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ON A CERTAIN CLASSES OF MEROMORPHIC FUNCTIONS WITH POSITIVE COEFFICIENTS

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Abstract. In this paper certain classes of meromorphic functions in punctured unit disk are defined. Some properties including coefficient inequalities, convolution and other interesting results are investigated.

1. Introduction and preliminaries

Let Σ denote the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n,$$

which are analytic in $D = \{z : 0 < |z| < 1\}$, having a simple pole at the origin. Motivated by M. L. Mogra [1] we define the following class of meromorphic functions and investigate some properties of this class.

A function $f \in \Sigma$ is said to be in the class $\Sigma(A, B, \lambda)$ if it satisfies the condition

$$\frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f(z) + \lambda z f'(z)} = -\frac{1 + A\omega(z)}{1 + B\omega(z)}, \tag{1.1}$$

where $\omega(z)$ is analytic and $|\omega(z)| \leq |z|$ in the unit disc U; A and B are real constants satisfying $0 < -A \leq B < 1$ and λ is a real constant satisfying $0 \leq \lambda \leq 1$, $\lambda \neq 1/2$. From (1.1), we have that $f(z) \in \Sigma(A, B, \lambda)$ if and only if

$$\frac{zF'(z)}{F(z)} = -\frac{1 + A\omega(z)}{1 + B\omega(z)}, \tag{1.2}$$

where

$$F(z) = \frac{1}{1-2\lambda}\{(1-\lambda)f(z) + \lambda z f'(z)\} = \frac{1}{z} + \cdots. \tag{1.3}$$

Let $C(A, B, \lambda)$ be the class of functions $f \in \Sigma$ such that $-zf'(z) \in \Sigma(A, B, \lambda)$. Also let Σ_p be the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n, a_n \geq 0, \tag{1.4}$$

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which are analytic in D. We define $\Sigma_p(A,B,\lambda) = \Sigma_p \cap \Sigma(A,B,\lambda)$ and $C_p(A,B,\lambda) = \Sigma_p \cap C_p(A,B,\lambda)$. The convolution or Hadamard product of two meromorphic functions $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n$ and $g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n$ with $a_n, b_n \geq 0$ is defined by

$$f(z) * g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n b_n z^n. \tag{1.5}$$

The main aim of the present paper is to establish a certain result concerning the convolution of meromorphic functions analogous to Padmanabhan and Ganesan [3]. Also M. L. Mogra et. al [2] have studied some convolution properties of a special class of meromorphic univalent functions which is close to our class and we extend their results in some directions. On the other hand we extend some corresponding results in A. Schild and H. Silverman [4] for meromorphic functions with positive coefficients in our class.

In the sequel for real constants A, B and λ satisfying $0 < -A \leq B < 1$, $0 \leq \lambda \leq 1$, $\lambda \neq 1/2$, we define

$$U_{n,\lambda}(A,B) = \frac{(1+\lambda(n-1))(n(B+1)+A+1)}{|1-2\lambda|(B-A)}. \tag{1.6}$$

2. Main results

Theorem 2.1. *If a univalent function $f(z)$ is in $\Sigma_p(A,B,\lambda)$ with $0 < -A \leq 1/3$, $-A \leq B \leq (1+A)/2$, then $G(z) = z^2F(z)$ is starlike univalent in $|z| < 1$, where $F(z)$ is given in (1.3). Moreover,*

$$\frac{zG'(z)}{G(z)} \prec \frac{1+(2B-A)z}{1+Bz}, \tag{2.1}$$

where $\prec$ denotes the subordination.

Proof. If $G(z) = z^2F(z)$, then

$$\frac{zG'(z)}{G(z)} = \frac{zF'(z)}{F(z)} + 2.$$

Applying (1.2), we obtain

$$\frac{zG'(z)}{G(z)} = \frac{1+(2B-A)\omega(z)}{1+B\omega(z)}.$$

This gives (2.1) because under the assumptions, we have $2B - A \leq 1$. Moreover, in this case we have

$$\mathfrak{Re}\left\{\frac{1+(2B-A)\omega(z)}{1+B\omega(z)}\right\} > 0 \quad |z| < 1,$$

then $G(z) = z^2F(z)$ is starlike univalent in $|z| < 1$. □

Theorem 2.2. *A function $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n, a_n \geq 0$ is in $\Sigma_p(A,B,\lambda)$ if and only if*

$$\sum_{n=1}^{\infty} U_{n,\lambda}(A,B)a_n \leq 1 \tag{2.2}$$

also f is in $C_p(A,B,\lambda)$ if and only if

$$\sum_{n=1}^{\infty} nU_{n,\lambda}(A,B)a_n \leq 1. \tag{2.3}$$

Proof. Let $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n, a_n \geq 0$ and (2.2) holds. We show that $f \in \Sigma(A, B, \lambda)$. It is sufficient to show that the function

$$\omega(z) = \frac{\sum_{n=1}^{\infty}(n+1)(1+\lambda(n-1))a_n z^{n+1}}{(1-2\lambda)(B-A) - \sum_{n=1}^{\infty}(A+nB)(1+\lambda(n-1))a_n z^{n+1}} \quad (z \in U) \qquad (2.4)$$

is analytic, $\omega(0) = 0$ and $|\omega(z)| \leq 1$.

We show that ω is analytic, i.e the denominator in (2.4) is not zero. By the assumption (2.2) we have

$$\begin{aligned} 0 &\leq |1-2\lambda|(B-A) - \sum_{n=1}^{\infty}(A+1+n(B+1))(1+\lambda(n-1))a_n \\ &< |1-2\lambda|(B-A) - \sum_{n=1}^{\infty}(A+nB)(1+\lambda(n-1))a_n. \end{aligned}$$

So

$$\begin{aligned} &|(1-2\lambda)(B-A) - \sum_{n=1}^{\infty}(A+nB)(1+\lambda(n-1))a_n z^{n+1}| \\ \geq\ & |1-2\lambda|(B-A) - |\sum_{n=1}^{\infty}(A+nB)(1+\lambda(n-1))a_n z^{n+1}| \\ \geq\ & |1-2\lambda|(B-A) - \sum_{n=1}^{\infty}(A+nB)(1+\lambda(n-1))a_n |z|^{n+1} \\ \geq\ & |1-2\lambda|(B-A) - \sum_{n=1}^{\infty}(A+nB)(1+\lambda(n-1))a_n > 0 \quad (z \in U). \end{aligned}$$

This shows that the denominator in (2.4) is not zero.

By (2.2) we have

$$|\omega(z)| \leq \frac{\sum_{n=1}^{\infty}(n+1)(1+\lambda(n-1))a_n}{|1-2\lambda|(B-A) - \sum_{n=1}^{\infty}(A+nB)(1+\lambda(n-1))a_n} \leq 1.$$

Conversely let $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \in \sum_p(A, B, \lambda)$. From (2.4) ω satisfies $\omega(0) = 0$ and $|\omega(z)| \leq 1$, also ω is analytic in the unit disk U. Since $\mathfrak{Re}\,\omega(z) \leq |\omega(z)| \leq 1 (z \in U)$, so for $z = r (0 < r < 1)$, we have

$$\omega(r) = \mathfrak{Re}\,\omega(r) \leq |\omega(r)| \leq 1,$$

thus

$$\frac{\sum_{n=1}^{\infty}(n+1)(1+\lambda(n-1))a_n r^{n+1}}{|(1-2\lambda)|(B-A) - \sum_{n=1}^{\infty}(A+nB)(1+\lambda(n-1))a_n r^{n+1}} \leq 1.$$

Letting $r \to 1^-$, we get

$$\frac{\sum_{n=1}^{\infty}(n+1)(1+\lambda(n-1))a_n}{|(1-2\lambda)|(B-A) - \sum_{n=1}^{\infty}(A+nB)(1+\lambda(n-1))a_n} \leq 1.$$

Therefore (2.2) is now obtained. For the proof of the second part of the theorem we apply the first part for the function $g(z) = -zf'(z)$. $\square$

Theorem 2.3. *If* $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n$ *and* $g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n$ *are in* $\Sigma_p(A, B, \lambda)$*, then the Hadamard product* $f(z) * g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n b_n z^n$ *is in* $\Sigma_p(A_1, B_1, \mu)$ *with* $0 < -A_1 \leq B_1 < 1$, $0 \leq \mu \leq \mu_0$, *where*

$$\mu_0 = \frac{\alpha - \sqrt{\alpha^2 - 2\beta\gamma}}{\beta}$$

$$\alpha = 4U_{2,\lambda}^2(A, B) - 3U_{1,\lambda}^2(A, B) + 1,$$

$$\beta = 12U_{1,\lambda}^2(A, B),$$

$$\gamma = 2U_{2,\lambda}^2(A, B) - 3U_{1,\lambda}^2(A, B) - 1$$

$$-A_1 \leq \frac{K(\lambda, \mu_0)}{2 - K(\lambda, \mu_0)}, \frac{K(\lambda, \mu_0) + A_1}{1 - K(\lambda, \mu_0)} \leq B_1$$

$$K(\lambda, \mu_0) = \frac{2|1 - 2\lambda|^2(B - A)^2}{|1 - 2\lambda|^2(B - A)^2 + (1 - 2\mu_0)(B + A + 2)^2}.$$

The bounds for A_1 *and* B_1 *cannot be improved.*

Proof. Suppose $f(z)$ and $g(z)$ are in $\Sigma_p(A, B, \lambda)$. In view of Theorem 2.2, we have

$$\sum_{n=1}^{\infty} U_{n,\lambda}(A, B) a_n \leq 1 \tag{2.5}$$

and

$$\sum_{n=1}^{\infty} U_{n,\lambda}(A, B) b_n \leq 1. \tag{2.6}$$

We wish to find the values of A_1, B_1 and μ for which $f(z) * g(z) \in \Sigma_p(A_1, B_1, \mu)$. Equivalently we want to determine A_1, B_1 and μ satisfying

$$\sum_{n=1}^{\infty} U_{n,\mu}(A_1, B_1) a_n b_n \leq 1. \tag{2.7}$$

Using Cauchy Schwarz inequality together with (2.5) and (2.6) we get

$$\sum_{n=1}^{\infty} U_{n,\lambda}(A, B)\sqrt{a_n b_n} \leq \left(\sum_{n=1}^{\infty} U_{n,\lambda}(A, B) a_n\right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} U_{n,\lambda}(A, B) b_n\right)^{\frac{1}{2}}. \tag{2.8}$$

From (2.5), (2.6) and (2.8), we get

$$\sum_{n=1}^{\infty} U_{n,\lambda}(A, B)\sqrt{a_n b_n} \leq 1.$$

So the inequality (2.7) is satisfied if

$$U_{n,\mu}(A_1, B_1) a_n b_n \leq U_{n,\lambda}(A, B)\sqrt{a_n b_n}$$

for $n \geq 1$.
That is

$$U_{n,\mu}(A_1, B_1)\sqrt{a_n b_n} \leq U_{n,\lambda}(A, B).$$

Since $U_{n,\lambda}(A, B) \geq 1$ so from (2.8), we have

$$\sqrt{a_n b_n} \leq \frac{1}{U_{n,\lambda}(A, B)}.$$

Thus it is enough to find $U_{n,\mu}(A_1, B_1)$ such that

$$U_{n,\mu}(A_1, B_1) \le U^2_{n,\lambda}(A, B). \tag{2.9}$$

The inequality (2.9) is equivalent to

$$\frac{(1+\mu(n-1))(n(B_1+1)+A_1+1)}{|1-2\mu|(B_1-A_1)} \le \left(\frac{(1+\lambda(n-1))(n(B+1)+A+1)}{|1-2\lambda|(B-A)}\right)^2 := u^2.$$

This yields

$$A_1 \le \frac{u^2|1-2\mu|B_1 + (1+\mu(n-1))(n(B_1+1)+1)}{1+\mu(n-1)+u^2|1-2\mu|}. \tag{2.10}$$

Now (2.10) gives on simplification

$$\frac{B_1 - A_1}{B_1+1} \ge \frac{(1+\mu(n-1))(n+1)}{1+\mu(n-1)+U^2_{n,\lambda}(A,B)|1-2\mu|}. \tag{2.11}$$

It is easy to see that the right hand of (2.11) decreases as n increases and it is maximum for $n = 1$, provided that $0 \le \mu \le \mu_0$ and

$$\frac{B_1 - A_1}{B_1+1} \ge \frac{2|1-2\lambda|^2(B-A)^2}{|1-2\lambda|^2(B-A)^2 + (1-2\mu_0)(B+A+2)^2} := K(\lambda, \mu_0), \tag{2.12}$$

where

$$\mu_0 = \frac{\alpha - \sqrt{\alpha^2 - 2\beta\gamma}}{\beta}, \quad \alpha = (4U^2_{2,\lambda}(A,B) - 3U^2_{1,\lambda}(A,B) + 1)$$

and where

$$\beta = 12U^2_{1,\lambda}(A,B), \quad \gamma = 2U^2_{2,\lambda}(A,B) - 3U^2_{1,\lambda}(A,B) - 1.$$

It is clear that $K(\lambda, \mu_0) < 1$. Fixing A_1 in (2.12), we get $B_1 \ge \frac{K(\lambda,\mu_0)+A_1}{1-K(\lambda,\mu_0)}$. It is easy to verify that $0 < -A_1 \le B_1 < 1$. If we take

$$f(z) = g(z) = \frac{1}{z} + |1-2\lambda|\frac{B-A}{B+A+2}z,$$

then

$$U_{n,\mu_0}(A_1, B_1) = \frac{(1-2\mu_0)K(\lambda,\mu_0)}{2-K(\lambda,\mu_0)}.$$

So we get $f(z) * g(z) \in \Sigma_p\left(-\frac{K(\lambda,\mu_0)}{2-K(\lambda,\mu_0)}, \frac{K(\lambda,\mu_0)}{2-K(\lambda,\mu_0)}\right)$ with $K(\lambda, \mu_0)$ as in (2.12). □

Corollary 2.4. *Let $f(z)$ and $g(z)$ be as in Theorem 2.3. Then*

$$h(z) = \frac{1}{z} + \sum_{n=1}^{\infty} u_{n,\lambda}(A,B)\sqrt{a_n b_n} z^n \in \Sigma_p(A, B, \lambda).$$

Proof. The result follows immediately from (2.8) using the Cauchy-Schwarz inequality. For the same functions as in Theorem 2.3, the result is the best possible. □

Theorem 2.5. *If $f(z) \in \Sigma_p(A, B, \lambda)$ and $g(z) \in \Sigma_p(A', B', \theta)$ then $f(z) * g(z) \in \Sigma_p(A_1, B_1, \mu)$ with $0 < -A_1 \le B_1 < 1$, $0 \le \mu \le \mu_0$, where*

$$\mu_0 = \frac{\alpha - \sqrt{\alpha^2 - 2\beta\gamma}}{\beta}$$

$$\alpha = 4U_{2,\lambda}(A,B)U_{2,\theta}(A',B') - 3U_{1,\lambda}(A,B)U_{1,\theta}(A',B') + 1$$

$$\beta = 12U_{1,\lambda}(A,B)U_{1,\theta}(A',B')$$

$$\gamma = 2U_{2,\lambda}(A,B)U_{2,\theta}(A',B') - 3U_{1,\lambda}(A,B)U_{1,\theta}(A',B') - 1$$

$$-A_1 \leq \frac{K(\lambda,\theta,\mu_0)}{2-K(\lambda,\theta,\mu_0)}, \frac{K(\lambda,\theta,\mu_0)+A_1}{1-K(\lambda,\theta,\mu_0)} \leq B_1$$

$$K(\lambda,\theta,\mu_0) = \frac{2|1-2\lambda||1-2\theta|(B-A)(B'-A')}{|1-2\lambda||1-2\theta|(B-A)(B'-A') + (1-2\mu_0)(B+A+2)(B'+A'+2)}.$$

The bounds for A_1 and B_1 cannot be improved.

Proof. Proceeding exactly as in Theorem 2.3, we require to show that

$$U_{n,\mu}(A_1,B_1) \leq U_{n,\theta}(A',B')U_{n,\lambda}(A,B)$$

for all $n \geq 1$. This on simplification yields

$$\frac{B_1-A_1}{B_1+1} \geq \frac{(1+\mu(n-1))(n+1)}{1+\mu(n-1)+U_{n,\lambda}(A,B)U_{n,\theta}(A',B')|1-2\mu|}. \tag{2.13}$$

The right hand of (2.13) decreases as n increases and it is maximum for $n=1$ provided that $0 \leq \mu \leq \mu_0$ and

$$\frac{B_1-A_1}{B_1+1} \geq \frac{2|1-2\lambda||1-2\theta|(B-A)(B'-A')}{|1-2\lambda||1-2\theta|(B-A)(B'-A') + (1-2\mu_0)(B+A+2)(B'+A'+2)}, \tag{2.14}$$

where

$$\mu_0 = \frac{\alpha - \sqrt{\alpha^2 - 2\beta\gamma}}{\beta}$$

$$\alpha = 4U_{2,\lambda}(A,B)U_{2,\theta}(A',B') - 3U_{1,\lambda}(A,B)U_{1,\theta}(A',B') + 1$$

$$\beta = 12u_{1,\lambda}(A,B)u_{1,\theta}(A',B')$$

$$\gamma = 2U_{2,\lambda}(A,B)U_{2,\theta}(A',B') - 3U_{1,\lambda}(A,B)U_{1,\theta}(A',B') - 1.$$

Clearly $K(\lambda,\theta,\mu_0) < 1$. Fixing A_1 in (2.14) we get

$$\frac{K(\lambda,\theta,\mu_0)+A_1}{1-K(\lambda,\theta,\mu_0)} \leq B_1.$$

It is easily seen that the result is the best possible for the functions

$$f(z) = \frac{1}{z} + |1-2\lambda|\frac{B-A}{B+A+2}z,$$

$$g(z) = \frac{1}{z} + |1-2\theta|\frac{B-A}{B+A+2}z.$$

□

Corollary 2.6. *If $f(z), g(z), h(z) \in \sum_p(A,B,\lambda)$ then $f(z)*g(z)*h(z) \in \sum_p(A_1,B_1,\mu)$ with $0 \leq \mu \leq \mu_0$ where μ_0 is as in Theorem 2.5, $0 \leq \theta \leq \theta_0$ and*

$$\theta_0 = \frac{\alpha - \sqrt{\alpha^2 - 2\beta\gamma}}{\beta}$$

$\alpha = 4U^2_{2,\lambda}(A,B) - 3U^2_{1,\lambda}(A,B) + 1,\ \beta = 12U^2_{1,\lambda}(A,B),$

$$\gamma = 2U^2_{2,\lambda}(A,B) - 3U^2_{1,\lambda}(A,B) - 1$$

$$-A_1 \leq \frac{K(\lambda,\theta,\mu_0)}{2-K(\lambda,\theta,\mu_0)}, \frac{K(\lambda,\theta,\mu_0)+A_1}{1-K(\lambda,\theta,\mu_0)} \leq B_1$$

$$K(\lambda,\theta,\mu_0) = \frac{2|1-2\lambda||1-2\theta|(B-A)(B'-A')}{|1-2\lambda||1-2\theta|(B-A)(B'-A') + (1-2\mu_0)(B+A+2)(B'+A'+2)}$$

$$-A' \leq \frac{K(\lambda,\theta_0)}{2-K(\lambda,\theta_0)}, \frac{K(\lambda,\theta_0)+A}{1-K(\lambda,\theta_0)} \leq B'$$

$$K(\lambda,\theta_0) = \frac{2|1-2\lambda|^2(B-A)^2}{|1-2\lambda|^2(B-A)^2+(1-2\theta_0)(B+A+2)^2}.$$

Proof. Since $f(z), g(z) \in \sum_p(A,B,\lambda)$ by Theorem 2.5, we have $f(z)*g(z) \in \sum_p(A',B',\theta)$, where $-A' \leq \frac{K(\lambda,\theta_0)}{2-K(\lambda,\theta_0)}, \frac{K(\lambda,\theta_0)+A}{1-K(\lambda,\theta_0)} \leq B'$ with

$$K(\lambda,\theta_0) = \frac{2|1-2\lambda|^2(B-A)^2}{|1-2\lambda|^2(B-A)^2+(1-2\theta_0)(B+A+2)^2}.$$

Now letting $f(z)*g(z) \in \sum_p(A',B',\theta)$ and $h(z) \in \sum_p(A,B,\lambda)$ the result follows by Theorem 2.5. □

Theorem 2.7. *If $f(z) \in C_p(A,B,\lambda)$ and $g(z) \in C_p(A',B',\theta)$ then $f(z)*g(z) \in C_p(A_1,B_1,\theta)$, where*

$$-A_1 \leq \frac{K(\lambda,\theta,\mu_0)}{2-K(\lambda,\theta,\mu_0)}, \frac{K(\lambda,\theta,\mu_0)+A_1}{1-K(\lambda,\theta,\mu_0)} \leq B_1$$

with $0 \leq \mu \leq \mu_0$ and μ_0 as in Theorem 2.5 the result is the best possible.

Theorem 2.8. *If $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n, a_n \geq 0$ belongs to $\Sigma_p(A,B,\lambda)$ and $g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n$ with $|b_n| \leq 1, n \geq 1$, then $f(z)*g(z) \in \Sigma(A,B,\lambda)$.*

Proof. Since $f(z) \in \Sigma_p(A,B,\lambda)$, we have

$$\Sigma_{n=1}^{\infty} U_{n,\lambda}(A,B)a_n \leq 1.$$

Furthermore $|b_n| \leq 1, n \geq 1$. Therefore,

$$\Sigma_{n=1}^{\infty} U_{n,\lambda}(A,B)|a_n b_n| = \sum_{n=1}^{\infty} U_{n,\lambda}(A,B)a_n|b_n| \leq 1,$$

this shows that $f(z)*g(z) \in \Sigma(A,B,\lambda)$. □

Corollary 2.9. *If $f(z) \in \sum_p(A,B,\lambda)$ and $g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n; 0 \leq b_n \leq 1$ for $n \geq 1$ then $f(z)*g(z) \in \sum_p(A,B,\lambda)$.*

Theorem 2.10. *If $f(z)$ and $g(z)$ are in $\sum_p(A,B,\lambda)$, then $h(z) = \frac{1}{z} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)z^n \in \sum_p(A_1,B_1,\mu)$, where*

$$0 \leq \mu \leq \mu_0 = \frac{\alpha - \sqrt{\alpha^2 - 2\beta\gamma}}{\beta},$$

$\alpha = 4U_{2,\lambda}^2(A,B) - 3U_{1,\lambda}^2(A,B) + 2,\ \beta = 12U_{1,\lambda}^2(A,B),$

$$\gamma = 2U_{2,\lambda}^2(A,B) - 3U_{1,\lambda}^2(A,B) - 2,$$

$$-A_1 \leq \frac{K(\lambda,\mu_0)}{2-K(\lambda,\mu_0)}, \frac{K(\lambda,\mu_0)+A_1}{1-K(\lambda,\mu_0)} \leq B_1$$

$$K(\lambda,\mu_0) = \frac{4|1-2\lambda|^2(B-A)^2}{2|1-2\lambda|^2(B-A)^2+(1-2\mu_0)(B+A+2)^2}.$$

The result is the best possible.

Proof. Since $f(z), g(z) \in \sum_p(A, B, \lambda)$, then

$$\sum_{n=1}^{\infty} U_{n,\lambda}(A,B)a_n \leq 1$$

and

$$\sum_{n=1}^{\infty} U_{n,\lambda}(A,B)b_n \leq 1.$$

Therefore,

$$\sum_{n=1}^{\infty} U_{n,\lambda}^2(A,B)a_n^2 \leq 1$$

and

$$\sum_{n=1}^{\infty} U_{n,\lambda}^2(A,B)b_n^2 \leq 1.$$

Hence

$$\sum_{n=1}^{\infty} \frac{1}{2}U_{n,\lambda}^2(A,B)(a_n^2+b_n^2) \leq 1. \tag{2.15}$$

We want to find the values of A_1, B_1 and μ such that

$$\sum_{n=1}^{\infty} U_{n,\mu}^2(A_1,B_1)(a_n^2+b_n^2) \leq 1. \tag{2.16}$$

Comparing (2.16) with (2.15) we see that (2.16) is true if

$$2U_{n,\mu}(A_1,B_1) \leq U_{n,\lambda}^2(A,B)$$

or

$$\frac{B_1-A_1}{B_1+1} \geq \frac{2(1+\mu(n-1))(n+1)}{2(1+\mu(n-1))+U_{n,\lambda}^2(A,B)|1-2\mu|} \tag{2.17}$$

for all $n \geq 1$. The right hand side of (2.17) is a decreasing function of n and is maximum for $n = 1$ provided that $0 \leq \mu \leq \mu_0$ and

$$\frac{B_1-A_1}{B_1+1} \geq \frac{4|1-2\lambda|^2(B-A)^2}{2|1-2\lambda|(B-A)^2+(1-2\mu_0)(B+A+2)^2} := K(\lambda,\mu_0). \tag{2.18}$$

Keeping A_1 fixed in (2.18) we get $\frac{K(\lambda,\mu_0)+A_1}{1-K(\lambda,\mu_0)} \leq B_1$ and $-A_1 \leq \frac{K(\lambda,\mu_0)}{2-K(\lambda,\mu_0)}$ with $K(\lambda,\mu_0)$ given as in (2.18). The functions $f(z) = g(z) = \frac{1}{z} + |1-2\lambda|\frac{B-A}{B+A+2}z$ show that our result is the best possible. □

References

[1] Mogra, M. L. Convolutions of certain classes of meromorphic univalent functions with positive coefficients, Mathematica Pannonica 9(1)(1998), 47–55.

[2] Mogra, M.L, Reddy, T.R and Juneja, O. P: Meromorphic univalent functions with positive coefficients, Bull Austral. Math Soc 32 (1985), 161–176.

[3] Padmanabhan, K. S. and Ganesan, M. S. Convolutions of certain classes of univalent functions with negative coefficients, Indian J. Pure Appl. Math. 19(9)(1998), 880–889.

[4] Schild, A. and Silverman, H. Convolutions of univalent functions with negative coefficients, Ann. Univ. Marie Curie-Skłodowska Sect A 29(1975), 99–107.

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