A MONOTONICITY RESULT FOR THE $q$–FRACTIONAL OPERATOR

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Abstract. In this article we prove that if the $q$–fractional operator $(q\nabla_{qa}^\alpha y)(t)$ of order $0 < \alpha \leq 1$, $0 < q < 1$ and starting at some $qa \in T_q = \{q^n : n \in \mathbb{Z}\} \cup \{0\}$, $a > 0$ is positive such that $y(a) \geq 0$, then $y(t)$ is $c_q(\alpha)$–increasing, $c_q(\alpha) = \frac{1-q\alpha}{1-q}q^{1-\alpha}$. Conversely, if $y(t)$ is increasing and $y(a) \geq 0$, then $(q\nabla_{qa}^\alpha y)(t) \geq 0$. As an application, we proved a $q$–fractional version of the Mean-Value Theorem.

1. Introduction and Preliminaries

Fractional calculus \cite{25, 28, 30} has recently occupied the minds of many researchers either theoretically or in different fields of applications \cite{17, 29}. The theory of $q$–fractional calculus was initiated in early of fifties of last century \cite{9, 11, 22, 6, 7, 8}. Then, this theory has started to be developed in the last decade or so \cite{3-5, 10, 11, 23, 29}. For some recent interesting extensions and generalization in the theory of $q$–polynomials and functions we guide the reader to \cite{15, 31, 32}. For the preliminaries about $q$–fractional calculus given here shortly, we refer the reader to the survey \cite{19} and the recent book \cite{10}. On the other hand the theory of discrete fractional calculus started to develop rapidly specially in the last decade \cite{1, 2, 12-16, 20, 21, 27}. Very recently, some monotonicity results have been reported for fractional difference type operators of order $0 < \alpha \leq 1$ \cite{13}, and of order $1 < \alpha < 2$ \cite{20, 24}. Motivated by what mentioned above and the fact that monotonicity results are of interest in usual calculus itself we obtain some monotonicity results for the $q$–fractional type operators of order $0 < \alpha \leq 1$ in Section 2. An application is also given in Section 3 by giving a $q$–fractional mean value theorem version.

For $0 < q < 1$, let $T_q$ be the time scale

$$T_q = \{q^n : n \in \mathbb{Z}\} \cup \{0\}.$$
For a function \( f : T_q \to \mathbb{R} \), the nabla \( q \)-derivative of \( f \) is given by
\[
\nabla_q f(t) = \frac{f(t) - f(qt)}{(1-q)t}, \quad t \in T_q - \{0\} \tag{1.1}
\]
The nabla \( q \)-integral of \( f \) is given by
\[
\int_0^t f(s) \nabla_q s = (1-q)t \sum_{i=0}^{\infty} q^i f(tq^i) \tag{1.2}
\]
and for \( 0 \leq a \in T_q \)
\[
\int_a^t f(s) \nabla_q s = \int_0^t f(s) \nabla_q s - \int_0^a f(s) \nabla_q s
\]
Alternatively, let \( a = q^{n_0} \in T_q \) where \( n < n_0 \), the nabla \( q \)-integral of \( f \) is given by
\[
\int_a^t f(s) \nabla_q s = (1-q) \sum_{i=n}^{n_0-1} q^i f(q^i) \tag{1.3}
\]
From the theory of \( q \)-calculus and the theory of time scale more generally, the following product rule is valid
\[
\nabla_q (f(t)g(t)) = f(qt) \nabla_q g(t) + (\nabla_q f(t))g(t) \tag{1.4}
\]
The \( q \)-factorial function for \( n \in \mathbb{N} \) is defined by
\[
(t-s)_q^n = \prod_{i=0}^{n-1} (t - q^i s) \tag{1.5}
\]
More generally, when \( \alpha \) is a not a positive integer, the \( q \)-factorial fractional function is defined by
\[
(t-s)_q^\alpha = t^\alpha \prod_{i=0}^{\infty} \left( 1 - \frac{s}{t} q^i \right)^\alpha \tag{1.6}
\]
The following properties will be used inside the proofs of the main results.

- \((t-s)_q^{\beta+\gamma} = (t-s)_q^{\beta}(t-q^\beta s)_q^{\gamma}\)
- \((at-as)_q^\alpha = a^\alpha (t-s)_q^\alpha\)
- The nabla \( q \)-derivative of the \( q \)-factorial function with respect to \( t \) is
  \[
  \nabla_q (t-s)_q^{\alpha} = \frac{1 - q^\alpha}{1-q} (t-s)_q^{\alpha-1}
  \]
- The nabla \( q \)-derivative of the \( q \)-factorial function with respect to \( s \) is
  \[
  \nabla_q (t-s)_q^{\alpha} = \frac{1 - q^\alpha}{1-q} (t-qs)_q^{\alpha-1}
  \]
where \( \alpha, \beta, \gamma \in \mathbb{R} \).

Moreover, the \( q \)-fractional integral of order \( \alpha \neq 0, -1, -2, \ldots \) is defined by
\[
_qI^\alpha f(t) = \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{\alpha-1} f(s) \nabla_q s. \tag{1.7}
\]
Let \( \alpha > 0 \). If \( \alpha \notin \mathbb{N} \), then the \( \alpha \)-order Liouville-Caputo (left) \( q \)-fractional derivative of a function \( f \) is defined by
\[
_qC_\alpha^\alpha f(t) \triangleq _qI_{a+}^{(n-\alpha)} \nabla_q^n f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-qs)^{n-\alpha-1} \nabla_q^n f(s) \nabla_q s \tag{1.8}
\]
where \( n = [\alpha] + 1 \) and \( [\alpha] \) denotes the greatest integer less than or equal to \( \alpha \). If \( \alpha \in \mathbb{N} \), then \( q^{\alpha} f(t) \triangleq \nabla_q^n f(t) \).

Let \( \alpha > 0 \). If \( \alpha \notin \mathbb{N} \), then the \( \alpha \)-order Riemann (left) \( q \)-fractional derivative of a function \( f \) is defined by

\[
q^{\alpha} f(t) = \nabla_q^n \mathcal{I}^{(n-\alpha)}_a f(t)
\]

(1.9)

The following identity is useful to transform Caputo \( q \)-fractional difference equation into \( q \)-fractional integrals.

Assume \( \alpha > 0 \) and \( f \) is defined in suitable domains. Then \( ^\mathbb{E} \)

\[
q^\alpha \mathcal{I}^\alpha_a q^{-\alpha} f(t) = f(t) - \sum_{k=0}^{n-1} \frac{(t-a)_{q}^k}{\Gamma_q(k+1)} \nabla_q^k f(a)
\]

(1.10)

and if \( 0 < \alpha \leq 1 \) then

\[
q^\alpha \mathcal{I}^\alpha_a q^{-\alpha} f(t) = f(t) - f(a)
\]

(1.11)

Also, the identity \( q^{\alpha} \mathcal{I}^\alpha_a q^{-\alpha} f(t) = f(t) \) is known. The following identity is essential to solve linear \( q \)-fractional equations

\[
q^\alpha \mathcal{I}^\alpha_a (x-a)_{q}^\mu = \frac{\Gamma_q(\mu+1)}{\Gamma_q(\alpha+\mu+1)} (x-a)_{q}^{\mu+\alpha} \quad (0 \leq a < x < b)
\]

(1.12)

where \( \alpha \in \mathbb{R}^+ \) and \( \mu \in (-1, \infty) \).

For more about \( q \)-Gamma functions and other \( q \)-calculus concepts we refer, for example, to \( ^\text{19} \). The following lemma is used in the proof of the main result. The proof follows from \( ^\text{16} \).

**Lemma 1.1.**

- \( (1-q^t)^{-\alpha} = \frac{1-q^t}{1-q^{[t]-1}} (1-q^{[t]+1})^{-\alpha} \)
- \( (q^{\alpha+1} - 1)^{-\alpha} = \frac{1-q^{-1}}{q-q^{-1}} (q^n - 1)^{-\alpha} \)
- \( (q^{m} - q^n)^{-\alpha} = \frac{1-q^{-m}}{q-q^{-m}} (q^{m-1} - q^n)^{-\alpha} \)
- \( (q^{m} - q^n)^{-\alpha} = \frac{1-q^{-m}}{q-q^{-m}} (q^{m-1} - q^n)^{-\alpha} \)

**Definition 1.2.** Fix \( \alpha \geq 0 \) and define

\[
c_q(\alpha) = \frac{1-q^\alpha}{1-q^{1-\alpha}}
\]

**Definition 1.3.** Let \( y : T_q \to \mathbb{R} \) be a function. \( y \) is called a \( c_q(\alpha) \)-increasing on \( T_q \), if

\[
y(q^{n-1}) \geq c_q(\alpha) y(q^n) \text{ for all } q^n \in T_q.
\]

**Definition 1.4.** Let \( y : T_q \to \mathbb{R} \) be a function. \( y \) is called a \( c_q(\alpha) \)-decreasing on \( T_q \), if

\[
y(q^{n-1}) \leq c_q(\alpha) y(q^n) \text{ for all } q^n \in T_q.
\]

Notice that if \( \alpha \geq 1 \), then \( c_q(\alpha) \geq 1 \) and if \( 0 \leq \alpha < 1 \), then \( 0 \leq c_q(\alpha) \leq 1 \). Hence, if \( y \) is increasing (decreasing) on \( T_q \) and \( 0 < \alpha < 1 \) then \( y \) is \( c_q(\alpha) \)-increasing (decreasing) on \( T_q \). Also, if \( y \) is \( c_q(\alpha) \)-increasing (decreasing) on \( T_q \) and \( \alpha > 1 \), then, \( y \) is increasing (decreasing) on \( T_q \). If \( \alpha = 1 \) then \( y \) is \( c_q(\alpha) \)-increasing (decreasing) if and only if \( y \) is increasing (decreasing).

Our main Theorems that we will present their proofs in Section 2 are
Theorem 1.5. Let \( y : T_q \to \mathbb{R} \) be a function satisfying \( y(a) \geq 0, a = q^{n_0} > 0 \). Suppose that
\[
q^{-\alpha}y(t) \geq 0 \text{ for each } t = q^n, n < n_0.
\]
Then, \( y \) is \( c_\alpha(\alpha) \)-increasing on \( \{ t \in T_q : t \geq a = q^{n_0} \} \).

And conversely,

Theorem 1.6. Let \( y : T_q \to \mathbb{R} \) be a function satisfying \( y(q^{n_0}) \geq 0, a = q^{n_0} \). Suppose that \( y \) is an increasing function on \( T_q \). Then,
\[
q^{-\alpha}y(t) \geq 0 \text{ for each } t = q^n, n < n_0.
\]

2. Main Results

In this Section we present the proof of our main results stated in Theorem 1.5 and Theorem 1.6 and their direct consequences.

Proof of Theorem 1.5. Let \( q^{-\alpha}y(t) \geq 0 \) for each \( t \in T_q, \alpha \in (0, 1) \), then
\[
q^{-\alpha}y(t) = \nabla_q \nabla^{-\alpha}_{aq} y(t) = \nabla_q \left[ \frac{1}{\Gamma_q(1 - \alpha)} \int_{aq}^t (t - qs)^{-\alpha} y(s) \nabla_q(s) \right] \geq 0
\]

Let \( s(t) = \frac{1 - q^t}{1 - q} \sum_{i=0}^{n_0} q^i (q^{n_0} - q^{n+1})^{-\alpha} y(q^i) \). Since \( \nabla_q s(t) \geq 0, s(t) \) is an increasing function on \( T_q \). This implies that
\[
s(q^{n_0+1}) - s(q^{n_0}) = \frac{1 - q^{n_0+1}}{\Gamma_q(1 - \alpha)} [(q^{n_0} - q^{n_0})^{-\alpha} y(q^{n_0}) - q((q^{n+1})^{n_0+1})^{-\alpha} y(q^{n_0})]
\]
\[
= \frac{1 - q^{n_0+1}}{\Gamma_q(1 - \alpha)} [q^\alpha (q^{n_0} - q^{n_0+1})^{-\alpha} y(q^{n_0}) - q((q^{n+1})^{n_0+1})^{-\alpha} y(q^{n_0})]
\]
\[
\geq 0
\]

therefore, we have
\[
q \left[ \frac{q^\alpha - q}{1 - q} - 1 \right] y(q^{n_0}) + q^\alpha y(q^{n_0+1}) \geq 0
\]

which implies that
\[
y(q^{n_0+1}) \geq \frac{1 - q^\alpha}{1 - q} q^{1-\alpha} y(q^{n_0})
\]

Now, we assume that the hypothesis is true for \( n = k \), i.e.
\[
y(q^{n_0-k}) \geq \frac{1 - q^\alpha}{1 - q} q^{1-\alpha} y(q^{n_0-k+1})
\]

hence, we have
\[
y(q^{n_0-k}) \geq c_q(\alpha) y(q^{n_0-k+1}) \geq c_q^2(\alpha) y(q^{n_0-k+2}) \geq ... \geq c_q^{k-1}(\alpha) y(q^{n_0-1}) \geq c_q^k(\alpha) y(q^{n_0})
\]

We want to prove that
\[
y(q^{n_0-k+1}) \geq c_q(\alpha) y(q^{n_0-k})
\] (2.1)
We start by calculating,

\[
\begin{align*}
\frac{1 - q}{\Gamma_q(1 - \alpha)} \left[ \sum_{i=n_0-k}^{n_0} q^i (q_{n_0-k-1} - q_{n_0-k})^{\alpha} y(q^i) - \sum_{i=n_0-k}^{n_0} q^i (q_{n_0-k} - q_{n_0-k+1})^{\alpha} y(q^i) \right] \\
= \frac{1 - q}{\Gamma_q(1 - \alpha)} [q^{n_0-k-1}(q^{n_0-k-1} - q^{n_0-k})^{\alpha} y(q^{n_0-k-1}) \\
+ q^{n_0-k}(q^{n_0-k-1} - q^{n_0-k+1})^{\alpha} y(q^{n_0-k}) - q^{n_0-k}(q^{n_0-k} - q^{n_0-k+1})^{\alpha} y(q^{n_0-k}) \\
+ q^{n_0-k+1}(q^{n_0-k-1} - q^{n_0-k+2})^{\alpha} y(q^{n_0-k+1}) - q^{n_0-k+1}(q^{n_0-k} - q^{n_0-k+2})^{\alpha} y(q^{n_0-k+1}) \\
+ \ldots \\
+ q^{n_0-1}(q^{n_0-k} - q^{n_0})^{\alpha} y(q^{n_0}) - q^{n_0-1}(q^{n_0-k} - q^{n_0})^{\alpha} y(q^{n_0}) \\
+ q^{n_0}(q^{n_0-k-1} - q^{n_0+1})^{\alpha} y(q^{n_0}) - q^{n_0}(q^{n_0-k} - q^{n_0+1})^{\alpha} y(q^{n_0}) ] \\
= \frac{1 - q}{\Gamma_q(1 - \alpha)} [q^{n_0-k-1}(q^{n_0-k-1} - q^{n_0-k})^{\alpha} y(q^{n_0-k-1}) \\
+ q^{n_0-k}(q^{n_0-k} - q^{n_0-k+1})^{\alpha} y(q^{n_0-k}) (\frac{q^\alpha - q}{1 - q} - 1) \\
+ q^{n_0-k+1}(q^{n_0-k} - q^{n_0-k+2})^{\alpha} y(q^{n_0-k+1}) (\frac{q^\alpha - q^2}{1 - q^2} - 1) \\
+ \ldots \\
+ q^{n_0-1}(q^{n_0-k} - q^{n_0})^{\alpha} y(q^{n_0}) (\frac{q^\alpha - q^k}{1 - q^k} - 1) \\
+ q^{n_0}(q^{n_0-k} - q^{n_0+1})^{\alpha} y(q^{n_0}) (\frac{q^\alpha - q^{k+1}}{1 - q^{k+1}} - 1)] \\
\end{align*}
\]

Since \( s(t) \) is increasing, we get

\[
q^{n_0-k-1}(q^{n_0-k-1} - q^{n_0-k})^{\alpha} y(q^{n_0-k-1}) + q^{n_0-k}(q^{n_0-k} - q^{n_0-k+1})^{\alpha} y(q^{n_0-k}) (\frac{q^\alpha - 1}{1 - q}) \\
\geq q^{n_0-k+1}(q^{n_0-k} - q^{n_0-k+2})^{\alpha} y(q^{n_0-k+1}) (\frac{1 - q^\alpha}{1 - q^2}) \\
+ \ldots \\
+ q^{n_0-1}(q^{n_0-k} - q^{n_0})^{\alpha} y(q^{n_0}) (\frac{1 - q^\alpha}{1 - q^k}) \\
+ q^{n_0}(q^{n_0-k} - q^{n_0+1})^{\alpha} y(q^{n_0}) (\frac{1 - q^\alpha}{1 - q^{k+1}})
\]
Using the induction assumption \([2.1]\), we get
\[
q^{n_0-k-1}(q^{n_0-k} - q^{n_0-k-1})q^{-\alpha} \left[ q^{\alpha} y(q^{n_0-k-1}) + q^{\alpha} \frac{1 - q^\alpha}{1 - q} y(q^{n_0-k}) \right] \\
\geq q^{n_0-k+1}(q^{n_0-k} - q^{n_0-k+1})q^{-\alpha} \left( \frac{1 - q^\alpha}{1 - q^k} (c_q(\alpha)) y(q^{n_0}) \right) \\
+ \ldots \ldots \ldots \\
+ q^{n_0-1}(q^{n_0-k} - q^{n_0})^{-\alpha} \left( \frac{1 - q^\alpha}{1 - q^k} (c_q(\alpha)) y(q^{n_0}) \right) \\
+ q^{n_0}(q^{n_0-k} - q^{n_0+1})q^{-\alpha} \left( \frac{1 - q^\alpha}{1 - q^{k+1}} y(q^{n_0}) \right)
\]

Since \( y(q^{n_0}) \geq 0 \), we conclude that \( q^\alpha y(q^{n_0-k-1}) + q^{\alpha} \frac{1 - q^\alpha}{1 - q} y(q^{n_0-k}) \geq 0 \) which implies that \( y(q^{n_0-k}) \geq c_q(\alpha)y(q^{n_0-k}) \), which completes the proof.

Using Theorem 1.5 and the following identity that relates (Riemann) \( q \)-fractional derivative \( \frac{d}{d_q} \) and the Caputo \( q \)-fractional derivative \( C^\alpha_q \) of order \( 0 < \alpha < 1 \),

\[
( \frac{d}{d_q} f)(t) = ( \frac{d}{d_q} f)(t) - \frac{(t-a)_q^{-\alpha}}{\Gamma_q(1-\alpha)} y(a),
\]

we can state the following Caputo monotonicity result:

**Corollary 2.1.** Let \( y : T_q \to \mathbb{R} \) be a function satisfying \( y(a) \geq 0 \), \( a = q^{n_0} > 0 \). Suppose that

\[
qC^\alpha_q y(t) \geq - \frac{(t-q)a}_q^{-\alpha} y(qa) \text{ for each } t = q^n, n < n_0.
\]

Then, \( y \) is \( c_q(\alpha) \)-increasing on \( \{ t \in T_q : t \geq a = q^{n_0} \} \).

**Proof of Theorem 1.6** We want to prove that

\[
qD^\alpha_q y(t) = \nabla_q \nabla_q^{-(1-\alpha)} y(t) = \nabla_q \left[ \frac{1}{\Gamma_q(1-\alpha)} \int_{aq}^t (t-q) \frac{n}{\alpha} y(s) \nabla_q(s) \right] \geq 0.
\]

Let

\[
s(t) = \left[ \frac{1}{\Gamma_q(1-\alpha)} \int_{aq}^t (t-q) \frac{n}{\alpha} y(s) \nabla_q(s) \right] = \frac{1 - q}{\Gamma_q(1-\alpha)} \sum_{i=n_0}^{n} q^i(q^n - q^{i+1})^{-\alpha} y(q^i).
\]
Since $\nabla_q s(t) \geq 0$. We need to show that $s(t)$ is increasing on $T_q$, i.e. we need to show that $s(q^{n_0-k-1}) - s(q^{n_0-k}) \geq 0$ for any natural number $k$ with $k \geq 1$. In fact,

$$s(q^{n_0-k-1}) - s(q^{n_0-k}) = \frac{(1-q)}{\Gamma_q(1-\alpha)} \left[ \sum_{i=n_0-k}^{n_0} q^i(q^{n_0-k-1} - q^{i+1})^\alpha y(q^i) - \sum_{i=n_0-k}^{n_0} q^i(q^{n_0-k} - q^{i+1})^\alpha y(q^i) \right]$$

$$= \frac{(1-q)}{\Gamma_q(1-\alpha)} [q^{n_0-k-1}(q^{n_0-k-1} - q^{n_0-k})^\alpha y(q^{n_0-k-1}) + \sum_{i=n_0-k}^{n_0} q^i(q^{n_0-k-1} - q^{i+1})^\alpha y(q^i) - \sum_{i=n_0-k}^{n_0} q^i(q^{n_0-k} - q^{i+1})^\alpha y(q^i) ]$$

$$= \frac{(1-q)}{\Gamma_q(1-\alpha)} [q^{n_0-k-1}(q^{n_0-k-1} - q^{n_0-k})^\alpha y(q^{n_0-k-1}) + \sum_{i=n_0-k}^{n_0} q^i[q^{\alpha - q^{i+1-n_0+k}} \frac{\alpha - 1}{1-q^{i+1-n_0+k}} - 1](q^{n_0-k} - q^{i+1})^\alpha y(q^i) ]$$

$$\geq \frac{(1-q)}{\Gamma_q(1-\alpha)} [q^{n_0-k-1}(q^{n_0-k-1} - q^{n_0-k})^\alpha y(q^{n_0-k-1}) + \sum_{i=n_0-k}^{n_0} q^i[q^{\alpha - q^{i+1-n_0+k}} \frac{\alpha - 1}{1-q^{i+1-n_0+k}} - 1](q^{n_0-k} - q^{i+1})^\alpha y(q^{n_0-k-1}) ] \geq 0.$$

Which concludes the proof.

**Theorem 2.2.** Let $y : T_q \rightarrow \mathbb{R}$ be a function satisfying $y(q^{n_0}) > 0$, $\alpha = q^{n_0}$. Suppose that $y$ is a strictly increasing function on $T_q$. Then,

$$q^{\alpha} y(t) > 0 \text{ for each } t = q^n, n < n_0.$$

In a similar way, the above results can be obtained for the function which takes negative value at the initial point of its domain.

**Theorem 2.3.** Let $y : T_q \rightarrow \mathbb{R}$ be a function satisfying $y(q^{n_0}) \leq 0$. Suppose that

$$q^{\alpha} y(t) \leq 0 \text{ for each } t = q^n, n < n_0.$$

Then, $y$ is $c_q(\alpha)$–decreasing on $T_q$

**Theorem 2.4.** Let $y : T_q \rightarrow \mathbb{R}$ be a function satisfying $y(q^{n_0}) \leq 0$. Suppose that $y$ is a decreasing function on $T_q$. Then,

$$q^{\alpha} y(t) \leq 0 \text{ for each } t = q^n, n < n_0.$$
3. An Application

In this section, we wish to prove a Mean-Value Theorem in $q$–fractional calculus. First, we need the following result:

**Theorem 3.1.** Let $f$ be defined on $T_q$ and $a, b \in T_q$ with $a < b$. Then the following equality holds:

$$q\nabla^{-\alpha} q\nabla^\alpha f(t)|_b = f(b) - \frac{(1-q)a^{1-\alpha}(b-a)^{\alpha-1}(1-q)^{-\alpha}}{\Gamma_q(\alpha)\Gamma_q(1-\alpha)}f(a) \text{ where } \alpha \in (0, 1).$$

**Proof.**

$$q\nabla^{-\alpha} q\nabla^\alpha f(t)|_b = q\nabla^{-\alpha}(q\nabla^{-1-\alpha}f(t))|_b = q\nabla^{-\alpha} q\nabla^{-1-\alpha}f(t)|_b - \frac{(t-a)^{\alpha-1}}{\Gamma_q(\alpha)}q\nabla^{-1-\alpha}f(a)|_b = q\nabla^{-\alpha} q\nabla^{-1}f(t)|_b - \frac{(t-a)^{\alpha-1}}{\Gamma_q(\alpha)}q\nabla^{-1-\alpha}f(a)|_b = f(b) - \frac{(1-q)a^{1-\alpha}(b-a)^{\alpha-1}(1-q)^{-\alpha}}{\Gamma_q(\alpha)\Gamma_q(1-\alpha)}f(a).$$

\[\square\]

**Theorem 3.2.** Assume $f$ and $g$ are defined on $T_q$, $a, b \in T_q$, $a > 0$ and $g$ is strictly increasing on $\Delta = \{t \in T_q : t \geq a, t \leq b\}$ and satisfying $g(a) > 0$. Then, there exists $r_1, r_2 \in \Delta$ such that

$$q\nabla_{aq}^\alpha f(r_1) \leq \frac{f(b) - M_q(\alpha, a, b)f(a)}{g(b) - M_q(\alpha, a, b)g(a)} \leq q\nabla_{aq}^\alpha f(r_2).$$

Where $M_q(\alpha, a, b) = \frac{(1-q)a^{1-\alpha}(b-a)^{\alpha-1}(1-q)^{-\alpha}}{\Gamma_q(\alpha)\Gamma_q(1-\alpha)}$.

**Proof.** Suppose without loss of generality to the contrary that

$$\frac{f(b) - M_q(\alpha, a, b)f(a)}{g(b) - M_q(\alpha, a, b)g(a)} > \frac{q\nabla_{aq}^\alpha f(t)}{q\nabla_{aq}^\alpha g(t)}.$$  

Since $g$ is strictly increasing, Theorem 2.2 implies that $q\nabla_{aq}^\alpha g(t) > 0$. Hence

$$\frac{f(b) - M_q(\alpha, a, b)f(a)}{g(b) - M_q(\alpha, a, b)g(a)} q\nabla_{aq}^\alpha g(t) > q\nabla_{aq}^\alpha f(t).$$

We apply $q\nabla_{aq}^{-\alpha}$ at $t = b$ to both sides of the above inequality and make use of Theorem 3.1 to get

$$\frac{f(b) - M_q(\alpha, a, b)f(a)}{g(b) - M_q(\alpha, a, b)g(a)} [g(b) - M_q(\alpha, a, b)g(a)] > f(b) - M_q(\alpha, a, b)f(a).$$

This leads to $g(b) > g(a)$, which is a contradiction. \[\square\]

**Remark.**

- Notice that if the edge point $a = 0$ then $M_q(\alpha, a, b) = 0$ and hence no Mean-Value Theorem.
- Notice that since $M_q(\alpha, a, b) < 1$ and $g(t)$ is strictly decreasing then $g(b) - M_q(\alpha, a, b)g(a)$ in the statement of Theorem 3.2 is not equal to zero.
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References


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