HERMITE-HADAMARD TYPE INEQUALITIES FOR PRODUCT
OF HARMONICALLY CONVEX FUNCTIONS VIA
RIEMANN-LIOUVILLE FRACTIONAL INTEGRALS

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ABSTRACT. In this paper, some Hermite-Hadamard type inequalities for products of two harmonically convex functions via Riemann-Liouville fractional integrals are established. Our results about harmonically convex functions are analogous generalizations for some other results proved by Pachpette, Chan and Noor for convex and harmonically $h$-convex functions.

1. INTRODUCTION

It is known that Hermite-hadamard integral inequality was built on a convex function. In [16], one can find general notations of convexity.

In recent years, very large number of studies of error estimations have been done for Hermite-Hadamard type inequalities. For some results which generalize, improve, and extend the Hermite-Hadamard inequality see [2, 7, 8, 10, 12, 16, 17, 18, 19, 20, 23, 26] and references therein.

Hermite-Hadamura type inequalities for products of two convex functions are interesting problem and firstly developed by Pachpatte in [21]. In [22], Pachpette also established Hermite-hadamard type inequalities involving two log-convex functions. In [13], Kırmacı et. al. proved several Hermite-Hadamard type inequalities for products of two convex and $s$-convex functions. In [19], Özdemir et al. established Hadamard-type inequalities for product of $s$-convex functions on co-ordinates. In [24], Sarıkaya et al. proved some Hermite-Hadamard type inequalities for products of two $h$-convex functions. In [1], Bakula et al. established Hermite-Hadamard type inequalities for products of two $m$-convex and $(\alpha, m)$-convex functions. In [4], Chen and Wu obtained some Hermite-Hadamard type inequalities for products of two convex and harmonically $s$-convex functions. In [25], Varošanec, introduced concept of $h$-convex functions, in [17, 18], Noor et al. established some Hadamard's type inequalities for product of relative convex functions, relative $h$-convex functions and harmonically convex functions. In [27], Yin and Qi established some Hermite-Hadamard type inequalities for products of two convex functions. In [5], Chen obtained some new Hermite-Hadamard type inequalities for products of two
convex functions via Riemann-Liouville fractional integrals and in [3] he extended this problem to $m$-convex and $(\alpha, m)$-convex functions.

In this work, we establish Hermite-Hadamard type inequalities for products of two harmonically convex functions via Riemann-Liouville fractional integrals. Our results are analogous generalization for some results in [4, 5, 17, 21].

2. Preliminaries

Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a convex function defined on the interval $I$ of real numbers and $a, b \in I$ with $a < b$. The inequality

$$f\left(\frac{a + b}{2}\right) \leq \frac{1}{b - a} \int_a^b f(x) \, dx \leq \frac{f(a) + f(b)}{2}$$

(2.1)

is well known in the literature as Hermite-Hadamard’s inequality [9].

In [21], Pachpette established following two Hermite-Hadamard type inequalities for products of convex functions as follows:

**Theorem 2.1.** Let $f$ and $g$ be real-valued, non-negative and convex functions on $[a, b]$. Then

$$\frac{1}{b - a} \int_a^b f(x) \, g(x) \, dx \leq \frac{1}{3} M(a, b) + \frac{1}{6} N(a, b)$$

(2.2)

and

$$2f\left(\frac{a + b}{2}\right) g\left(\frac{a + b}{2}\right) \leq \frac{1}{b - a} \int_a^b f(x) \, g(x) \, dx$$

$$+ \frac{1}{6} M(a, b) + \frac{1}{3} N(a, b)$$

(2.3)

where $M(a, b) = f(a) \, g(a) + f(b) \, g(b)$ and $N(a, b) = f(a) \, g(b) + f(b) \, g(a)$.

In [5], Chan established following two Hermite-Hadamard type inequalities for products of convex functions via Riemann-Liouville fractional integrals as follows:

**Theorem 2.2.** Let $f$ and $g$ be real-valued, non-negative and convex functions on $[a, b]$. Then

$$\frac{\Gamma(\alpha + 1)}{2(b - a)^\alpha} \left[ J_+^\alpha f(b) \, g(b) + J_-^\alpha f(a) \, g(a) \right]$$

$$\leq \left( \frac{\alpha}{\alpha + 2} - \frac{\alpha}{\alpha + 1} + \frac{1}{2} \right) M(a, b) + \frac{\alpha}{(\alpha + 1)(\alpha + 2)} N(a, b)$$

(2.4)

and

$$2f\left(\frac{a + b}{2}\right) g\left(\frac{a + b}{2}\right)$$

$$\leq \frac{\Gamma(\alpha + 1)}{2(b - a)^\alpha} \left[ J_+^\alpha f(b) \, g(b) + J_-^\alpha f(a) \, g(a) \right]$$

$$+ \frac{\alpha}{(\alpha + 1)(\alpha + 2)} M(a, b) + \left( \frac{\alpha}{\alpha + 2} - \frac{\alpha}{\alpha + 1} + \frac{1}{2} \right) N(a, b)$$

(2.5)

where $M(a, b) = f(a) \, g(a) + f(b) \, g(b)$ and $N(a, b) = f(a) \, g(b) + f(b) \, g(a)$.

In [12], İşcan gave definition of harmonically convex functions as follows:
Definition 1. Let $I \subset \mathbb{R} \setminus \{0\}$ be a real interval. A function $f : I \rightarrow \mathbb{R}$ is said to be harmonically convex, if

$$f \left( \frac{xy}{tx + (1-t)y} \right) \leq tf(y) + (1-t)f(x)$$

(2.6)

for all $x, y \in I$ and $t \in [0, 1]$. If the inequality in (2.6) is reversed, then $f$ is said to be harmonically concave.

We will now give definitions of the right-hand side and left-hand side Riemann-Liouville fractional integrals which are used throughout this paper.

Definition 2. Let $f \in L[a, b]$. The Riemann-Liouville integrals $J^\alpha_{a+}f$ and $J^\alpha_{b-}f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J^\alpha_{a+}f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a$$

and

$$J^\alpha_{b-}f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x < b$$

respectively, where $\Gamma(\alpha)$ is the Gamma function defined by $\Gamma(\alpha) = \int_0^\infty e^{-t}t^{\alpha-1}dt$ and $J^\alpha_{a+}f(x) = J^\alpha_{b-}f(x) = f(x)$.

Some results about Hermite-Hadamard type inequality for some class of convex functions via Riemann-Liouville fractional integrals see [3, 5, 11, 14, 18, 20] and references therein.

In [11], İsçan and Wu presented Hermite-Hadamard’s inequalities for harmonically convex functions in fractional integral forms as follows:

Theorem 2.3. Let $f : I \subset (0, \infty) \rightarrow \mathbb{R}$ be a function such that $f \in L[a, b]$, where $a, b \in I$ with $a < b$. If $f$ is a harmonically convex function on $[a, b]$, then the following inequalities for fractional integrals hold:

$$f \left( \frac{2ab}{a+b} \right) \leq \frac{\Gamma(\alpha+1)}{2} \left( \frac{ab}{b-a} \right)^\alpha \left\{ J^\alpha_{1/a-} (f \circ h)(1/b) \right\} + J^\alpha_{1/b+} (f \circ h)(1/a)$$

$$\leq \frac{f(a) + f(b)}{2}$$

(2.7)

with $\alpha > 0$ and $h(x) = 1/x$.

In [14], Kunt et al. presented, Hermite-Hadamard inequality in fractional integral forms for harmonically convex functions as follows:

Theorem 2.4. Let $f : I \subseteq (0, \infty) \rightarrow \mathbb{R}$ be a function such that $f \in L[a, b]$, where $a, b \in I$ with $a < b$. If $f$ is a harmonically convex function on $[a, b]$, then the following inequalities for fractional integrals hold:

$$f \left( \frac{2ab}{a+b} \right) \leq \frac{\Gamma(\alpha+1)}{2^{1-\alpha}} \left( \frac{ab}{b-a} \right)^\alpha \left\{ J^{\frac{\alpha}{2^{\alpha+1}}}_{a+b} (f \circ h)(1/a) \right\} + J^{\frac{\alpha}{2^{\alpha-1}}}_{1/b} (f \circ h)(1/b)$$

$$\leq \frac{f(a) + f(b)}{2}$$

(2.8)

with $\alpha > 0$ and $h(x) = 1/x$, $x \in \left[ \frac{1}{b}, \frac{1}{a} \right]$. 

3. General Results

Theorem 3.1. Let \( f \) and \( g : [a, b] \subseteq (0, \infty) \to \mathbb{R} \) be non-negative and harmonically convex functions with \( a < b \) and \( f \in L[a, b] \), then the following inequality for fractional integrals holds:

\[
\frac{\Gamma (\alpha + 1)}{2} \left( \frac{ab}{b - a} \right)^\alpha \left\{ J_{1/a}^\alpha (f \circ h) (1/b) (g \circ h) (1/b) + J_{1/b}^\alpha (f \circ h) (1/a) (g \circ h) (1/a) \right\} \\
\leq \left( \frac{\alpha}{\alpha + 2} - \frac{\alpha}{\alpha + 1} + \frac{1}{2} \right) M (a, b) + \frac{\alpha}{(\alpha + 1)(\alpha + 2)} N (a, b) \quad (3.1)
\]

where \( \alpha > 0 \), \( h(x) = 1/x \), \( x \in \left[ \frac{1}{b}, \frac{1}{a} \right] \), \( M (a, b) = f (a) g (a) + f (b) g (b) \) and \( N (a, b) = f (a) g (b) + f (b) g (a) \).

Proof. Theorem 3.1 is direct result of (2.4) in Theorem 2.1. In (2.4), if we replace the convex functions \( f \) and \( g \) with the convex functions \( F \) defined \( F := f \circ h \) and \( G \) defined \( G := g \circ h \), \( h (x) = \frac{1}{x} \) and if we replace \( a \) and \( b \) with \( \frac{1}{b} \) and \( \frac{1}{a} \) respectively, we have (3.1).

Corollary 3.2. In Theorem 3.1 if we take \( g : [a, b] \to \mathbb{R} \) as \( g (x) = 1 \) for all \( x \in [a, b] \), then we have

\[
\frac{\Gamma (\alpha + 1)}{2} \left( \frac{ab}{b - a} \right)^\alpha \left\{ J_{1/a}^\alpha (f \circ h) (1/b) + J_{1/b}^\alpha (f \circ h) (1/a) \right\} \leq \frac{f (a) + f (b)}{2}
\]

which is the right hand side of (2.7).

Corollary 3.3. In Theorem 3.1 if we take \( \alpha = 1 \), then we recapture

\[
\frac{ab}{b - a} \int_a^b f (x) g (x) \, dx \leq \frac{1}{3} M (a, b) + \frac{1}{6} N (a, b)
\]

for harmonically convex functions which is given in [6] Remark 11.

Remark. Corollary 3.3 about harmonically convex functions, we have an analogue result of (2.2) for convex functions by Pachpette in [21]. If one take \( h (t) = t \) in [17] Theorem 3.6., one has the same inequality.

Theorem 3.4. Let \( f \) and \( g : [a, b] \subseteq (0, \infty) \to \mathbb{R} \) be non-negative and harmonically convex functions with \( a < b \) and \( f \in L[a, b] \), then the following inequality for fractional integrals holds:

\[
2f \left( \frac{2ab}{a + b} \right) g \left( \frac{2ab}{a + b} \right) \leq \frac{\Gamma (\alpha + 1)}{2} \left( \frac{ab}{b - a} \right)^\alpha \left\{ J_{1/a}^\alpha (f \circ h) (1/b) (g \circ h) (1/b) + J_{1/b}^\alpha (f \circ h) (1/a) (g \circ h) (1/a) \right\} \\
+ \frac{\alpha}{(\alpha + 1)(\alpha + 2)} M (a, b) + \left( \frac{\alpha}{\alpha + 2} - \frac{\alpha}{\alpha + 1} + \frac{1}{2} \right) N (a, b) \quad (3.2)
\]

where \( M (a, b) = f (a) g (a) + f (b) g (b) \) and \( N (a, b) = f (a) g (b) + f (b) g (a) \).
Proof. Theorem 3.4 is direct result of (2.3) in Theorem 2.1. In (2.5), if we replace the convex functions \( f \) and \( g \) with the convex functions \( F \) defined \( F := f \circ h \) and \( G \) defined \( G := g \circ h \), \( h(x) = \frac{1}{x} \) and if we replace \( a \) and \( b \) with \( \frac{1}{b} \) and \( \frac{1}{a} \) respectively, we have (3.2).

Corollary 3.5. In Theorem 3.4 if we take \( g : [a, b] \rightarrow \mathbb{R} \) as \( g(x) = 1 \) for all \( x \in [a, b] \), then we have

\[
2f \left( \frac{2ab}{a + b} \right) \leq \frac{\Gamma (\alpha + 1)}{2} \left( \frac{ab}{b - a} \right)^{\alpha} \left\{ J_{1/a}^{\alpha} (f \circ h) \left( \frac{1}{b} \right) + J_{1/b}^{\alpha} (f \circ h) \left( \frac{1}{a} \right) \right\} + \frac{f (a) + f (b)}{2}.
\]

Corollary 3.6. In Theorem 3.4 if we take \( \alpha = 1 \), then we recapture

\[
2f \left( \frac{2ab}{a + b} \right) g \left( \frac{2ab}{a + b} \right) \leq \frac{ab}{b - a} \int_{a}^{b} f (x) g (x) \frac{dx}{x^2} + \frac{1}{6} M (a, b) + \frac{1}{3} N (a, b)
\]

for harmonically convex functions which is given in [6, Remark 14].

Remark. Corollary 3.6 about harmonically convex functions, we have an analogue result of (2.3) for convex functions by Pachpette in [21].

Theorem 3.7. Let \( f \) and \( g \) be real-valued, non-negative and convex functions on \([a, b]\) with \( a < b \) and \( f, g \in L [a, b] \), then the following inequality for the fractional integrals holds:

\[
\frac{\Gamma (\alpha + 1)}{2^{1 - \alpha} (b - a)^\alpha} \left\{ J_{\alpha/b}^{\alpha} f (b) g (b) + J_{\alpha/a}^{\alpha} f (a) g (a) \right\} \leq \left( \frac{\alpha}{4(\alpha + 2)} - \frac{\alpha}{2(\alpha + 1)} \right) M (a, b) + \frac{\alpha^2 + 3\alpha}{4(\alpha + 2)(\alpha + 1)} N (a, b).
\]

(3.3)

where \( M (a, b) = f (a) g (a) + f (b) g (b) \) and \( N (a, b) = f (a) g (b) + f (b) g (a) \).

Proof. Since \( f \) and \( g \) are convex functions on \([a, b]\), then for all \( t \in [0, 1] \) we have

\[
\begin{align*}
f \left( t a + (1 - t) b \right) & \leq \ t f (a) + (1 - t) f (b), \\
g \left( t a + (1 - t) b \right) & \leq \ t g (a) + (1 - t) g (b), \\
f \left( t b + (1 - t) a \right) & \leq \ t f (b) + (1 - t) f (a), \\
g \left( t b + (1 - t) a \right) & \leq \ t g (b) + (1 - t) g (a).
\end{align*}
\]

A composition of this inequalities, we have

\[
\begin{align*}
& \quad f \left( t a + (1 - t) b \right) g \left( t a + (1 - t) b \right) + f \left( t b + (1 - t) a \right) g \left( t b + (1 - t) a \right) \\
& \quad \leq \ \left( 2t^2 - 2t + 1 \right) M (a, b) + 2t (1 - t) N (a, b).
\end{align*}
\]

(3.4)
Multiplying both sides of (3.3) by $t^{\alpha-1} \frac{\alpha}{2^{1-\alpha}}$, then integrating the obtained inequality with respect to $t$ over $[0, \frac{1}{2}]$, we have

$$\frac{\alpha}{2^{1-\alpha}} \left\{ \int_0^{\frac{1}{2}} t^{\alpha-1} f (ta + (1 - t) b) g (ta + (1 - t) b) \, dt + \int_0^{\frac{1}{2}} t^{\alpha-1} f (tb + (1 - t) a) g (tb + (1 - t) a) \, dt \right\} =\frac{\alpha}{2^{1-\alpha} (b - a)^{\alpha}} \left\{ \int_a^b (b - u)^{\alpha-1} f (u) g (u) \, du + \int_a^{\frac{a+b}{2}} (u - a)^{\alpha-1} f (u) g (u) \, du \right\} = \frac{\Gamma (\alpha + 1)}{2^{1-\alpha} (b - a)^{\alpha}} \left\{ J_{\frac{a+b}{2}+\alpha}^\alpha f (b) g (b) + J_{\frac{a+b}{2}+\alpha}^\alpha f (a) g (a) \right\} \leq \frac{\alpha}{2^{1-\alpha}} \left[ M (a, b) \int_0^{\frac{1}{2}} t^{\alpha-1} (2t^2 - 2t + 1) \, dt + N (a, b) \int_0^{\frac{1}{2}} t^{\alpha-1}2t (1 - t) \, dt \right] = \left( \frac{\alpha}{4 (\alpha + 2)} - \frac{\alpha}{2 (\alpha + 1)} + \frac{1}{2} \right) M (a, b) + \frac{\alpha^2 + 3\alpha}{4 (\alpha + 2) (\alpha + 1)} N (a, b).$$

This completes the proof. \(\square\)

**Theorem 3.8.** Let $f$ and $g : [a, b] \subseteq (0, \infty) \rightarrow \mathbb{R}$ be non-negative and harmonically convex functions with $a < b$ and $f, g \in L[a, b]$, then the following inequality for fractional integrals holds:

$$\frac{\Gamma (\alpha + 1)}{2^{1-\alpha}} \left( \frac{ab}{b-a} \right)^{\alpha} \left\{ J_{\frac{a+b}{2}+\alpha}^\alpha f (b) (g \circ h) (1/a) + J_{\frac{a+b}{2}+\alpha}^\alpha f (a) (g \circ h) (1/b) \right\} \leq \left( \frac{\alpha}{4 (\alpha + 2)} - \frac{\alpha}{2 (\alpha + 1)} + \frac{1}{2} \right) M (a, b) + \frac{\alpha^2 + 3\alpha}{4 (\alpha + 2) (\alpha + 1)} N (a, b) \tag{3.5}$$

where $M (a, b) = f (a) g (a) + f (b) g (b)$ and $N (a, b) = f (a) g (b) + f (b) g (a)$.

**Proof.** Theorem 3.8 is a direct result of (3.3) in Theorem 3.7. In (3.3), if we replace the convex functions $f$ and $g$ with the convex functions $F$ defined $F := f \circ h$ and $G$ defined $G := g \circ h$, $h (x) = \frac{x}{2}$ and if we replace $a$ and $b$ with $\frac{1}{2}$ and $\frac{b}{2}$ respectively, we have (3.5). \(\square\)

**Corollary 3.9.** In Theorem 3.8 if we take $g : [a, b] \rightarrow \mathbb{R}$ as $g (x) = 1$ for all $x \in [a, b]$, then we have

$$\frac{\Gamma (\alpha + 1)}{2^{1-\alpha}} \left( \frac{ab}{b-a} \right)^{\alpha} \left\{ J_{\frac{a+b}{2}+\alpha}^\alpha (f \circ h) (1/a) + J_{\frac{a+b}{2}+\alpha}^\alpha (f \circ h) (1/b) \right\} \leq \frac{f (a) + f (b)}{2}$$

which is the right hand side of (2.8).
Corollary 3.10. In Theorem 3.8, if we take \( \alpha = 1 \), then we recapture
\[
\frac{ab}{b-a} \int_a^b f(x) g(x) \frac{dx}{x^2} \leq \frac{1}{3} M(a,b) + \frac{1}{6} N(a,b)
\]
for harmonically convex functions which is given in [8, Remark 11].

Remark. In Corollary 3.10, about harmonically convex functions, we have an analogue result of [22] for convex functions by Pachpette in [21]. If one take \( h(t) = t \) in [17, Theorem 3.6.], one has the same inequality.

Theorem 3.11. Let \( f \) and \( g \) be real-valued, non-negative and convex functions on \([a,b]\) with \( a < b \) and \( f, g \in L[a,b] \), then the following inequality for the fractional integrals holds:
\[
2f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \leq \frac{\Gamma (\alpha + 1)}{2^{1-\alpha} (b-a)^\alpha} \left\{ J^{\alpha}_{\frac{a}{2}} + f(b) g(b) + J^{\alpha}_{\frac{a}{2}} f(a) g(a) \right\} \\
+ \frac{\alpha^2 + 3\alpha}{4(\alpha + 1)(\alpha + 2)} M(a,b) \\
+ \left( \frac{\alpha}{4(\alpha + 2)} - \frac{\alpha}{2(\alpha + 1)} + \frac{1}{2} \right) N(a,b) \tag{3.6}
\]
where \( M(a,b) = f(a) g(a) + f(b) g(b) \) and \( N(a,b) = f(a) g(b) + f(b) g(a) \).

Proof. Since \( f \) and \( g \) are convex functions on \([a,b]\), then for all \( t \in [0,1] \) we have
\[
f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) \\
= f\left(\frac{ta + (1-t)b}{2} + \frac{(1-t)a + tb}{2}\right) g\left(\frac{ta + (1-t)b}{2} + \frac{(1-t)a + tb}{2}\right) \\
\leq \frac{1}{4} [f(ta + (1-t)b) + f((1-t)a + tb)] [g(ta + (1-t)b) + g((1-t)a + tb)] \\
= \frac{1}{4} [f(ta + (1-t)b) g(ta + (1-t)b) + f((1-t)a + tb) g((1-t)a + tb)] \\
+ \frac{1}{4} [f(ta + (1-t)b) g((1-t)a + tb) + f((1-t)a + tb) g(ta + (1-t)b)] \\
\leq \frac{1}{4} [f(ta + (1-t)b) g(ta + (1-t)b) + f((1-t)a + tb) g((1-t)a + tb)] \\
\frac{1}{4} \left\{ 2t(1-t) M(a,b) + \left[ (1-t)^2 + t^2 \right] N(a,b) \right\} \tag{3.7}
\]

Multiplying both sides of (3.7) by \( 2^{1+\alpha} \alpha t^{\alpha-1} \), then integrating the obtained inequality with respect to \( t \) over \([0,\frac{1}{2}]\), we have the desired result. \( \square \)

Theorem 3.12. Let \( f \) and \( g : [a,b] \subseteq (0,\infty) \to \mathbb{R} \) be non-negative and harmonically convex functions with \( a < b \) and \( f \in L[a,b] \), then the following inequality for
fractional integrals holds:

\[
2f \left( \frac{2ab}{a+b} \right) g \left( \frac{2ab}{a+b} \right) \leq \frac{\Gamma (\alpha +1)}{2^{1-\alpha}} \left( \frac{ab}{b-a} \right)^{\alpha} \times \left\{ \frac{J_{\alpha+\frac{1}{2}}^{\alpha}(f \circ h)(1/a)}{a,b} + \frac{J_{\alpha+\frac{3}{2}}^{\alpha}(f \circ h)(1/b)}{a,b} \right\} + \frac{\alpha^2 + 3\alpha}{2(\alpha + 2)} M(a,b) + \left( \frac{\alpha}{4(\alpha + 2)} - \frac{\alpha}{2(\alpha + 1)} + \frac{1}{2} \right) N(a,b)
\]

where \( M(a,b) = f(a) g(a) + f(b) g(b) \) and \( N(a,b) = f(a) g(b) + f(b) g(a) \).

Proof. Theorem 3.12 is direct result of (3.6) in Theorem 3.11. In (3.6), if we replace the convex functions \( f \) and \( g \) with the convex functions \( F \) defined \( F := f \circ h \) and \( G \) defined \( G := g \circ h \), \( h(x) = \frac{x}{2} \) and if we replace \( a \) and \( b \) with \( \frac{a}{2} \) and \( \frac{b}{2} \) respectively, we have (3.8).

Corollary 3.13. In Theorem 3.12 if we take \( g : [a,b] \to \mathbb{R} \) as \( g(x) = 1 \) for all \( x \in [a,b] \), then we have

\[
2f \left( \frac{2ab}{a+b} \right) g \left( \frac{2ab}{a+b} \right) \leq \frac{\Gamma (\alpha +1)}{2^{1-\alpha}} \left( \frac{ab}{b-a} \right)^{\alpha} \left\{ \frac{J_{\alpha+\frac{1}{2}}^{\alpha}(f \circ h)(1/a)}{a,b} + \frac{J_{\alpha+\frac{3}{2}}^{\alpha}(f \circ h)(1/b)}{a,b} \right\} + \frac{f(a) + f(b)}{2}.
\]

Corollary 3.14. In Theorem 3.12 if we take \( \alpha = 1 \), then we recapture

\[
2f \left( \frac{2ab}{a+b} \right) g \left( \frac{2ab}{a+b} \right) \leq \frac{ab}{b-a} \int_{a}^{b} f(x) g(x) \frac{dx}{x^2} + \frac{1}{6} M(a,b) + \frac{1}{3} N(a,b)
\]

for harmonically convex functions which is given in [6, Remark 14].

Remark. In Corollary 3.14, about harmonically convex functions, we have an analogue result of (2.3) for convex functions by Pachpette in [21].

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