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FIXED POINTS OF AUTOMORPHISMS OF $\mathbb{Z}_p \times \mathbb{Z}_{p^3}$

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Abstract. Let $G = \mathbb{Z}_p \times \mathbb{Z}_{p^3}$ be a finite abelian group of order p^4. Suppose d is a divisor of order of group G. In this article, we find the number of automorphisms of G fixing d elements of G, and denote it by $\theta(G, d)$. We prove that

$$\theta(\mathbb{Z}_p \times \mathbb{Z}_{p^3}, d) = \begin{cases} p^4(p^2 - 4p + 4) & ; for\ d = 1 \\ p^3(2p^2 - 5p + 2) & ; for\ d = p \\ p(2p^3 - 3p^2 + 1) & ; for\ d = p^2 \\ p(p^2 - 1) - 1 & ; for\ d = p^3 \\ 1 & ; for\ d = p^4 \end{cases}$$

1. Introduction

The concept of fixed point theory is very useful tool, which is not only used in Mathematics, but also in other branches of science, such as Economics, Biology, Chemistry, and Physics. More precisely linearisation near an unstable fixed point has led to Wilson's Nobel prize-winning work inventing the renormalization group, see [1, 5, 6, 9] for further details.

Automorphisms of various structures play important role in many areas such as Groups, Algebra, and Lie algebra. Some authors derived explicitly the fixed points of the automorphism groups in different branches of mathematics, see for example [2, 3, 6, 7, 8, 10].

In [3], authors discussed the automorphism group and their fixed points for finite cyclic groups. In this article, we determine the fixed points of the automorphism group of the group $\mathbb{Z}_p \times \mathbb{Z}_{p^3}$. We recall some basic definitions and results, these will be used in the rest of this paper.

Definition 1.1. [3] *For any finite group G, the function $\Delta_G : \{Aut(G) \to \psi(G)\}$ is defined as*
$\Delta_G(g) = \{\mathrm{x} \in G : g(\mathrm{x}) = \mathrm{x}\}$ *for any $g \in Aut(G)$, where $\psi(G)$ is the set of all subgroups of group G. We call Δ_G the fixed point group of g.*

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Definition 1.2. [3] *Let d be a divisor of the order of a group G. Then d-fixers set is*
$S_d^G = \{g \in Aut(G) : |\Delta_G(g)| = d\}$, *we denote* $|S_d^G|$ *by* $\theta(G, d)$.

Definition 1.3. [3] *Let G be a finite group, for every non empty subset U of G, we define*

$$U_\Delta = \{g \in Aut(G) : U \subseteq \Delta(g)\}$$

to be the set of all automorphisms of G fixing at least set U.

Definition 1.4. [3] *Let G be a finite group, and suppose H be any subgroup of G, then H-fixers is defined by*

$$\Delta_G^{-1}(H) = \{g \in Aut(G) : \Delta_G(g) = H\}.$$

This is the set of all automorphisms fixing exactly subgroup H.

Remarks:[3]

- Let G be a finite group, then $|\mathrm{Aut}(G)| = \sum_{d||G|} \theta(G, d)$.
- Obviously for any subset U of G, $U_\Delta \leq \mathrm{Aut}(G)$.
- If $\Delta_G^{-1}(H) \subseteq H_\Delta$ for any subgroup V of G, then $\Delta_G^{-1}(H)$ need not be a group.

2. STRUCTURE OF $\mathbb{Z}_p \times \mathbb{Z}_{p^3}$ AND ITS AUTOMORPHISM GROUP

For any prime p, the direct product $G = \mathbb{Z}_p \times \mathbb{Z}_{p^3}$ of two cyclic groups $\mathbb{Z}_p$ and $\mathbb{Z}_{p^3}$ is a finite abelian group. In this case the converse of the Lagrange theorem is true for each divisor d of G, we get a subgroup of order d.

In [4] Keith Conrad derived all the possible subgroups of the group $\mathbb{Z}_{p^a} \times \mathbb{Z}_{p^b}$, where $a, b \geq 1$.

If $a = 1$, $b = 3$, then the following are proper subgroups of G.

There are total $(p+1)$ cyclic subgroups of order p. These subgroups are denoted by $H_k = \langle \begin{pmatrix} k \\ p^2 \end{pmatrix} \rangle$, where $k \in \mathbb{Z}_p$, and $H_p = \langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rangle$.

There are total $(p + 1)$ subgroups of order p^2. One of them is non-cyclic and rest of those are cyclic subgroups, and these subgroups are denoted by K_k, where $k \in \mathbb{Z}_p$ and K_p, $K_k = \langle \begin{pmatrix} k \\ p \end{pmatrix} \rangle$, where $k \in \mathbb{Z}_p$ and the non-cyclic subgroup of order p^2 is denoted by $K_p = \langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ p^2 \end{pmatrix} \rangle$.

Furthermore, there are total $(p + 1)$ subgroups of order p^3. Among those, p are cyclic subgroups and one is non-cyclic subgroups. The cyclic subgroups are $Y_k = \langle \begin{pmatrix} k \\ 1 \end{pmatrix} \rangle$, where $k \in \mathbb{Z}_p$ and the non-cyclic subgroup is $Y_p = \langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ p \end{pmatrix} \rangle$.

Finally there is only one non-cyclic subgroup of order p^4, denoted by Z and is given by $Z = \langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rangle$.

Now we discuss automorphisms of $\mathbb{Z}_p \times \mathbb{Z}_{p^3}$. Suppose f is an automorphism, i.e

$$f : \mathbb{Z}_p \times \mathbb{Z}_{p^3} \to \mathbb{Z}_p \times \mathbb{Z}_{p^3}$$

is an isomorphism. Let $\alpha, \beta \in \mathbb{Z}_p$ and $\mu, \gamma \in \mathbb{Z}_{p^3}$ s.t $f(1,0) = (\alpha, \mu)$ and $f(0,1) = (\beta, \gamma)$.

Since $(1,0) \in \mathbb{Z}_p \times \mathbb{Z}_{p^3}$ has order p so is (α, μ), this implies that order of α is p and $p\mu = 0$. This means that $\mu = p^2 t$ for some $t \in \mathbb{Z}_p$. Similarly order of $(0,1)$ is p^3, so does (β, γ) hence $\gamma \in \mathbb{Z}_{p^3}^{\times}$, where $\mathbb{Z}_{p^3}^{\times}$ be the set of unit elements with respect to multiplication.

As f is an automorphism so is one-to-one correspondence. Let $(x, y) \in \mathrm{Aut}(\mathbb{Z}_p \times \mathbb{Z}_{p^3})$ so we get,
$f(x,y) = xf(1,0) + yf(0,1) = (x\alpha + y\beta, xp^2t + y\gamma)$.

If $f(x,y) = (0,0)$, then

$$x\alpha + y\beta \equiv 0 \ (mod\ p) \qquad (*)$$

and

$$xp^2t + y\gamma \equiv 0 \ (mod\ p^3). \qquad (**)$$

Since $\gamma \in \mathbb{Z}_{p^3}^{\times}$, so we can write y form $(\star\star)$ as follows

$$y = -xp^2t\gamma^{-1}.$$

After substituting y in $(\star)$, we get

$$x\alpha \equiv 0 \ (mod\ p^2),$$

thus we conclude that $x = 0$ it means $\alpha \neq 0$, that is $\alpha \in \mathbb{Z}_p^{\times}$.

Finally we can write $f(1,0) = (\alpha, p^2t)$ and $f(0,1) = (\beta, \gamma)$, s.t f is an automorphism iff $\alpha \in \mathbb{Z}_p^{\times}$ $\gamma \in \mathbb{Z}_{p^3}^{\times}$. So in general automorphisms can be defined by matrices notation.

$\mathrm{Aut}(\mathbb{Z}_p \times \mathbb{Z}_{p^3}) = \{ \begin{pmatrix} \alpha & \beta \\ p^2t & \gamma \end{pmatrix} : \alpha \in \mathbb{Z}_p^{\times}, t, \beta \in \mathbb{Z}_p, \gamma \in \mathbb{Z}_{p^3}^{\times} \}$.

We will apply matrix $\begin{pmatrix} \alpha & \beta \\ p^2t & \gamma \end{pmatrix}$ on the elements of $\mathbb{Z}_p \times \mathbb{Z}_{p^3}$ from the left.

3. PROOF OF THE MAIN RESULT: FIXED POINTS OF $\mathrm{Aut}(\mathbb{Z}_p \times \mathbb{Z}_{p^3})$

We prove our main result i.e

$$\theta(\mathbb{Z}_p \times \mathbb{Z}_{p^3}, d) = \begin{cases} p^4(p^2 - 4p + 4) & ; for\ d = 1 \\ p^3(2p^2 - 5p + 2) & ; for\ d = p \\ p(2p^3 - 3p^2 + 1) & ; for\ d = p^2 \\ p(p^2 - 1) - 1 & ; for\ d = p^3 \\ 1 & ; for\ d = p^4 \end{cases}$$

for distinct divisors d in the form of following lemmas.

Remark:

For any non-cyclic subgroup of the group $G = \mathbb{Z}_p \times \mathbb{Z}_{p^3}$, we get

$$\langle a, b \rangle_{\Delta} = \{a\}_{\Delta} \cap \{b\}_{\Delta}.$$

Lemma 3.1. *For the subgroup* $\langle\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\rangle$ *of* G*, we have*
$(Z)_\Delta=\{\begin{pmatrix}1&0\\0&1\end{pmatrix}\}$*, and the order of* $(Z)_\Delta$ *is* 1.

Proof. From remark(3.1) $(Z)_\Delta=\langle\begin{pmatrix}1\\0\end{pmatrix}\rangle_\Delta\cap\langle\begin{pmatrix}0\\1\end{pmatrix}\rangle_\Delta$
first we find $\langle\begin{pmatrix}1\\0\end{pmatrix}\rangle_\Delta$ and then $\langle\begin{pmatrix}0\\1\end{pmatrix}\rangle_\Delta$.

$$\begin{aligned}
\langle\begin{pmatrix}1\\0\end{pmatrix}\rangle_\Delta &= \{\begin{pmatrix}\alpha&\beta\\p^2t&\gamma\end{pmatrix}:\begin{pmatrix}\alpha&\beta\\p^2t&\gamma\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix}\}\\
&= \{\begin{pmatrix}\alpha&\beta\\p^2t&\gamma\end{pmatrix}:\alpha\equiv 1\ (\mathrm{mod}\ p),\ p^2t\equiv 0\ (\mathrm{mod}\ p^3)\}\\
&= \{\begin{pmatrix}\alpha&\beta\\p^2t&\gamma\end{pmatrix}:p|\alpha-1,\ p|t\}\\
&= \{\begin{pmatrix}\alpha&\beta\\p^2t&\gamma\end{pmatrix}:\alpha=1,\ t=0\}\\
&= \{\begin{pmatrix}1&\beta\\0&\gamma\end{pmatrix}:\beta\in\mathbb{Z}_p,\ \gamma\in\mathbb{Z}_{p^3}^\times\}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\langle\begin{pmatrix}0\\1\end{pmatrix}\rangle_\Delta &= \{\begin{pmatrix}\alpha&\beta\\p^2t&\gamma\end{pmatrix}:\begin{pmatrix}\alpha&\beta\\p^2t&\gamma\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix}\}\\
&= \{\begin{pmatrix}\alpha&\beta\\p^2t&\gamma\end{pmatrix}:\beta\equiv 0\ (\mathrm{mod}\ p),\ \gamma\equiv 1\ (\mathrm{mod}\ p^3)\}\\
&= \{\begin{pmatrix}\alpha&\beta\\p^2t&\gamma\end{pmatrix}:p\mid\beta,\ p^3\mid\gamma-1\}\\
&= \{\begin{pmatrix}\alpha&\beta\\p^2t&\gamma\end{pmatrix}:\alpha\in\mathbb{Z}_p^\times,\ t\in\mathbb{Z}_p,\ \beta=0,\ \gamma=1\}\\
&= \{\begin{pmatrix}\alpha&0\\p^2t&1\end{pmatrix}:\alpha\in\mathbb{Z}_p^\times,\ t\in\mathbb{Z}_p\}.
\end{aligned}$$

Thus $(Z)_\Delta=\langle\begin{pmatrix}1\\0\end{pmatrix}\rangle_\Delta\cap\langle\begin{pmatrix}0\\1\end{pmatrix}\rangle_\Delta$,
therefore $(Z)_\Delta=\{\begin{pmatrix}1&0\\0&1\end{pmatrix}\}$ and hence $|(Z)_\Delta|=1$. □

Above lemma shows that there is only identity automorphism fixing the whole group G.

Next, we calculate $\theta(G,p^3)$.

Lemma 3.2. *For the cyclic subgroup* $Y_k=\langle\begin{pmatrix}k\\1\end{pmatrix}\rangle$ *for* $k\in\mathbb{Z}_p$*, of the group* G*, there are total* $p^2(p-1)$ *automorphisms fixing all* Y_k*, where* $k\in\mathbb{Z}_p$.

Proof. We know that
$(Y_k)_\Delta=\langle\begin{pmatrix}0\\1\end{pmatrix}\rangle_\Delta\cup\langle\begin{pmatrix}n\\1\end{pmatrix}\rangle_\Delta$ for $n\in\mathbb{Z}_p\setminus\{0\}$.

Now

$$\begin{aligned}
\langle\begin{pmatrix} 0 \\ 1 \end{pmatrix}\rangle_\Delta &= \{\begin{pmatrix} \alpha & \beta \\ p^2t & \gamma \end{pmatrix} : \begin{pmatrix} \alpha & \beta \\ p^2t & \gamma \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\} \\
&= \{\begin{pmatrix} \alpha & \beta \\ p^2t & \gamma \end{pmatrix} : \beta \equiv 0 \pmod p,\ \gamma \equiv 1 \pmod{p^3}\} \\
&= \{\begin{pmatrix} \alpha & \beta \\ p^2t & \gamma \end{pmatrix} : p \mid \beta,\ p^3 \mid \gamma - 1\} \\
&= \{\begin{pmatrix} \alpha & \beta \\ p^2t & \gamma \end{pmatrix} : \alpha \in \mathbb{Z}_p^\times,\ t \in \mathbb{Z}_p,\ \beta = 0,\ \gamma = 1\} \\
&= \{\begin{pmatrix} \alpha & 0 \\ p^2t & 1 \end{pmatrix} : \alpha \in \mathbb{Z}_p^\times,\ t \in \mathbb{Z}_p\}.
\end{aligned}$$

So, $|\langle\begin{pmatrix} 0 \\ 1 \end{pmatrix}\rangle_\Delta| = p\Phi(p)$, where Φ is the Euler phi function. Also

$$\begin{aligned}
\langle\begin{pmatrix} n \\ 1 \end{pmatrix}\rangle_\Delta &= \{\begin{pmatrix} \alpha & \beta \\ p^2t & \gamma \end{pmatrix} : \begin{pmatrix} \alpha & \beta \\ p^2t & \gamma \end{pmatrix}\begin{pmatrix} n \\ 1 \end{pmatrix} = \begin{pmatrix} n \\ 1 \end{pmatrix}\} \\
&= \{\begin{pmatrix} \alpha & \beta \\ p^2t & \gamma \end{pmatrix} : n\alpha + \beta \equiv n \pmod p,\ p^2nt + \gamma \equiv 1 \pmod{p^3}\} \\
&= \{\begin{pmatrix} \alpha & \beta \\ p^2t & \gamma \end{pmatrix} : p|n(\alpha - 1) + \beta,\ p^3|p^2nt + (\gamma - 1)\} \\
&= \{\begin{pmatrix} \alpha & \beta \\ p^2t & \gamma \end{pmatrix} : \text{for each } \alpha \text{ values there must } \beta, \\
&\quad \text{and for each } t \text{ values there must exist } \gamma\}.
\end{aligned}$$

Thus, $|\langle\begin{pmatrix} n \\ 1 \end{pmatrix}\rangle_\Delta| = p\Phi(p)(p-1)$.
Therefore $|(Y_k)_\Delta| = p\Phi(p) + p\Phi(p)(p-1) = p^2(p-1)$. □

Lemma 3.3. *For the subgroup* $Y_p = \langle\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ p \end{pmatrix}\rangle$ *of* G*, we have* $(Y_p)_\Delta = \{\begin{pmatrix} 1 & \beta \\ 0 & \gamma \end{pmatrix} : \beta \in \mathbb{Z}_p,\ \gamma \in \{sp^2 + 1 : s \in \mathbb{Z}_p\}\}$ *and* $|(Y_p)_\Delta| = p^2$.

Proof. By definition,

$$\begin{aligned}
\langle\begin{pmatrix} 1 \\ 0 \end{pmatrix}\rangle_\Delta &= \{\begin{pmatrix} \alpha & \beta \\ p^2t & \gamma \end{pmatrix} : \begin{pmatrix} \alpha & \beta \\ p^2t & \gamma \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\} \\
&= \{\begin{pmatrix} \alpha & \beta \\ p^2t & \gamma \end{pmatrix} : \alpha \equiv 1 \pmod p,\ p^2t \equiv 0 \pmod{p^3}\} \\
&= \{\begin{pmatrix} \alpha & \beta \\ p^2t & \gamma \end{pmatrix} : p|\alpha - 1,\ p|t\} \\
&= \{\begin{pmatrix} \alpha & \beta \\ p^2t & \gamma \end{pmatrix} : \alpha = 1,\ t = 0\} \\
&= \{\begin{pmatrix} 1 & \beta \\ 0 & \gamma \end{pmatrix} : \beta \in \mathbb{Z}_p,\ \gamma \in \mathbb{Z}_{p^3}^\times\}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\langle\begin{pmatrix} 0 \\ p \end{pmatrix}\rangle_\Delta &= \{\begin{pmatrix} \alpha & \beta \\ p^2t & \gamma \end{pmatrix} : \begin{pmatrix} \alpha & \beta \\ p^2t & \gamma \end{pmatrix}\begin{pmatrix} 0 \\ p \end{pmatrix} = \begin{pmatrix} 0 \\ p \end{pmatrix}\} \\
&= \{\begin{pmatrix} \alpha & \beta \\ p^2t & \gamma \end{pmatrix} : p\beta \equiv 0 \pmod p,\ p\gamma \equiv p \pmod{p^3}\} \\
&= \{\begin{pmatrix} \alpha & \beta \\ p^2t & \gamma \end{pmatrix} : 1|\beta,\ p^2|\gamma - 1\} \\
&= \{\begin{pmatrix} \alpha & \beta \\ p^2t & \gamma \end{pmatrix} : \alpha \in \mathbb{Z}_p^\times,\ t \in \mathbb{Z}_p,\ \beta \in \mathbb{Z}_p,\ \gamma \in \{sp^2+1 : s \in \mathbb{Z}_p\}\}.
\end{aligned}$$

We know from remark(3.1)

$$\begin{aligned}
(Y_p)_\Delta &= \langle\begin{pmatrix} 1 \\ 0 \end{pmatrix}\rangle_\Delta \cap \langle\begin{pmatrix} 0 \\ p \end{pmatrix}\rangle_\Delta \\
&= \{\begin{pmatrix} 1 & \beta \\ 0 & \gamma \end{pmatrix} : \beta \in \mathbb{Z}_p,\ \gamma \in \{sp^2+1 : s \in \mathbb{Z}_p\}\}.
\end{aligned}$$

Therefore order of $(Y_p)_\Delta$ is p^2. □

Now we are in the position to calculate $\theta(G, p^3)$ for $\mathrm{Aut}(\mathbb{Z}_p \times \mathbb{Z}_{p^3})$. We need to calculate all those automorphisms fixing at least p^3 elements, then we subtract identity automorphism, because identity automorphism fixes all the elements of the group.

Lemma 3.4. *For the group $\mathbb{Z}_p \times \mathbb{Z}_{p^3}$, we have $\theta(G, p^3) = p(p^2 - 1) - 1$.*

Proof. We know that $\sum_{k=0}^{p-1} |(Y_k)_\Delta| = p^2(p-1)$, $|(Y_p)_\Delta| = p^2$
$\sum_{k=0}^{p-1}(|\Delta^{-1}_{\mathbb{Z}_p\times\mathbb{Z}_{p^3}}(Y_k)|) = (p^2(p-1)-(p-1)) = (p+1)(p-1)^2$, $|\Delta^{-1}_{\mathbb{Z}_p\times\mathbb{Z}_{p^3}}(Y_p)| = p^2-1$. Thus

$$\begin{aligned}
\theta(G, p^3) &= \sum_{k=0}^{p-1}(|\Delta^{-1}_{\mathbb{Z}_p\times\mathbb{Z}_{p^3}}(Y_k)|) + |\Delta^{-1}_{\mathbb{Z}_p\times\mathbb{Z}_{p^3}}(Y_p)| \\
&= (p+1)(p-1)^2 + p^2 - 1 \\
&= p(p^2-1).
\end{aligned}$$

Now we subtract identity automorphism, so we get

$$\theta(G, p^3) = p(p^2-1) - 1.$$

Which is required. □

Next, we calculate all those automorphisms which fix at least p^2 elements, then we subtract all those automorphisms fixing more then p^2 elements. For this first we prove the following lemmas.

Lemma 3.5. *For the subgroup $K_p = \langle\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ p^2 \end{pmatrix}\rangle$ of $\mathbb{Z}_p \times \mathbb{Z}_{p^3}$, we have $(K_p)_\Delta = \{\begin{pmatrix} 1 & \beta \\ 0 & \gamma \end{pmatrix} : \beta \in \mathbb{Z}_p,\ \gamma \in \{sp+1 : s \in \mathbb{Z}_{p^2}\}\}$ and $|(K_p)_\Delta| = p^3$.*

Proof. By definition

$$\begin{aligned}
\langle\begin{pmatrix} 1 \\ 0 \end{pmatrix}\rangle_\Delta &= \{\begin{pmatrix} \alpha & \beta \\ p^2t & \gamma \end{pmatrix} : \begin{pmatrix} \alpha & \beta \\ p^2t & \gamma \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\} \\
&= \{\begin{pmatrix} \alpha & \beta \\ p^2t & \gamma \end{pmatrix} : \alpha \equiv 1 \pmod p,\ p^2t \equiv 0 \pmod{p^3}\} \\
&= \{\begin{pmatrix} \alpha & \beta \\ p^2t & \gamma \end{pmatrix} : p|\alpha - 1,\ p|t\} \\
&= \{\begin{pmatrix} \alpha & \beta \\ p^2t & \gamma \end{pmatrix} : \alpha = 1,\ t = 0\} \\
&= \{\begin{pmatrix} 1 & \beta \\ 0 & \gamma \end{pmatrix} : \beta \in \mathbb{Z}_p,\ \gamma \in \mathbb{Z}_{p^3}^\times\}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\langle\begin{pmatrix} 0 \\ p^2 \end{pmatrix}\rangle_\Delta &= \{\begin{pmatrix} \alpha & \beta \\ p^2t & \gamma \end{pmatrix} : \begin{pmatrix} \alpha & \beta \\ p^2t & \gamma \end{pmatrix}\begin{pmatrix} 0 \\ p^2 \end{pmatrix} = \begin{pmatrix} 0 \\ p^2 \end{pmatrix}\} \\
&= \{\begin{pmatrix} \alpha & \beta \\ p^2t & \gamma \end{pmatrix} : p^2\beta \equiv 0 \pmod p,\ p^2\gamma \equiv p^2 \pmod{p^3}\} \\
&= \{\begin{pmatrix} \alpha & \beta \\ p^2t & \gamma \end{pmatrix} : 1|p\beta,\ p|\gamma - 1\} \\
&= \{\begin{pmatrix} \alpha & \beta \\ p^2t & \gamma \end{pmatrix} : \alpha \in \mathbb{Z}_p^\times,\ \beta,\ t \in \mathbb{Z}_p,\ \gamma \in \{sp + 1 : s \in \mathbb{Z}_{p^2}\}\}.
\end{aligned}$$

By remark (3.1)

$$\begin{aligned}
(K_p)_\Delta &= \langle\begin{pmatrix} 1 \\ 0 \end{pmatrix}\rangle_\Delta \cap \langle\begin{pmatrix} 0 \\ p^2 \end{pmatrix}\rangle_\Delta \\
&= \{\begin{pmatrix} 1 & \beta \\ 0 & \gamma \end{pmatrix} : \beta \in \mathbb{Z}_p,\ \gamma \in \{sp + 1 : s \in \mathbb{Z}_{p^2}\}\}.
\end{aligned}$$

Therefore order of $(K_p)_\Delta$ is p^3. □

Lemma 3.6. *For the subgroup* $K_k = \langle\begin{pmatrix} k \\ p \end{pmatrix}\rangle$ *where* $k \in \mathbb{Z}_p$, *of* G *there are total* $2p^3(p-1)$ *automorphisms fixing all* K_k, *where* $k \in \mathbb{Z}_p$.

Proof. Let $(K_k)_\Delta = \langle\begin{pmatrix} 0 \\ p \end{pmatrix}\rangle_\Delta \cup \langle\begin{pmatrix} n \\ p \end{pmatrix}\rangle_\Delta$, $n \in \mathbb{Z}_p \setminus \{0\}$.
First we need to calculate $\langle\begin{pmatrix} 0 \\ p \end{pmatrix}\rangle_\Delta$.

$$\begin{aligned}
\langle\begin{pmatrix} 0 \\ p \end{pmatrix}\rangle_\Delta &= \{\begin{pmatrix} \alpha & \beta \\ p^2t & \gamma \end{pmatrix} : \begin{pmatrix} \alpha & \beta \\ p^2t & \gamma \end{pmatrix}\begin{pmatrix} 0 \\ p \end{pmatrix} = \begin{pmatrix} 0 \\ p \end{pmatrix}\} \\
&= \{\begin{pmatrix} \alpha & \beta \\ p^2t & \gamma \end{pmatrix} : p\beta \equiv 0 \pmod p,\ p\gamma \equiv p \pmod{p^3}\} \\
&= \{\begin{pmatrix} \alpha & \beta \\ p^2t & \gamma \end{pmatrix} : 1|\beta,\ p^2|\gamma - 1\} \\
&= \{\begin{pmatrix} \alpha & \beta \\ p^2t & \gamma \end{pmatrix} : \alpha \in \mathbb{Z}_p^\times,\ t \in \mathbb{Z}_p,\ \beta \in \mathbb{Z}_p,\ \gamma \in \{sp^2 + 1 : s \in \mathbb{Z}_p\}\}.
\end{aligned}$$

$|\langle\begin{pmatrix} 0 \\ p \end{pmatrix}\rangle_\Delta| = p^3(p-1)$.
Similarly,

$$\begin{aligned}
\langle\begin{pmatrix} n \\ p \end{pmatrix}\rangle_\Delta &= \{\begin{pmatrix} \alpha & \beta \\ p^2t & \gamma \end{pmatrix} : \begin{pmatrix} \alpha & \beta \\ p^2t & \gamma \end{pmatrix}\begin{pmatrix} n \\ p \end{pmatrix} = \begin{pmatrix} n \\ p \end{pmatrix}\} \\
&= \{\begin{pmatrix} \alpha & \beta \\ p^2t & \gamma \end{pmatrix} : n\alpha + p\beta \equiv n \pmod p,\ p^2nt + p\gamma \equiv p \pmod{p^3}\} \\
&= \{\begin{pmatrix} \alpha & \beta \\ p^2t & \gamma \end{pmatrix} : p|n(\alpha - 1) + p\beta,\ p^2|pnt + (\gamma - 1)\} \\
&= \{\begin{pmatrix} \alpha & \beta \\ p^2t & \gamma \end{pmatrix} : \alpha = 1,\ \beta \in \mathbb{Z}_p, \text{and for any } t \text{ there must} \\
&\qquad \text{exist } p \text{ elements for } \gamma\}.
\end{aligned}$$

And $|\langle\begin{pmatrix} n \\ p \end{pmatrix}\rangle_\Delta| = p^3(p-1)$.
Therefore $\sum_{k=0}^{p-1} |(K_k)_\Delta| = p^3(p-1) + p^3(p-1) = 2p^3(p-1)$. □

Lemma 3.7. *For the group $G = \mathbb{Z}_p \times \mathbb{Z}_{p^3}$,*

$$\theta(G, p^2) = p(2p^3 - 3p^2 + 1).$$

Proof. We know that, $|(K_p)_\Delta| = p^3$, $\sum_{k=0}^{p-1} |(K_k)_\Delta| = 2p^3(p-1)$.
Therefore
$|\Delta^{-1}_{\mathbb{Z}_p \times \mathbb{Z}_{p^3}}(K_p)| = p^3 - p^2$, and $\sum_{k=0}^{p-1} |\Delta^{-1}_{\mathbb{Z}_p \times \mathbb{Z}_{p^3}}(K_k)| = 2p^3(p-1) - p^2(p-1)$.
Hence

$$\begin{aligned}
\theta(G, p^2) &= |\Delta^{-1}_{\mathbb{Z}_p \times \mathbb{Z}_{p^3}}(K_p)| + \sum_{k=0}^{p-1}(|\Delta^{-1}_{\mathbb{Z}_p \times \mathbb{Z}_{p^3}} K_k|) \\
&= p^3 - p^2 + (2p^3(p-1) - p^2(p-1)) \\
&= (p-1)(p^3 + p^3 - p^2 + p^2) \\
&= 2p^4 - 2p^3.
\end{aligned}$$

Now we subtract $\theta(G,p^3)$ and $\theta(G,p^4)$ form $\theta(G,p^2)$, so we get

$$\begin{aligned}\theta(G,p^2) &= 2p^4-2p^3-(p(p^2-1)-1)-1\\ &= 2p^4-3p^3+p\\ &= p(2p^3-3p^2+1).\end{aligned}$$

Which is required. □

Next we calculate all those automorphisms fixing at least p elements. Then we subtract $\theta(G,p^2)$, $\theta(G,p^3)$ and $\theta(G,p^4)$ form $\theta(G,p)$.

Lemma 3.8. *For the cyclic subgroup $H_k = \langle\begin{pmatrix} k \\ p^2\end{pmatrix}\rangle$, for $k \in \mathbb{Z}_p$ of the group $G=\mathbb{Z}_p\times\mathbb{Z}_{p^3}$, there are total $p^3(p-1)(2p-1)$ automorphisms fixing H_k.*

Proof. Since know that

$$(H_k)_\Delta = \langle\begin{pmatrix} 0 \\ p^2\end{pmatrix}\rangle \cup \langle\begin{pmatrix} n \\ p^2\end{pmatrix}\rangle,\ \text{for } n\in\mathbb{Z}_p\setminus\{0\}.$$

Now

$$\begin{aligned}\langle\begin{pmatrix} 0 \\ p^2\end{pmatrix}\rangle_\Delta &= \{\begin{pmatrix}\alpha & \beta\\ p^2t & \gamma\end{pmatrix} : \begin{pmatrix}\alpha & \beta\\ p^2t & \gamma\end{pmatrix}\begin{pmatrix} 0 \\ p^2\end{pmatrix} = \begin{pmatrix} 0 \\ p^2\end{pmatrix}\}\\ &= \{\begin{pmatrix}\alpha & \beta\\ p^2t & \gamma\end{pmatrix} : \beta p^2 \equiv 0 \pmod p,\ p^2\gamma \equiv p^2 \pmod{p^3}\}\\ &= \{\begin{pmatrix}\alpha & \beta\\ p^2t & \gamma\end{pmatrix} : 1 \mid p\beta,\ p \mid \gamma-1\}\\ &= \{\begin{pmatrix}\alpha & \beta\\ p^2t & \gamma\end{pmatrix} : \alpha\in\mathbb{Z}_p^\times,\ t\in\mathbb{Z}_p,\ \beta\in\mathbb{Z}_p,\ \gamma\in\{sp+1 : s\in\mathbb{Z}_{p^2}\}\}.\end{aligned}$$

Thus

$|\langle\begin{pmatrix} 0 \\ p^2\end{pmatrix}\rangle_\Delta| = p^4(p-1)$. Also

$$\begin{aligned}\langle\begin{pmatrix} n \\ p^2\end{pmatrix}\rangle_\Delta &= \{\begin{pmatrix}\alpha & \beta\\ p^2t & \gamma\end{pmatrix} : \begin{pmatrix}\alpha & \beta\\ p^2t & \gamma\end{pmatrix}\begin{pmatrix} n \\ p^2\end{pmatrix} = \begin{pmatrix} n \\ p^2\end{pmatrix}\}\\ &= \{\begin{pmatrix}\alpha & \beta\\ p^2t & \gamma\end{pmatrix} : n(\alpha-1)+\beta p^2 \equiv 0 \pmod p,\ p^2nt+\gamma p^2 \equiv p^2 \pmod{p^3}\}\\ &= \{\begin{pmatrix}\alpha & \beta\\ p^2t & \gamma\end{pmatrix} : p|n(\alpha-1)+\beta p^2,\ p|nt+(\gamma-1)\}\\ &= \{\begin{pmatrix}\alpha & \beta\\ p^2t & \gamma\end{pmatrix} : \alpha=1,\ \beta\in\mathbb{Z}_p,\ \text{if } t\in\{1,p-1\} \text{ then } \gamma \text{ does not exist,}\\ &\qquad \text{for remaining } t \text{ values there must exist } p^2 \text{ elements for each } \gamma\}.\end{aligned}$$

And $|\langle\begin{pmatrix} n \\ p^2\end{pmatrix}\rangle_\Delta| = p^3(p-1)^2$.
Therefore $|(H_k)_\Delta| = p^4(p-1)+p^3(p-1)^2 = p^3(p-1)(2p-1)$. □

Lemma 3.9. *For the group $\mathbb{Z}_p\times\mathbb{Z}_{p^3}$, we have that*

$$\theta(G,p) = p^3(2p^2-5p+2).$$

Proof. We know, $|(H_p)_\Delta| = p^3(p-1)$, and $\sum_{k=0}^{p-1} |(H_k)_\Delta| = p^3(p-1)(2p-1)$.
So $|\Delta^{-1}_{\mathbb{Z}_p\times\mathbb{Z}_{p^3}}(H_p)| = p^3(p-1) - p^3$, and
$\sum_{k=0}^{p-1} |\Delta^{-1}_{\mathbb{Z}_p\times\mathbb{Z}_{p^3}}(H_k)| = p^3(p-1)(2p-1) - p^3(p-1)$.
Therefore

$$\begin{aligned}\theta(G,p) &= |\Delta^{-1}_{\mathbb{Z}_p\times\mathbb{Z}_{p^3}}(H_p)| + \sum_{k=0}^{p-1} |\Delta^{-1}_{\mathbb{Z}_p\times\mathbb{Z}_{p^3}}(H_k)| \\ &= p^3(p-1) - p^3 + (p^3(p-1)(2p-1) - p^3(p-1)) \\ &= 2p^5 - 3p^4.\end{aligned}$$

Now we subtract $\theta(G,p^4)$, $\theta(G,p^3)$ and $\theta(G,p^2)$ from $\theta(G,p)$, to get

$$\begin{aligned}\theta(G,p) &= 2p^5 - 3p^4 - \theta(G,p^4) - \theta(G,p^3) - \theta(G,p^2) \\ &= 2p^5 - 3p^4 - 1 - (p^3 - p - 1) - (2p^4 - 3p^3 + p) \\ &= 2p^5 - 5p^4 + 2p^3 \\ &= p^3(2p^2 - 5p + 2).\end{aligned}$$

This completes the proof. □

Finally we calculate all automorphisms fixing only identity element. To prove this result we subtract $\theta(G,p)$, $\theta(G,p^2)$, $\theta(G,p^3)$ and $\theta(G,p^4)$ form the order of the group $\mathrm{Aut}(\mathbb{Z}_p \times \mathbb{Z}_{p^3})$.

Lemma 3.10. *For the group $\mathbb{Z}_p \times \mathbb{Z}_{p^3}$, we have*

$$\theta(G, d = 1) = p^4(p^2 - 4p + 4).$$

Proof.

$$\begin{aligned}\theta(G = \mathbb{Z}_p \times \mathbb{Z}_{p^3}, 1) &= |\mathrm{Aut}(\mathbb{Z}_p \times \mathbb{Z}_{p^3})| - \theta(G,p) - \theta(G,p^2) - \theta(G,p^3) - \theta(G,p^4) \\ &= p^4(p-1)^2 - p^3(2p^2 - 5p + 2) - (2p^4 - 3p^3 + p) - (p^3 - p - 1) - 1 \\ &= p^6 - 4p^5 + 4p^4 \\ &= p^4(p^2 - 4p + 4).\end{aligned}$$

Hence proved. □

The main result of this paper i.e to find the fixed points of automorphism of the group $\mathbb{Z}_p \times \mathbb{Z}_{p^3}$ is proved in lemmas 3.2, 3.5, 3.8, 3.10 and 3.11.

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