Journal of Mathematical Analysis
ISSN: 2217-3412, URL: http://ilirias.com/jma
Volume 7 Issue 6(2016), Pages 1-12.

GENERALIZED BERINDE-TYPE $(\eta, \xi, \vartheta, \theta)$ CONTRACTIVE MAPPINGS IN B-METRIC SPACES WITH AN APPLICATION

HABES ALSAMIR, MOHD SALMI, MD NOORANI, WASFI SHATANAWI, FAWZIA SHADDAD

Abstract. The purpose of this paper is to present a new class of generalized Berinde-type $(\eta, \xi, \vartheta, \theta)$-contractive in b-metric space and establish some fixed and common fixed point results for the class of mappings in partially ordered complete b-metric spaces. Our results generalize and extend several known results from the context of ordered metric spaces to the setting of ordered b-metric spaces.

1. Introduction and preliminaries

Some mathematicians developed and extended the Banach contraction principle (B.C.P) to different ways in some different spaces. Bakhtin [4] introduced the notion of a b-metric space as a generalization of the standard metric space and studied the Banach contraction principle in b-metric spaces. Then after, many researchers studied many fixed and common fixed point results in the b-metric spaces (see, e.g, [2],[8],[12],[13]).

In 1994, Khan et al.[7] introduced the notation of an altering distance function as follows:

Definition 1.1. The function $\eta : [0, \infty) \to [0, \infty)$ is called altering distance function, if the following conditions hold:

(1) η is continuous and nondecreasing.
(2) $\eta(k) = 0 \Leftrightarrow k = 0$

The following definition will be needed in our results:

Definition 1.2. [6] Let Y be a nonempty set and $s \geq 1$ be a given real number. A function $d : Y \times Y \to [0, \infty)$ is b-metric if for all $y, w, z \in Y$, the following conditions hold:

(1) $d(y, z) = 0$ iff $y = z$,
(2) $d(y, z)=d(z, y)$,
(3) $d(y, z)\leq s[d(y, w) + d(w, z)]$.

In this case, the pair (Y, d) is called a b-metric space.

Key words and phrases. fixed point; common fixed point; partially ordered set; b-metric spaces; integral equation.

Submitted June 8, 2016. Published August 31, 2016.

Notice that every metric space is a b-metric space is metric space with $s = 1$. Some authors introduced some b-metric space which is not metric space. For this instant, see ([10], p.264).

Definition 1.3. [5] Let (Y,d) be a b-metric space. Then a sequence $\{y_r\}$ in Y is called:

(1) b-convergent if and only if there exists $y \in Y$ such that $d(y_r, y) \to 0$, as $r \to \infty$. In this case, we write $\lim_{r\to\infty} y_r = y$.
(2) b-Cauchy if and only if $d(y_r, y_m) \to 0$ as $r, m \to +\infty$.
(3) The b-metric space (Y, d) is b-complete if every b-Cauchy sequence in Y is b-convergent.

Proposition 1.4. *(see remark 2.1 in* [5]*) In the* b*-metric space* (Y, d) *the following assertions holds:*

(1) *A* b*-convergent sequence has a unique limit.*
(2) *Each* b*-convergent is* b*-Cauchy.*
(3) *In general, a* b*-metric is not continuous.*

Hussain et al. (see Example 3 in[10]) introduced an example of a b-metric which is not continuous.

In 2014, the following Lemma about the convergent sequence in b-metric space was proved by Aghajani et. al [1].

Lemma 1.5. *Let* (Y, d) *be a* b*-metric space* $s \geq 1$*, suppose that* $\{y_r\}$ *and* $\{z_r\}$ b*-converge to* y, z *respectively. Then, we have*

$$\frac{1}{s}d(y, z) \leq \liminf_{r\to\infty} d(y_r, z_r) \leq \limsup_{r\to\infty} d(y_r, z_r) \leq sd(y, z).$$

In particular, if $y = z$*, then,* $\lim_{r\to\infty} d(y_r, z_r) = 0$*. Moreover, for each* $w \in Y$ *we have*

$$\frac{1}{s}d(y, w) \leq \liminf_{r\to\infty} d(y_r, w) \leq \limsup_{r\to\infty} d(y_r, w) \leq sd(y, w).$$

Definition 1.6. [3] Let $(Y, \preceq)$ be a partially ordered set. Then the two mappings $T, f : Y \to Y$ are said to be weakly increasing if $Ty \preceq fTy$ and $fy \preceq Tfy$, for all $y \in Y$.

Roshan et al. [9] proved the following interesting theorem for the frame of almost generalized (η, θ) contractive mappings in b-metric space.

Theorem 1.7. *Let* $(Y, \preceq, d)$ *be a partially ordered complete* b*-metric space with constant* $s \geq 1$*. Suppose* $\eta, \theta : [0, \infty) \to [0, \infty)$ *are altering distance functions. Let* $T, f : Y \to Y$ *be two weakly increasing mappings with respect to* $\preceq$*. Suppose that* T *and* f *satisfy the following condition:*

$$\eta(s^4 d(Ty, fz)) \leq \eta(M(y, z)) - \theta(M(y, z)) + L\eta(N(y, z)),$$

where

$$M(y, z) = \max\{d(y, z), d(y, Ty), d(z, fz), \frac{1}{2s}(d(y, fz) + d(z, Ty))\}$$

and

$$N(x, y) = \min\{d(y, Ty), d(z, Ty), d(y, fz)\}$$

for all comparable elements $y,z\in Y$. If either T or f is continuous, then T and f have a common fixed point.

Our goal in this manuscript is to present the notation of generalized Berinde-type $(\eta,\xi,\vartheta,\theta)$ contraction mappings include η and θ are altering distance functions and ξ,ϑ are upper semi-continuous function and lower semi-continuous function respectively with $\gamma\in[0,1]$. Then we derive fixed and common fixed point theorems in complete ordered b-metric spaces. Our results extend and generalize several well-known result in the literature.

2. MAIN RESULT

In the rest of this paper to facilitate our subsequent arguments, we introduce the following:
(a) The functions $\eta,\theta:[0,\infty)\to[0,\infty)$ are assumed to be altering distance functions.
(b) The function $\xi:[0,\infty)\to[0,\infty)$ is assumed to be upper semi-continuous with $\xi(k)<\eta(k)$ for each $k>0$ and $\xi(k)=0$ iff $k=0$.
(c) The function $\vartheta:[0,\infty)\to[0,\infty)$ is assumed to be a lower semi-continuous function with $\vartheta(k)=0$ iff $k=0$.

Now, we introduce the following type of contractive conditions:

Definition 2.1. Let $(Y,\preceq,d)$ be a partially ordered b-metric space with constant $s\geq 1$, and $T,f:Y\to Y$ be two mappings. We say that the pair (T,f) is generalized Berinde-type $(\eta,\vartheta,\xi,\theta)$-contractive if there exist $L\geq 0$ and $\gamma\in[0,1]$ such that

$$\eta(sd(Ty,fz))\leq\xi(\gamma v(y,z))-\vartheta(\gamma v(y,z))+L\theta(N(y,z)),\tag{2.1}$$

where

$$v(y,z)\in\{d(y,z),d(y,Ty),d(z,fz),\frac{1}{2s}(d(Ty,z)+d(y,fz))\}$$

and

$$N(y,z)=\min\{d(y,z),d(y,Ty),d(z,fz),d(Ty,z),d(y,fz)\}$$

holds for all comparable elements $y,z\in Y$.

Lemma 2.2. *Let $(Y,\preceq,d)$ be a partially ordered* b*-metric space with constant $s\geq 1$. Let $T,f:Y\to Y$ be two mappings such that the pair (T,f) is generalized Berinde-type $(\eta,\vartheta,\xi,\theta)$-contractive. Then u is a fixed point of T if and only if u is a fixed point of f.*

Proof. Since the pair (T,f) is generalized Berinde-type $(\eta,\vartheta,\xi,\theta)$-contractive, then there exist $L\geq 0$ and $\gamma\in[0,1]$ such that T and f satisfy the condition (2.1). Now, suppose that u is a fixed point of T. Since $u\preceq u$, by (2.1), we have

$$\begin{aligned}\eta(d(u,fu)) &= \eta(d(Tu,fu))\\ &\leq \eta(sd(Tu,fu))\\ &\leq \xi(\gamma v(u,u))-\vartheta(\gamma v(u,u))+L\theta(N(u,u))\end{aligned}\tag{2.2}$$

where

$$\begin{aligned} v(u,u) &\in \{d(u,u), d(u,Tu), d(u,fu), \frac{1}{2s}(d(Tu,u)+d(u,fu))\} \\ &= \{d(u,u), d(u,u), d(u,fu), \frac{1}{2s}(d(u,u)+d(u,fu))\} \\ &= \{0, d(u,fu), \frac{1}{2s}d(u,fu)\} \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} N(u,u) &= min\{d(u,u), d(u,Tu), d(u,fu), d(Tu,u), d(u,fu)\} \\ &= min\{d(u,u), d(u,u), d(u,fu), d(u,u), d(u,fu)\} \\ &= min\{0, d(u,fu), 0, d(u,fu)\} \\ &= 0. \end{aligned} \tag{2.4}$$

Now from (2.3) and (2.4), then (2.2) becomes:

$$\eta(d(u,fu)) \leq \xi(\gamma v(u,u)) - \vartheta(\gamma v(u,u)). \tag{2.5}$$

If $\gamma = 0$, then from (2.5) we get $\eta(d(u,fu)) = 0$. By using the properties of η, we conclude that $d(u,fu) = 0$. Hence $fu = u$. Therefore u is a fixed point of f.
If $\gamma > 0$, then we study the following three cases.
Case(1) If $v(u,u) = 0$, then

$$\begin{aligned} \eta(d(u,fu)) &\leq \xi(\gamma.0) - \vartheta(\gamma.0)) \\ &= \xi(0) - \vartheta(0) = 0. \end{aligned}$$

Thus we have $d(u,fu) = 0$, and hence $fu = u$. Therefore u is a fixed point of f.
Case(2) If $v(u,u) = d(u,fu)$, then

$$\begin{aligned} \eta(d(u,fu)) &\leq \xi(\gamma d(u,fu)) - \vartheta(\gamma d(u,fu)) \\ &\leq \eta(\gamma d(u,fu)) - \vartheta(\gamma d(u,fu)). \end{aligned}$$

The last inequality is true only if $\vartheta(\gamma d(u,fu)) = 0$. By using the property of ϑ, we get $\gamma d(u,fu) = 0$. Hence $d(u,fu) = 0$. Therefore $fu = u$. Thus u is a fixed point of f.
Case(3) If $v(u,u) = \frac{1}{2s}d(u,fu)$, then

$$\begin{aligned} \eta(d(u,fu)) &\leq \xi(\frac{\gamma}{2s}d(u,fu)) - \vartheta(\frac{\gamma}{2s}d(u,fu))) \\ &\leq \eta(\frac{\gamma}{2s}d(u,fu)) - \vartheta(\frac{\gamma}{2s}d(u,fu))). \end{aligned}$$

Since $\frac{\gamma}{2s} \leq \frac{1}{2}$, then the last inequality is true only if $d(u,fu) = 0$. Therefore $fu = u$ and hence u is a fixed point of f.
Similarly, we can prove that if u is a fixed point of f, then u is a fixed point of T. □

Theorem 2.3. *Let $(Y, \preceq, d)$ be a partially ordered complete* b*-metric space with constant $s \geq 1$. Let $T, f : Y \to Y$ be two weakly increasing mappings with respect to $\preceq$. Assume that the pair (T, f) is generalized Berinde-type $(\eta, \vartheta, \xi, \theta)$-contractive. If either T or f is continuous, then T and f have a common fixed point.*

Proof. Let $y_0 \in Y$ be an arbitrary. We construct $\{y_r\}$ in Y such that $y_{2r+1} = Ty_{2r}$ and $y_{2r+2} = fy_{2r+1}$ for all $r = 0,1,2,....$ Since T and f are weakly increasing with respect to $\preceq$, then we get the following:

$$y_1 = Ty_0 \preceq fTy_0 = y_2 = fy_1 \preceq Tfy_1 = Ty_2 \preceq ...y_{2r+1} = Ty_{2r} \preceq fTy_{2r} = y_{2r+2} \preceq ...$$

If $y_{2r} = y_{2r+1}$ for some $r \in \mathbb{N}$, then $y_{2r} = Ty_{2r}$. Thus y_{2r} is a fixed point of T. By Lemma 2.2, we conclude that y_{2r} is a fixed point of f.
If $y_{2r+1} = y_{2r+2}$ for some $r \in \mathbb{N}$, then $y_{2r+1} = fy_{2r+1}$. Thus y_{2r+1} is a fixed point of f. By Lemma 2.2, we deduce that y_{2r+1} is a fixed point of T.
Now, suppose that $y_r \neq y_{r+1}$ for all $r \in \mathbb{N}$.
step slowromancapi@: We shall prove that

$$\lim_{r\to\infty} d(y_r, y_{r+1}) = 0.$$

Since y_{2r+1} and y_{2r+2} are comparable, then by (2.1) we have

$$\begin{aligned}
\eta(d(y_{2r+1}, y_{2r+2})) &\leq \eta(sd(y_{2r+1}, y_{2r+2})) \\
&\leq \eta(sd(Ty_{2r}, fy_{2r+1})) \\
&\leq \xi(\gamma v(y_{2r}, y_{2r+1})) - \vartheta(\gamma v(y_{2r}, y_{2r+1})) + L\theta(N(y_{2r}, y_{2r+1})), \qquad (2.6)
\end{aligned}$$

where

$$\begin{aligned}
N(y_{2r}, y_{2r+1}) &= \min\{d(y_{2r}, y_{2r+1}), d(y_{2r}, Ty_{2r}), d(y_{2r+1}, fy_{2r+1}), d(Ty_{2r}, y_{2r+1}), d(y_{2r}, fy_{2r+1})\} \\
&= \min\{d(y_{2r}, y_{2r+1}), d(y_{2r}, y_{2r+1}), d(y_{2r+1}, y_{2r+2}), d(y_{2r+1}, y_{2r+1}), d(y_{2r}, y_{2r+2})\} \\
&= \min\{d(y_{2r}, y_{2r+1}), d(y_{2r+1}, y_{2r+2}), 0, d(y_{2r}, y_{2r+2})\} \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
v(y_{2r}, y_{2r+1}) &\in \{d(y_{2r}, y_{2r+1}), d(y_{2r}, Ty_{2r}), d(y_{2r+1}, fy_{2r+1}), \frac{1}{2s}(d(Ty_{2r}, y_{2r+1}) + d(y_{2r}, fy_{2r+1}))\} \\
&= \{d(y_{2r}, y_{2r+1}), d(y_{2r}, y_{2r+1}), d(y_{2r+1}, y_{2r+2}), \frac{1}{2s}(d(y_{2r+1}, y_{2r+1}) + d(y_{2r}, y_{2r+2}))\} \\
&= \{d(y_{2r}, y_{2r+1}), d(y_{2r+1}, y_{2r+2}), \frac{1}{2s}d(y_{2r}, y_{2r+2})\}. \qquad (2.7)
\end{aligned}$$

So, we have

$$\eta(d(y_{2r+1}, y_{2r+2})) \leq \xi(\gamma v(y_{2r}, y_{2r+1})) - \vartheta(\gamma v(y_{2r}, y_{2r+1})) \qquad (2.8)$$

and

$$\eta(sd(y_{2r+1}, y_{2r+2})) \leq \xi(\gamma v(y_{2r}, y_{2r+1})) - \vartheta(\gamma v(y_{2r}, y_{2r+1})). \qquad (2.9)$$

Now, we show that $\{d(y_r, y_{r+1})\}$ is a decreasing sequence. To do that, we study the following cases:
Case(1): If $v(y_{2r}, y_{2r+1}) = d(y_{2r}, y_{2r+1})$, then

$$\begin{aligned}
\eta(d(y_{2r+1}, x_{2r+2})) &\leq \xi(\gamma d(y_{2r}, y_{2r+1})) - \vartheta(\gamma d(y_{2r}, y_{2r+1})) \\
&< \xi(\gamma d(y_{2r}, y_{2r+1})) \\
&< \eta(\gamma d(y_{2r}, y_{2r+1})).
\end{aligned}$$

By using the properties of η, we have

$$d(y_{2r+1}, y_{2r+2}) < \gamma d(y_{2r}, y_{2r+1}) < d(y_{2r}, y_{2r+1}).$$

Case(2): If $v(y_{2r}, y_{2r+1}) = d(y_{2r+1}, y_{2r+2})$, then

$$\begin{aligned} \eta(d(y_{2r+1}, y_{2r+2})) &\leq \xi(\gamma d(y_{2r+1}, y_{2r+2}) - \vartheta(\gamma d(y_{2r+1}, y_{2r+2})) \\ &< \xi(\gamma d(y_{2r+1}, y_{2r+2})) \\ &< \eta(\gamma d(y_{2r+1}, y_{2r+2})). \end{aligned}$$

By using the properties of η, we have $d(y_{2r+1}, y_{2r+2}) < d(y_{2r+1}, y_{2r+2})$, which is impossible.

Case(3): If $v(y_{2r}, y_{2r+1}) = \frac{1}{2s} d(y_{2r}, y_{2r+2})$, then

$$\begin{aligned} \eta(d(y_{2r+1}, y_{2r+2})) &\leq \xi(\frac{\gamma}{2s} d(y_{2r}, y_{2r+2})) - \vartheta(\frac{\gamma}{2s} d(y_{2r}, y_{2r+2})) \\ &< \xi(\frac{\gamma}{2s} d(y_{2r}, y_{2r+2})) \\ &< \eta(\frac{\gamma}{2s} d(y_{2r}, y_{2r+2})) \\ &< \eta(\frac{\gamma}{2}(d(y_{2r}, y_{2r+1}) + d(y_{2r+1}, y_{2r+2}))). \end{aligned}$$

By using the properties of η, we can show that

$$d(y_{2r+1}, y_{2r+2}) < \frac{\gamma}{2-\gamma} d(y_{2r}, y_{2r+1}),$$

Since $\frac{\gamma}{2-\gamma} \leq 1$, we conclude that $d(y_{2r+1}, y_{2r+2}) < d(y_{2r}, y_{2r+1})$.
Hence from the previous cases we get

$$d(y_{2r+1}, y_{2r+2}) \leq d(y_{2r+1}, y_{2r}). \tag{2.10}$$

Similarly, we can show that:

$$d(y_{2r+1}, y_{2r}) \leq d(y_{2r}, y_{2r-1}) \tag{2.11}$$

From (2.10)and (2.11), the sequence$\{d(y_r, y_{r+1}) : r \in \mathbb{N}\}$ is nonnegative decreasing. Thus, there exists $d \geq 0$ such that:

$$\lim_{r\to\infty} d(y_r, y_{r+1}) = d \tag{2.12}$$

Next, we show that $d = 0$. Assume that $d \neq 0$. By referring to (2.7), we have the following cases:

Case(1): If $v(y_{2r}, y_{2r+1}) \in \{d(y_{2r}, y_{2r+1}), d(y_{2r+1}, y_{2r+2})\}$, then $\lim_{n\to+\infty} v(y_{2r}, y_{2r+1}) = d$.

By taking $r \to \infty$ in (2.8) and using the properties of η, ξ and ϑ, we have:

$$\begin{aligned} \eta(d) &\leq \xi(\gamma d) - \vartheta(\gamma d) \\ &< \eta(\gamma d) - \vartheta(\gamma d) \\ &< \eta(\gamma d) \\ &\leq \eta(d). \end{aligned}$$

If $\gamma = 0$, then we obtain that $\eta(d) = 0$ and hence $d = 0$; which is, a contradiction. So, we assume that $d \neq 0$. Therefore the above inequalities are true only if

$\vartheta(\gamma d) = 0$. Hence $\gamma d = 0$. So $d = 0$; which is, a contradiction.

Case(2): If $v(y_{2r}, y_{2r+1}) = \frac{1}{2s} d(y_{2r}, y_{2r+2})$, then $v(y_{2r}, y_{2r+1}) \leq \frac{1}{2} d(y_{2r}, y_{2r+1}) + \frac{1}{2} d(y_{2r+1}, y_{2r+2})$. So

$$\begin{aligned} \eta(d(y_{2r+1}, y_{2r+2})) &\leq \xi(\frac{\gamma}{2s} d(y_{2r}, y_{2r+2})) - \vartheta(\frac{\gamma}{2s} d(y_{2r}, y_{2r+2})) \\ &< \eta(\frac{\gamma}{2s}(d(y_{2r}, y_{2r+2})) - \vartheta(\frac{\gamma}{2s} d(y_{2r}, y_{2r+2})). \quad (2.13) \end{aligned}$$

Letting $r \to \infty$ in (2.13), we have the following subcases: Subcase(1): If $\gamma = 0$, then $\eta(d) = 0$. Hence $d = 0$ a contradiction.
Subcase(2): $\gamma \neq 0$. Here, we obtain $\vartheta(\gamma d) < \eta(\gamma d) - \eta(d) \leq 0 \Rightarrow \gamma d \leq 0$ which leads to contradiction. From the above we get $d = 0$. Therefore

$$\lim_{r\to\infty} d(y_{2r+1}, y_{2r+2}) = 0 \quad (2.14)$$

Step slowromancapii@: (Cauchy sequence)
First, we shall prove that

$$\lim_{l\to\infty} d(y_{2r_l}, y_{2m_l}) = 0. \quad (2.15)$$

Assume (2.15) is incorrect. Thus there exists $\epsilon > 0$ and subsequences $\{y_{2m_l}\}$ and $\{y_{2r_l}\}$ of $\{y_{2r}\}$ with $r_l > m_l > l$ such that:

$$d(y_{2r_l}, y_{2m_l}) \geq \frac{\epsilon}{l} \quad (2.16)$$

This means that:

$$d(y_{2r_l-1}, y_{2m_l}) < \frac{\epsilon}{l} \quad (2.17)$$

From the definition of b-metric space, (2.16) and (2.17) we get:

$$\begin{aligned} \frac{\epsilon}{l} \leq d(y_{2r_l}, y_{2m_l}) &\leq sd(y_{2r_l}, y_{2r_l-1}) + sd(y_{2r_l-1}, y_{2m_l}) \\ &\leq sd(y_{2r_l}, y_{2r_l-1}) + \frac{s\epsilon}{l}. \quad (2.18) \end{aligned}$$

Taking the upper limit when $l \to \infty$, we will obtain

$$\lim_{l\to\infty} d(y_{2r_l}, y_{2m_l}) = 0. \quad (2.19)$$

Which is a contradiction with (2.16).Thus $\{y_r\}$ is a b-Cauchy sequence in b-metric space (Y, d).

Step slowromancapiii@: (We will show that T and f have a common fixed point)
Since $\{y_r\}$ is a b-Cauchy sequence in Y which is a b-complete b-metric space, then there exists $u \in Y$ such that $y_r \to x$ as $r \to \infty$, and

$$\lim_{r\to\infty} y_{2r+1} = \lim_{r\to\infty} Ty_{2r} = u$$

Without loss of generality, we may assume that T is continuous. By using the definition of b-metric space, we obtain:

$$d(u, Tu) \leq s(d(u, Ty_{2r}) + d(Ty_{2r}, Tu)).$$

By letting $r \to \infty$, we get

$$d(u,Tu) \quad \leq \quad s\lim_{r\to\infty} d(u,Ty_{2r}) + s\lim_{r\to\infty} d(Ty_{2r},Tu) = 0.$$

So, we have $d(u,Tu) = 0$ and hence $Tu = u$. So u is a fixed point of T. By using Lemma 2.2, we conclude that u is also a fixed point of f. So u is a common fixed point of T and f. □

The continuity in Theorem 2.3 can be dropped.

In the rest of this paper, we assume that Y satisfies the following conditions: (i) If $\{y_r\}$ is a nondecreasing sequence in Y such that $y_r \to y \in Y$, then $y_r \preceq y$, for all $r \in \mathbb{N}$.

Theorem 2.4. *Assume that the hypotheses of Theorem 2.3, without the continuity of one of functions T or f. If Y satisfies condition (i), then T and f have a common fixed point.*

Proof. By using the similar arguments of the proof of Theorem 2.3, we construct a nondecreasing sequence $\{y_r\}$ in Y such that $y_r \to u \in Y$ for some $u \in Y$. Using the assumption on Y, we have $y_r \preceq u$ for all $r \in \mathbb{N}$. Now, we show that $Tu = fu = u$. Suppose the contrary $fu \neq u$, by (2.1), we have:

$$\begin{aligned} \eta(d(y_{2r+1},fu)) \quad &\leq \quad \eta(sd(Ty_{2r},fu)) \\ &\leq \quad \xi(\gamma v(y_{2r},u)) - \vartheta(\gamma v(y_{2r},u)) + L\theta(N(y_{2r},u)) \end{aligned} \tag{2.20}$$

where

$$\begin{aligned} N(y_{2r},u) \quad &= \quad \min\{d(y_{2r},u), d(y_{2r},Ty_{2r}), d(u,fu), d(Ty_{2r},u), d(y_{2r},fu)\} \\ &= \quad \min\{d(y_{2r},u), d(y_{2r},y_{2r+1}), d(u,fu), d(y_{2r+1},u), d(y_{2r},fu)\} \end{aligned} \tag{2.21}$$

and

$$\begin{aligned} v(y_{2r},u) \quad &\in \quad \{d(y_{2r},u), d(y_{2r},Ty_{2r}), d(u,fu), \frac{1}{2s}(d(Ty_{2r},u) + d(y_{2r},fu))\} \\ &= \quad \{d(y_{2r},u), d(y_{2r},y_{2r+1}), d(u,fu), \frac{1}{2s}(d(y_{2r+1},u) + d(y_{2r},fu))\} \end{aligned} \tag{2.22}$$

Taking $r \to \infty$ in (2.21)and(2.22)and using Lemma 1.5, we get

$$\lim_{r\to\infty}\sup\, v(y_{2r+1},fu) \leq \{d(u,fu), \frac{d(u,fu)}{2s}\}, \tag{2.23}$$

and $N(y_{2r},u) \to 0$. Now, taking $\limsup$ as $r \to \infty$ in (2.20) and using Lemma 1.5 and (2.23), we get

$$\begin{aligned} \eta(d(u,fu)) \quad &= \quad \eta(s\frac{1}{s}d(u,fu)) \leq \eta(s\lim_{r\to\infty}\sup d(y_{2r+1},fu)) \\ &\leq \quad \xi(\gamma\lim_{r\to\infty}\sup v(y_{2r},u) - \vartheta(\gamma\lim_{r\to\infty}\inf v(y_{2r},u)) \\ &< \quad \eta(\gamma\{d(u,fu), \frac{d(u,fu)}{2}\}) - \vartheta(\gamma\lim_{r\to\infty}\inf v(y_{2r},u)) \end{aligned}$$

It follows that $\vartheta(\gamma\lim_{r\to\infty}\inf v(y_{2r},u)) = 0$, and from (2.23). Now, suppose that $d(u,fu) \neq 0$. From above inequalities, we discuss the following cases:

Case(1): If $\gamma\{d(u, fu), \frac{d(u,fu)}{2}\} = \gamma d(u, fu)$, then $\eta(d(u, fu)) < \eta(\gamma d(u, fu)) \leq \eta(d(u, fu))$, which is a contradiction.

Case (2) If $\gamma\{d(u, fu), \frac{d(u,fu)}{2}\} = \gamma\frac{d(u,fu)}{2}$, then $\eta(d(u, fu)) < \eta(\gamma\frac{d(u,fu)}{2}) \leq \eta(d(u, fu))$, which is a contradiction.
Hence, $fu = u$. By Lemma 2.2, we conclude that u is a fixed point of T. Therefore, u is a common fixed point of f and T. □

Putting $f = T$ in Theorem 2.3 and 2.4, we obtain the next two corollaries:

Corollary 2.5. *Let $(Y, \preceq, d)$ be a partially ordered complete* b*-metric space with constant $s \geq 1$. Let $T : Y \to Y$ be a nondecreasing continuous mapping with respect to $\preceq$. Suppose that T satisfies the following condition*

$$\eta(sd(Ty, Tz)) \leq \xi(\gamma v(y, z)) - \vartheta(\gamma v(y, z)) + L\theta(N(y, z)), \tag{2.24}$$

where

$$v(y, z) \in \{d(y, z), d(y, Ty), d(z, Tz), \frac{1}{2s}(d(Ty, z) + d(y, Tz))\}$$

and

$$N(y, z) = \min\{d(y, z), d(y, Ty), d(z, Tz), d(Ty, z), d(y, Tz)\}$$

for all comparable elements $y, z \in Y$. If there exists $y_0 \in Y$ such that $y_0 \preceq Ty_0$, then T has a fixed point.

Corollary 2.6. *Assume that the hypotheses of Corollary 2.5, without the continuity of T. If Y satisfies condition (i), then T has a fixed point.*

Corollary 2.7. *Let $(Y, \preceq, d)$ be a partially ordered complete* b*-metric space. Let $T, f : Y \to Y$ be two weakly increasing mappings with respect to $\preceq$. Suppose that T and f satisfying the following inequality:*

$$\eta(sd(Ty, fz)) \leq \xi(\gamma M(y, z)) - \vartheta(\gamma M(y, z)) + L\theta(N(y, z)).$$

where

$$M(y, z) = \max\{d(y, z), d(y, Ty), d(z, fz), \frac{1}{2s}(d(Ty, z) + d(y, fz))\}$$

and

$$N(y, z) = \min\{d(y, z), d(y, Ty), d(z, fz), d(Ty, z), d(y, fz)\}$$

for all comparable $y, z \in Y$ with $L \geq 0$. Then T have a fixed point.

Proof. Since $M(y, z) \in \{d(y, z), d(y, Ty), d(z, fz), \frac{1}{2s}(d(Ty, z)+d(y, fz))\}$, then the result follows from Theorem 2.3. □

Corollary 2.8. *Assume that the hypotheses of Corollary 2.7, without assuming the continuity of T or f. If Y satisfies condition (i), then T and f have a common fixed point in Y.*

Proof. Since $M(y, z) \in \max\{d(y, z), d(y, Ty), d(z, fz), \frac{1}{2s}(d(Ty, z) + d(y, fz))\}$, it Follows from Theorem 2.4 □

Remark 2.9. In Corollary (2.5):

(1) If θ, ξ, η and ϑ are altering distance functions, $\gamma = 1$ and $N(y, z) = \min\{d(y, Ty), d(z, Ty)\}$, then we obtain Theorem 3 of Roshan, J et al. [9].
(2) If θ, ξ, η and ϑ are an altering distance functions, $s = 1$, then we obtain a metric version of Theorem 2.1 of Shatanawi, W and Al-rawashdeh, A [11].

Example 2.10. *Let $Y = [0, \infty)$ be equipped with the* b-*metric* $d(y, z) = |y - z|^2$ *for all* $y, z \in Y$*, where* $s = 2^{3-1}$*. Define the relation* $\preceq$ *on* Y *by* $y \preceq z$ *iff* $z \leq y$*. Also, define the function* $T : Y \to Y$ *by*

$$Ty = \ln(1 + \frac{y}{3}),$$

and the functions $\eta, \xi, \vartheta : [0, \infty) \to [0, \infty)$ *by* $\eta(k) = \frac{4a}{s}t$, $\xi(k) = ak$ *and* $\vartheta(k) = (a - 1)k$*, where* $1 \leq a \leq \frac{9}{4}$*. Take* $\gamma = 1$ *and* $L = 0$*. Then, we have the following:*

1. $(Y, \preceq, d)$ *is a partially ordered complete* b-*metric.*
2. T *is nondecreasing with respect* $\preceq$*.*
3. T *is continuous.*
4. T *satisfies*

$$\eta(sd(Ty, Tz)) \leq \xi(\gamma d(y, z)) - \vartheta(\gamma d(y, z)) + L\theta(N(y, z))$$

for all comparable elements $y, z \in Y$*. That is, the pair* (T, T) *is generalized Berinde-type* $(\eta, \xi, \vartheta, \theta)$ *contractive.*

Proof. The proof of (1) is clear. To prove (2), given $y_1, y_2 \in Y$ be such that $y_1 \preceq y_2$. Then $y_2 \leq y_1$. So

$$1 + \frac{y_2}{3} \leq 1 + \frac{y_1}{3}.$$

Hence

$$\ln\left(1 + \frac{y_2}{3}\right) \leq \ln\left(1 + \frac{y_1}{3}\right).$$

So $Ty_2 \leq Ty_1$. Therefore $Ty_1 \preceq Ty_2$; that is, T is nondecreasing with respect to $\preceq$. It is an easy matter to see that T continuous.

To prove (4), let $y, z \in Y$ with $y \preceq z$. So $z \leq y$. By using the mean value theorem for $\ln(1 + k)$, for $k \in [\frac{z}{3}, \frac{y}{3}]$, we have

$$\begin{aligned}
\eta(sd(Ty, Tz) &= 4ad(Ty, Tz) \\
&= 4a\left(\ln(1 + \frac{y}{3}) - \ln(1 + \frac{z}{3})\right)^2 \\
&\leq 4a\left(\frac{y}{3} - \frac{z}{3}\right)^2 \\
&= \frac{4a}{9}(y - z)^2 \\
&\leq (y - z)^2 = d(y, z) \\
&= \xi(d(y, z)) - \vartheta(d(y, z)).
\end{aligned}$$

Since $L \geq 0$, then we have

$$\eta(sd((Ty, fz)) \leq \xi(\gamma d(y, z)) - \vartheta(\gamma d(y, z)) + L\theta(N(y, z))$$

holds for all comparable elements $y, z \in Y$.
Note that T satisfies all the hypothesis of Corollary 2.5. So T has a fixed point, Here 0 is the fixed point of T. ∎

Denote by Υ the set of all functions $\alpha:\mathbb{R}^+\to\mathbb{R}^+$ satisfying the following hypotheses:

(1) Every $\alpha\in\Upsilon$ is a Lebesgue integrable function on each compact $\subset[0,\infty)$;
(2) For every $\epsilon>0$, we have $\int_0^\epsilon \alpha(\omega)\,\mathrm{d}\omega>0$.

We have the following:

Corollary 2.11. *Let $(Y,\preceq)$ be a partially ordered complete* b*-metric space with constant $s\geq 1$. Let $T:Y\to Y$ be a nondecreasing continuous mappings with respect to $\preceq$. Suppose that there exist $\gamma\in[0,1]$, $p\in(0,1)$, $\alpha\in\Upsilon$ and $L\geq 0$ such that*

$$\int_0^{sd(Ty,Tz)}\alpha(\omega)d\omega \quad\leq\quad \frac{p}{2}\int_0^{\gamma v(y,z)}\alpha(\omega)\,\mathrm{d}\omega+L\int_0^{N(y,z)}\alpha(\omega)d\omega$$

where

$$v(y,z)\in\{d(y,z),d(y,Ty),d(z,Tz),\frac{1}{2s}(d(Ty,z)+d(y,Tz))\}$$

and

$$N(y,z)=\min\{d(y,z),d(y,Ty),d(z,Tz),d(Ty,z),d(y,Tz)\}$$

holds for all comparable elements $y,z\in Y$. If there exists $y_0\in Y$ such that $y_0\preceq Ty_0$, then T has a fixed point.

Proof. Define $\eta(k)=\int_0^k\alpha(\omega)\,\mathrm{d}\omega$, $\xi(k)=p\int_0^k\alpha(\omega)d\omega$, $\vartheta(k)=\frac{p}{2}\int_0^k\alpha(\omega)d\omega$ and $\theta(k)=\int_0^k\alpha(\omega)d\omega$ for all $k\in[0,\infty)$. Then the result follows from Corollary 2.5. □

Corollary 2.12. *Let $(X,\preceq,d)$ be a partially ordered complete* b*-metric space with constant $s\geq 1$. Let $T,f:X\to X$ be two weakly increasing mappings with respect to $\preceq$. Suppose that there exist $\gamma\in[0,1]$, $p\in(0,1)$, $\alpha\in\Upsilon$ and $L\geq 0$ such that*

$$\int_0^{sd(Ty,fz)}\alpha(\omega)\,d\omega \quad\leq\quad \frac{p}{2}\int_0^{\gamma v(y,z)}\alpha(\omega)\,\mathrm{d}\omega+L\int_0^{N(y,z)}\alpha(\omega)d\omega$$

where

$$v(y,z)\in\{d(y,z),d(y,Ty),d(z,fz),\frac{1}{2s}(d(Ty,z)+d(y,fz))\}$$

and

$$N(y,z)=\min\{d(y,z),d(y,Ty),d(z,fz),d(Ty,z),d(y,fz)\}$$

holds for all comparable elements $y,z\in Y$. If either $Torf$ is continuous, then T and f have a common fixed point.

Proof. Define $\eta(k)=\int_0^k\alpha(\omega)\,\mathrm{d}\omega$, $\xi(k)=p\int_0^k\alpha(\omega)d\omega$, $\vartheta(k)=\frac{p}{2}\int_0^k\alpha(\omega)d\omega$ and $\theta(k)=\int_0^k\alpha(\omega)d\omega$ for all $k\in[0,\infty)$. Then the result follows from Theorem 2.3. □

3. Acknowledgement

The authors are grateful for the reviewers for their valuable remarks on the paper, that lead us to rewrite our paper in very good version. Also, the authors would like to acknowledge the grant: UKM Grant DIP-2014-034 and Ministry of Education, Malaysia grant FRGS/1/2014/ST06/UKM/01/1 for financial support.

References

[1] A. Aghajani, M. Abbas, J. R. Roshan, *Common fixed point of generalized weak contractive mappings in partially ordered b-metric spaces*, Math. Slovaca, **64** (2014), 941960.

[2] R. Allahyari, R. Arab, A. S. Haghighi, *A generalization on weak contractions in partially ordered b-metric spaces and its application to quadratic integral equations*. J. Inequal. Appl. **2014** (2014), 1-15.

[3] I. Altun, B. Damjanovìc, D. Dorìc, *Fixed point and common fixed point theorems on ordered cone metric spaces*. Appl. Math. Lett.. **23**(2010), 310316.

[4] IA. Bakhtin, *The contraction principle in quasimetric spaces*, Funct. Anal. **30** (1989), 26-37.

[5] M. Boriceanu, *Strict fixed point theorems for multivalued operators in b-metric spaces*. Int. J. Mod. Math. **4**(2009), 285-301. 1

[6] S. Czerwik, *Nonlinear set-valued contraction mappings in b-metric spaces*. Atti Semin. Mat. Fis. Univ. Modena **46**(1998), 263-276. 1, 2, 2, 3.8

[7] M. S. Khan, M. Swaleh, S. Sessa, *Fixed point theorems by altering distances between the points*, Bull. Austral. Math. Soc., **30** (1984), 19. 2, 2.1

[8] J. R. Roshan, V. Parvaneh, I. Altun, *Some coincidence point results in ordered b-metric spaces and applications in a system of integral equations*. Appl. Math. Comput. **226**(2014), 725-737.

[9] J. R. Roshan, V. Parvaneh, S. Sedghi, N. Shobkolaei, W. Shatanawi, *Common fixed points of almost generalized $(\psi, \varphi)_s$-contractive mappings in ordered b-metric spaces*. Fixed Point Theory Appl. **2013**(2013), 159.

[10] S. L. Singh, B. Prasad, *Some coincidence theorems and stability of iterative procedures*. Comput. Math. Appl. **55**(2008), 2512-2520. 2.7

[11] W.Shatanawi, A. Alrawashdeh, *Common fixed points of almost generalized (ψ, φ)-contractive mappings in ordered metric spaces*. Fixed Point Theory Appl. **2012**(80)), 1687-1812.

[12] W. Shatanawi, A, Pitea, R. Lazovic, Contraction conditions using comparison functions on b-metric spaces, Fixed Point Theory and Applications 2014, 2014:135 doi:10.1186/1687-1812-2014-135

[13] F. Zabihi, A. Razani, *Fixed point theorems for hybrid rational Geraghty contractive mappings in ordered b-metric spaces*. J. Appl. Math. **2014**(2014), Article ID 929821.

Habes Alsamir

School of mathematical Sciences, Faculty of Science and Technology, University Kebangsaan Malaysia, 43600 UKM, Selangor Darul Ehsan, Malaysia.

E-mail address: h.alsamer@gmail.com

Mohd Salmi MD Noorani

School of mathematical Sciences, Faculty of Science and Technology,University Kebangsaan Malaysia, 43600 UKM, Selangor Darul Ehsan, Malaysia

E-mail address: msn@ukm.my

Wasfi Shatanawi

Department of Mathematics, Hashemite University, Zarqa 1315, Jordan

Department of Mathematics and general courses, Prince Sultan University, Riyadh, Saudi Arabia

E-mail address: wshatanawi@yahoo.com

E-mail address: wshatanawi@psu.edu.sa

Fawzia Shaddad

Department of Mathematics, Sanaa University,Yemen

E-mail address: fzsh99@gmail.com