soliton solutions of shallow water wave equations by means of (G'/G)-expansion method, Journal of Applied Analysis and Computation, **4(3)** (2014) 221-229.

[12] M. Alquran, H.M. Jaradat, S. Al-Shara', F. Awawdeh, A New Simplified Bilinear Method for the N-Soliton Solutions for a Generalized FmKdV Equation with Time-Dependent Variable Coefficients, International Journal of Nonlinear Sciences and Numerical Simulation, **16(6)** (2015) 259-269.

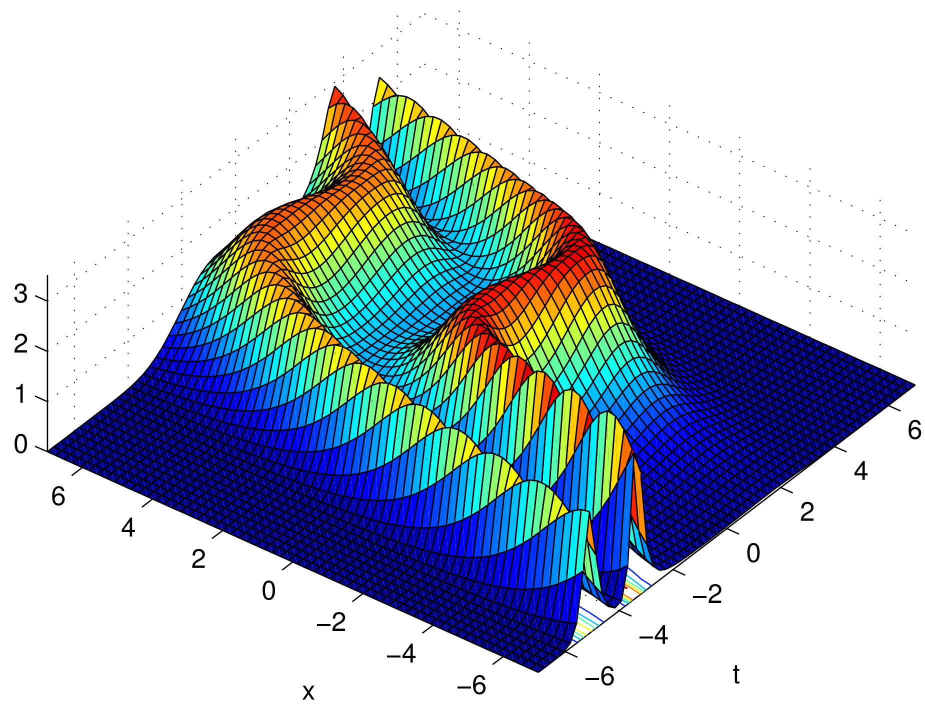

FIGURE 3. The propagation of the two-soliton wave via solution (2.11) with $h_1 = 0.7$, $h_2 = 1$, $f(t) = \frac{1}{2\Gamma(0.9)}(t^2 - t - 1)$, $g(t) = \frac{1}{4\Gamma(0.9)}(t^2 - t - 1)$.

[13] O. Alsayyed, H.M. Jaradat, M.M.M. Jaradat, Zead Mustafa, Feras Shatat, Multi-soliton solutions of the BBM equation arisen in shallow water, Journal of Nonlinear Sciences and Applications, **9(4)** (2016) 1807-1814.

[14] F. Awawdeh, H.M. Jaradat, S. Al-Shara', Applications of a simplified bilinear method to ion-acoustic solitary waves in plasma, Eur. Phys. J. D, **66** (2012) 1-8.

[15] F. Awawdeh, S. Al-Shara', H.M. Jaradat, A.K. Alomari, R. Alshorman, Symbolic computation on soliton solutions for variable coefficient quantum Zakharov-Kuznetsov equation in magnetized dense plasmas, International Journal of Nonlinear Sciences and Numerical Simulation, **15(1)** (2014) 35-45.

[16] R. Hirota, Direct methods in soliton theory, in: R. K. Bullough, P.J. Caudrey (Eds.), Solitons, Springer, Berlin, (1980).

[17] R. Hirota, Exact N-soliton solutions of a nonlinear wave equation, J. Math. Phys., **14(7)** (1973) 805-809.

[18] R. Hirota, The direct method in soliton theory, Cambridge University Press, Cambridge, (2004).

[19] R. Hirota, Exact solutions of the Korteweg-de Vries equation for multiple collisions of solitons, Phys. Rev. Lett., **27** (1971) 1192-1194.

[20] R. Hirota and M. Ito, Resonance of solitons in one dimension, J. Phys. Soc. Jpn., **52** (1983) 744-748.

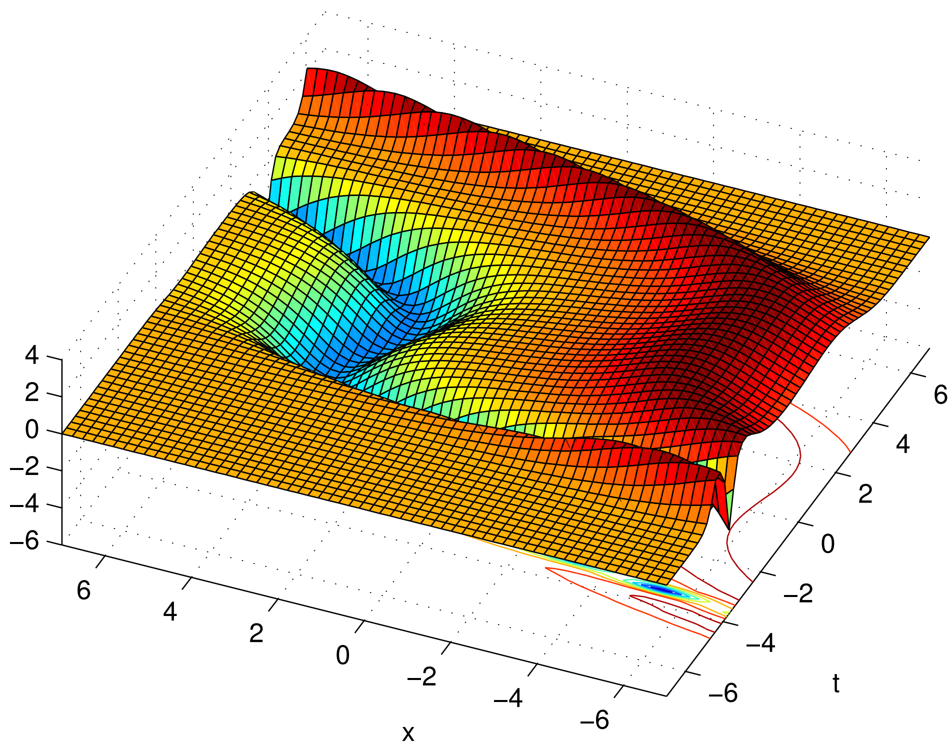

FIGURE 4. The propagation of the two-soliton wave via solution (2.11) with $h_1 = 0.7$, $h_2 = -1$, $f(t) = \frac{1}{2\Gamma(0.9)}(t^2 - t - 1)$, $g(t) = \frac{1}{4\Gamma(0.9)}(t^2 - t - 1)$.

[21] R. Hirota, Exact solution of the modified Korteweg-de Vries equation for multiple collisions of solitons, J. Phys. Soc. Jpn., **33** (1972) 1456-1458.

[22] H.M. Jaradat, S. Al-Shara', F. Awawdeh, M. Alquran, Variable coefficient equations of the Kadomtsev-Petviashvili hierarchy: multiple soliton solutions and singular multiple soliton solutions, Phys. Scr., **85** (2012) 035001.

[23] H.M. Jaradat, F. Awawdeh, S. Al-Shara', M. Alquran, S. Momani, Controllable dynamical behaviors and the analysis of fractal burgers hierarchy with the full effects of inhomogeneities of media, Romanian Journal of Physics, **60(3-4)** (2015) 324-343.

[24] H.M. Jaradat, New solitary wave and multiple soliton solutions for the time-space fractional boussinesq equation, Italian journal of pure and applied mathematics, **36** (2016) 367-376.

[25] H.M. Jaradat, Dynamic behavior of traveling wave solutions for a class for the time-space coupled fractional kdV system with time-dependent coefficients, Italian journal of pure and applied mathematics, **36** (2016), 945-958.

[26] H.M. Jaradat, M.M.M. Jaradat, F. Awawdeh, Z. Mustafa, O. Alsayyed, A new numerical method for heat equation subject to integral specifications, Journal of Nonlinear Sciences and Applications, **9(5)** (2016) 2117-2125.

[27] A. Qawasmeh, M. Alquran, Reliable study of some new fifth-order nonlinear equations by means of (G'/G)-expansion method and rational sine-cosine method, Applied Mathematical Sciences, **8(120)** (2014) 5985-5994.

[28] A. Qawasmeh, M. Alquran, Soliton and periodic solutions for $(2+1)$-dimensional dispersive long water-wave system, Applied Mathematical Sciences, **8(50)** (2014) 2455-2463.
[29] K.R. Raslan, The Application of He's Exp-function Method for MKdV and Burgers' Equations with Variable Coefficients, International Journal of Nonlinear Science, **7(2)** (2009) 174-181.
[30] S. Shukri, K. Al-khaled, The extended tanh method for solving systems of nonlinear wave equations, Applied Mathematics and Computations, **217(5)** (2010) 1997-2006.
[31] N. Taghizadeh, M. Mirzazadeh, A.S. Paghaleh, The First Integral Method to Nonlinear Partial Differential Equations, Applications and Applied Mathematics, **7(1)** (2012) 117-132.
[32] A. Veksler, Y. Zarmi, Wave interactions and the analysis of the perturbed Burgers equation, Physica D., **211** (2005) 57-73.
[33] A.M. Wazwaz, Multiple soliton solutions for the (2+1)-dimensional asymmetric Nizhnik Novikov Veselov equation, Nonlinear Analysis., **72** (2010) 1314-1318.
[34] A.M. Wazwaz, Completely integrable coupled KdV and coupled KP systems, Commun Nonlinear Sci Numer Simulat., **15** (2010) 2828-2835.
[35] A.M. Wazwaz, Multiple-soliton solutions for the Boussinesq equation, Appl. Math. Comput., **192** (2007) 479-486.
[36] A.M. Wazwaz, Multiple soliton solution for two integrable couplings of the modified korteweg-de vries equation, J. The Romanian academy, series A., **14(3)** (2013) 219-225.
[37] A.M. Wazwaz, A variety of distinct kinds of multiple soliton solutions for a (3+1)-dimensional nonlinear evolution equation, Math. Meth. Appl. Sci., **36** (2013) 349-357.
[38] C. Yan, A simple transformation for nonlinear waves, Physics Letters A., **224** (1996) 77-84.
[39] Z.Y. Yan, H.Q. Zhang, New explicit and exact travelling wave solutions for a system of variant Boussinesq equations in mathematical physics, Physics Letters A., **252(6)** (1999) 291-296.
[40] Z.Y. Yan, H.Q. Zhang, On a new algorithm of constructing solitary wave solutions for systems of nonlinear evolution equations in mathematical physics, Appl. Math. Mech., **21** (2000) 383-388.
[41] X. Yu, Y.T Gao, Z.Y Sun, Y. Liu, N-soliton solutions, Bäcklund transformation and Lax pair for a generalized variable-coefficient fifth-order Korteweg-de Vries equation, Phys. Scr., **81** (2010) 045402.

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