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FIXED POINT AND BEST PROXIMITY POINT THEOREMS ON PARTIAL METRIC SPACES

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Abstract. The aim of the paper is to study some generalizations of weakly Kannan and weakly Chatterjea-type contractive mappings in the setting of partial metric spaces. Some important outcomes are stated and proved, extending classic fixed point and best proximity point theorems. Examples are given in order to emphasize the utility of the main results.

1. Introduction and preliminaries

1.1. **Partial metric spaces.** In 1994, the famous work of Mathews [10] introduced the notion of partial metric space, directly connected with theoretical computer science. From the mathematical point of view, the relevance of the concept is emphasized by fixed point and best proximity point results on partial metric spaces, see [1], and the references therein. Some generalizations of the concept have been stated, for example see [2], [4], [11], [13] and the references therein. This section recalls basic definitions and properties on partial metric spaces.

Definition 1.1 ([10]). *A partial metric on a nonempty set X is a function $p\colon X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$:*

(p_1) $x = y \Leftrightarrow p(x,x) = p(x,y) = p(y,y)$;
(p_2) $p(x,x) \leq p(x,y)$;
(p_3) $p(x,y) = p(y,x)$;
(p_4) $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$.

A partial metric space is a pair (X,p) such that X is a nonempty set and p is a partial metric on X.

Each partial metric p generates a topology $\tau(p)$ on X, whose base is a family of open balls defined as it follows: for $x \in X$ and $\epsilon > 0$, $B_p(x,\epsilon) = \{y \in X \mid p(x,y) - p(x,x) < \epsilon\}$. Within this topological context, some adapted concepts of convergence, Cauchy sequence and completeness may be defined:

(d_1) A sequence $\{x_n\}$ in a partial metric space (X,p) converges to $u \in X$ if and only if $p(u,u) = \lim\limits_{n\to\infty} p(u,x_n)$.

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(d_2) A sequence $\{x_n\}$ in a partial metric space (X, p) is called a Cauchy sequence if $\lim_{n,m\to\infty} p(x_n, x_m)$ exists and is finite.

(d_3) A partial metric space (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges to a point $u \in X$ and $\lim_{n,m\to\infty} p(x_n, x_m) = p(u, u)$.

(d_4) A mapping $T\colon X \to X$ is continuous at $x_0 \in X$ if for each $\epsilon > 0$, there is $\delta > 0$, so that $T(B_p(x_0, \delta)) \subset B_p(Tx_0, \epsilon)$. Obviously, if T is continuous, $x_n \to u$ implies $Tx_n \to Tu$.

Remark ([10]). If p is a partial metric on a non-empty set X, then p induces a corresponding metric $d_p\colon X \times X \to \mathbb{R}^+$ defined by:

$$d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y).$$

A correlation between p and d_p is suggested by the following properties:

1) A sequence $\{x_n\}$ in a partial metric space (X, p) is a Cauchy sequence if and only if it is a Cauchy sequence with respect to the distance function d_p.

2) A partial metric space (X, p) is complete if and only if the corresponding metric space (X, d_p) is complete.

3) Given a sequence $\{x_n\}$ in a partial metric space (X, p) and $u \in X$,

$$\lim_{n\to\infty} d_p(x_n, u) = 0 \Leftrightarrow \lim_{n\to\infty} p(x_n, u) = \lim_{n,m\to\infty} p(x_n, x_m) = p(u, u).$$

Explicitly, the previous equivalence states that the convergence with respect to d_p involves convergence with respect to p; the converse though is not true.

Also, it has to be emphasized that a partial metric is not necessarily continuous. For example, consider $X = [0, \infty)$, $p\colon X \times X \to [0, \infty)$, $p(x, y) = \max\{x, y\}$. Let $x_n = 1$, for any $n \in \mathbb{N}$, and $u > 1$. Obviously, $x_n \to u$ with resect to the partial metric p, but $\lim_{n\to\infty} p(x_n, x_n) \neq p(u, u)$. On the other hand, the following lemma holds.

Lemma 1.2 ([14]). *Let (X, p) be a partial metric space and d_p the induced metric. If $\{x_n\}$, $\{y_n\} \subset X$ converge to $u \in X$, and $v \in X$, with respect to d_p, then $p(x_n, y_n) \to p(u, v)$, when $n \to \infty$.*

Recently, Haghi et al [7] proved that some results on partial metric spaces can be obtained from the metric space context. We emphasize that their approach cannot be applied here, since our Picard iterations do not refer to 0-Cauchy sequences or 0-completness.

1.2. **Weakly Kannan and weakly Chatterjea-type contractions on complete metric spaces.**

In 1968, Kannan [8] introduced a new type of contraction, the one we call now "the Kannan contraction".

Definition 1.3 ([8], [9]). *Let (X, d) be a metric space. A mapping $T\colon X \to X$ is called Kannan-type contractive mapping if there exists $\alpha \in [0, 1)$ such that, for all x, $y \in X$,*

$$d(Tx, Ty) \leq \frac{\alpha}{2}[d(x, Tx) + d(y, Ty)].$$

This definition was followed, in 1972, by another generalization of the Banach principle, this time due to Chatterjea [5].

Definition 1.4 ([5]). *Let (X, d) be a metric space. A mapping $T : X \to X$ is called Chatterjea-type contractive mapping if there exists $\alpha \in [0, 1)$ such that for all x, $y \in X$,*

$$d(Tx, Ty) \leq \frac{\alpha}{2}[d(y, Tx) + d(x, Ty)].$$

Recently, Razani and Parvaneh [12] defined the weakly T-Chatterjea and weakly T-Kannan-type contractive mappings, establishing their behavior on complete metric spaces. If T is set to be the identity map, it leads to the definitions of weakly Kannan (different from the weakly Kannan contractions analyzed in [3] and [6]) and weakly Chatterjea-type contractions. We recall these definitions and some fixed point properties on complete metric spaces related to them.

Let $\mu\colon [0, \infty) \to [0, \infty)$ be a monotone strictly increasing continuous function with $\mu(0) = 0$, that is an altering distance function, and $\psi\colon [0, \infty)^2 \to [0, \infty)$ be a continuous function such that $\psi(x, y) > 0$, $\forall x, y > 0$, and $\psi(0, 0) = 0$.

Definition 1.5 ([12]). *A selfmapping T on a metric space (X, d) is called generalized weakly Kannan-type contraction if for any x, $y \in X$,*

$$\mu(d(Tx, Ty)) \leq \mu\left(\frac{d(x, Tx) + d(y, Ty)}{2}\right) - \psi(d(x, Tx), d(y, Ty)).$$

Definition 1.6 ([12]). *A mapping T from a metric space (X, d) into itself is called generalized weakly Chatterjea-type contraction if for any x, $y \in X$,*

$$\mu(d(Tx, Ty)) \leq \mu\left(\frac{d(y, Tx) + d(x, Ty)}{2}\right) - \psi(d(y, Tx), d(x, Ty)).$$

They proved the following result with regard to these types of contractions.

Theorem 1.7 ([12]). *Let (X, d) be a complete metric space and $T : X \to X$ be a generalized weakly Kannan-type or Chatterjea-type contractive mapping. Then T has a unique fixed point x^* in X and the Picard sequence of iterates $\{T^n x\}_{n\in\mathbb{N}}$ converges, for each $x \in X$, to x^*.*

2. Fixed point results on partial metric spaces

2.1. **Weakly Kannan-type contractions on partial metric spaces.** In [6] the weakly Chatterjea and Kannan-type contractive mappings defined above were enforced with cyclic behavior and adapted fixed point properties were obtained.

In the following, we reconsider these definitions in the partial metric context and we state and prove fixed point results. Let (X, p) be a partial metric space and let $\mu\colon [0, \infty) \to [0, \infty)$ be a monotone strictly increasing continuous function with $\mu(0) = 0$, and $\psi\colon [0, \infty)^2 \to [0, \infty)$ be a continuous function such that

$$\lim_{n\to\infty} \psi(x_n, y_n) = 0 \Rightarrow \begin{cases} \lim\limits_{n\to\infty} x_n = 0; \\ \lim\limits_{n\to\infty} y_n = 0. \end{cases} \tag{2.1}$$

Definition 2.1. *A selfmapping T on a partial metric space (X, p) is called generalized weakly Kannan-type contraction if, for any x, $y \in X$, the following inequality holds*

$$\begin{aligned} \mu(p(Tx, Ty)) \quad \leq \quad & \mu\left(\frac{p(x, Tx) + p(y, Ty)}{2}\right) \\ & - \psi(p(x, Tx) - p(Tx, Tx), p(y, Ty) - p(Ty, Ty)). \end{aligned} \tag{2.2}$$

Theorem 2.2. *Let (X,p) be a complete partial metric space and $T\colon X \to X$ be a generalized weakly Kannan-type contractive mapping. Then T has a unique fixed point u in X and, for each initial point $x \in X$, the Picard sequence of iterates $\{T^n x\}_{n\in N}$ converges to u, with respect to d_p.*

Proof. STEP 1. Consider $x_0 \in X$ and the corresponding Picard sequence of iterates $x_{n+1} = Tx_n = T^{n+1}x_0$, $n \in \mathbb{N} \cup \{0\}$. Since T is a weakly Kannan contraction, inequality (2.2) leads to

$$\begin{aligned}\mu(p(x_n, x_{n+1})) &= \mu(Tx_{n-1}, Tx_n) \le \mu\left(\frac{p(x_{n-1}, x_n) + p(x_n, x_{n+1})}{2}\right)\\ &\quad -\psi(p(x_{n-1}, x_n) - p(x_n, x_n), p(x_n, x_{n+1}) - p(x_{n+1}, x_{n+1}))\\ &\le \mu\left(\frac{p(x_{n-1}, x_n) + p(x_n, x_{n+1})}{2}\right) \qquad (2.3)\end{aligned}$$

and, since μ is a non-decreasing function, it follows, for all $n = 1, 2, \ldots$,

$$p(x_n, x_{n+1}) \le \frac{p(x_{n-1}, x_n) + p(x_n, x_{n+1})}{2}.$$

Therefore, $p(x_n, x_{n+1}) \le p(x_{n-1}, x_n)$, and $\{p(x_n, x_{n+1})\}_{n\in N}$ is a decreasing sequence of non-negative real numbers, hence it is convergent. Denote by $r \in \mathbb{R}^+$ its limit.

Also, relation (2.3) implies

$$\begin{aligned}0 &\le \psi(p(x_{n-1}, x_n) - p(x_n, x_n), p(x_n, x_{n+1}) - p(x_{n+1}, x_{n+1}))\\ &\le \mu\left(\frac{p(x_{n-1}, x_n) + p(x_n, x_{n+1})}{2}\right) - \mu(p(x_n, x_{n+1})).\end{aligned}$$

Letting $n \to \infty$ leads to

$$\lim_{n\to\infty} \psi(p(x_{n-1}, x_n) - p(x_n, x_n), p(x_n, x_{n+1}) - p(x_{n+1}, x_{n+1})) = 0$$

and, based on property (2.1) of ψ, we may conclude that

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = \lim_{n\to\infty} p(x_n, x_n) = r. \qquad (2.4)$$

STEP 2. In the following, we shall prove that $\{x_n\}$ is a Cauchy sequence and $\lim\limits_{m,n\to\infty} p(x_n, x_m) = r$. Assume the contrary. Thus, there exists $\epsilon > 0$ such that, for all $n \in \mathbb{N}$, we can find $r_n > q_n \ge n$ with either $p(x_{r_n}, x_{q_n}) \le -\epsilon + r$, or $p(x_{r_n}, x_{q_n}) \ge \epsilon + r$. Let us eliminate each of these two possibilities.

First, suppose $p(x_{r_n}, x_{q_n}) \le -\epsilon + r$. Property (p_2) of partial metrics leads to $p(x_{r_n}, x_{r_n}) \le -\epsilon + r$ and, by letting $n \to \infty$, we obtain $r \le -\epsilon + r$, i.e. $\epsilon \le 0$, which is contradiction with $\epsilon > 0$.

Let us analyze the second case, that is $p(x_{r_n}, x_{q_n}) \ge \epsilon + r$. Since μ is a non-decreasing function and T is weakly Kannan contractive, it follows

$$\begin{aligned}\mu(\epsilon + r) &\le \mu(p(x_{r_n}, x_{q_n})) = \mu(p(Tx_{r_n-1}, Tx_{q_n-1}))\\ &\le \mu\left(\frac{p(x_{r_n-1}, x_{r_n}) + p(x_{q_n-1}, x_{q_n})}{2}\right)\\ &\quad -\psi(p(x_{r_n-1}, x_{r_n}) - p(x_{r_n}, x_{r_n}), p(x_{q_n-1}, x_{q_n}) - p(x_{q_n}, x_{q_n})).\end{aligned}$$

Letting $n \to \infty$ and using (2.4), we obtain

$$\mu(\epsilon + r) \le \mu(r) - \psi(0, 0) \le \mu(r) \Rightarrow \epsilon + r \le r \Rightarrow \epsilon \le 0,$$

contradicting again the hypotheses on ϵ.

Therefore,

$$\lim_{m,n\to\infty} p(x_n,x_m) = \lim_{n\to\infty} p(x_n,x_{n+1}) = \lim_{n\to\infty} p(x_n,x_n) = r \tag{2.5}$$

and $\{x_n\}$ is a Cauchy sequence in a complete partial metric space. In conclusion, there exists $u \in X$ such that

$$\lim_{m,n\to\infty} p(x_n,x_m) = \lim_{n\to\infty} p(x_n,u) = p(u,u). \tag{2.6}$$

Moreover, combining (2.5) and (2.6), we get $p(u,u) = r$.

Step 3. We shall prove next that u is a fixed point of T. Again, let us start with T being a weakly Kannan-type contractive mapping:

$$\begin{aligned} \mu(p(x_{n+1},Tu)) &\leq \mu\left(\frac{p(x_n,x_{n+1})+p(u,Tu)}{2}\right) \\ &\quad -\psi\left(p(x_n,x_{n+1})-p(x_{n+1},x_{n+1}),p(u,Tu)-p(Tu,Tu)\right) \\ &\leq \mu\left(\frac{p(x_n,x_{n+1})+p(u,Tu)}{2}\right). \end{aligned} \tag{2.7}$$

Letting $n \to \infty$, based on Lemma 1.2, on the continuity of the function μ, and using the monotonicity of μ, it follows

$$p(u,Tu) \leq \frac{p(u,u)+p(u,Tu)}{2} \Rightarrow p(u,Tu) \leq p(u,u).$$

On the other hand, property (p_2) of partial metrics ensures us about the converse inequality $p(u,Tu) \geq p(u,u)$. Hence $p(u,Tu) = p(u,u)$. Moreover, substituting these in relation (2.7), after letting $n \to \infty$, leads to

$$\psi(0,p(u,Tu)-p(Tu,Tu)) = 0,$$

resulting $p(u,Tu)-p(Tu,Tu) = 0$. Therefore, $p(u,Tu) = p(u,u) = p(Tu,Tu)$ and, consequence of (p_1), it follows $u = Tu$, that is u is a fixed point for T.

Step 4. Finally, let us prove the uniqueness of the fixed point. Suppose there exist two fixed points u and v of mapping T. Using again the properties of weakly Kannan-type contractive mappings, we find

$$\begin{aligned} \mu(p(Tu,Tv)) &\leq \mu\left(\frac{p(u,Tu)+p(v,Tv)}{2}\right) \\ &\quad -\psi\left(p(u,Tu)-p(Tu,Tu),p(v,Tv)-p(Tv,Tv)\right), \end{aligned}$$

that is

$$\mu(p(u,v)) \leq \mu\left(\frac{p(u,u)+p(v,v)}{2}\right) - \psi\left(0,0\right) \leq \mu\left(\frac{p(u,u)+p(v,v)}{2}\right),$$

$$p(u,v) \leq \frac{p(u,u)+p(v,v)}{2}.$$

It follows that $d_p(u,v) \leq 0$, that is $u = v$. □ □

2.2. **Weakly Chatterjea-type contractions on partial metric spaces.**

Definition 2.3. *A mapping T from a partial metric space (X,p) into itself is called generalized weakly Chatterjea-type contraction if, for each x, $y \in X$,*

$$\begin{aligned} \mu(p(Tx,Ty)) &\leq \mu\left(\frac{p(y,Tx)+p(x,Ty)}{2}\right) \\ &\quad -\psi(p(y,Tx)-p(Tx,Tx),p(x,Ty)-p(Ty,Ty)). \end{aligned} \tag{2.8}$$

Theorem 2.4. *Let (X,p) be a complete partial metric space and $T\colon X \to X$ be a generalized weakly Chatterjea-type contractive map. Then T has a unique fixed point u in X and, for each $x \in X$, the Picard sequence of iterates $\{T^n x\}_{n\in\mathbb{N}}$ converges to u, with respect to d_p.*

Proof. Step 1. Consider $x_0 \in X$ and the corresponding Picard sequence of iterates $x_{n+1} = Tx_n = T^{n+1}x_0$, $n \in \mathbb{N} \cup \{0\}$. Since T is weakly Chatterjea contractive, inequality (2.8) leads to

$$\begin{aligned}\mu(p(x_n,x_{n+1})) &= \mu(Tx_{n-1},Tx_n) \le \mu\left(\frac{p(x_n,x_n)+p(x_{n-1},x_{n+1})}{2}\right)\\ &\quad -\psi(0,p(x_{n-1},x_{n+1})-p(x_{n+1},x_{n+1}))\\ &\le \mu\left(\frac{p(x_n,x_n)+p(x_{n-1},x_{n+1})}{2}\right) \qquad (2.9)\end{aligned}$$

and, based on the monotonicity of μ and on property (p_4), it follows

$$\begin{aligned}p(x_n,x_{n+1}) &\le \frac{p(x_n,x_n)+p(x_{n-1},x_{n+1})}{2}\\ &\le \frac{p(x_n,x_n)+p(x_{n-1},x_n)+p(x_n,x_{n+1})-p(x_n,x_n)}{2}\\ &= \frac{p(x_{n-1},x_n)+p(x_n,x_{n+1})}{2}. \qquad (2.10)\end{aligned}$$

Therefore, $p(x_n,x_{n+1}) \le p(x_{n-1},x_n)$, and $\{p(x_n,x_{n+1})\}_{n\in N}$ is a monotone decreasing sequence of non-negative real numbers, hence is convergent. Let $r \in \mathbb{R}^+$ be its limit. Relation (2.10) leads to

$$2p(x_n,x_{n+1}) \le p(x_n,x_{n+1}) + p(x_{n-1},x_{n+1}).$$

Taking the $\liminf$ as n goes to ∞, it follows $r \le \liminf p(x_{n-1},x_{n+1})$.

Also, from (2.9), we obtain

$$\begin{aligned}0 &\le \psi(0,p(x_{n-1},x_{n+1})-p(x_{n+1},x_{n+1}))\\ &\le \mu\left(\frac{p(x_n,x_n)+p(x_{n-1},x_{n+1})}{2}\right) - \mu(p(x_n,x_{n+1}))\end{aligned}$$

and letting $n\to\infty$, leads to

$$\lim_{n\to\infty} \psi(0,p(x_{n-1},x_{n+1})-p(x_{n+1},x_{n+1})) = 0,$$

therefore

$$\lim_{n\to\infty} [p(x_{n-1},x_{n+1})-p(x_{n+1},x_{n+1})] = 0. \qquad (2.11)$$

Considering the $\liminf$ in relation (2.11), we obtain

$$0 \ge \liminf_{n\to\infty} p(x_{n-1},x_{n+1}) - \limsup_{n\to\infty} p(x_{n+1},x_{n+1}),$$

hence $\limsup_{n\to\infty} p(x_{n+1},x_{n+1}) \ge r$. Since the converse is obvious, we have proved that

$$\lim_{n\to\infty} p(x_n,x_n) = \lim_{n\to\infty} p(x_n,x_{n+1}) = r.$$

Step 2. In this section, we shall prove that $\{x_n\}$ is a Cauchy sequence and $\lim\limits_{m,n\to\infty} p(x_n,x_m) = r$. Assume the contrary. Thus, there exists $\epsilon > 0$ such that, for all $n \in \mathbb{N}$, we can find $r_n > q_n \ge n$ with either $p(x_{r_n},x_{q_n}) \le -\epsilon + r$, or

$p(x_{r_n}, x_{q_n}) \geq \epsilon + r$. In order to eliminate these two assumptions we use similar arguments as in the proof of Theorem 2.2.

First, suppose $p(x_{r_n}, x_{q_n}) \leq -\epsilon + r$. Property (p_2) of partial metrics leads to $p(x_{r_n}, x_{r_n}) \leq -\epsilon + r$ and, by letting $n \to \infty$, we obtain $r \leq -\epsilon + r$, that is $\epsilon \leq 0$, which is contradiction with $\epsilon > 0$.

In the following, suppose $p(x_{r_n}, x_{q_n}) \geq \epsilon + r$. Let r_n be the smallest integer such that $r_n > q_n \geq n$ and $p(x_{r_n}, x_{q_n}) \geq \epsilon + r$. Then either 1) $r_n - 1 = q_n$, or 2) $p(x_{r_n - 1}, x_{q_n}) < \epsilon + r$.

Case 1) $r_n - 1 = q_n \Rightarrow p(x_{r_n}, x_{q_n}) = p(x_{q_n}, x_{q_n+1}) \geq \epsilon + r$. Letting $n \to \infty$, we find $r \geq \epsilon + r$, contradiction with ϵ being non-negative!

Case 2) Suppose $p(x_{r_n - 1}, x_{q_n}) < \epsilon + r$. Then, based on (p_4), we have

$$\begin{aligned}\epsilon + r \quad &\leq p(x_{r_n}, x_{q_n}) \leq p(x_{r_n}, x_{r_n - 1}) + p(x_{r_n - 1}, x_{q_n}) - p(x_{r_n - 1}, x_{r_n - 1}) \\ &< \epsilon + r + p(x_{r_n}, x_{r_n - 1}) - p(x_{r_n - 1}, x_{r_n - 1}).\end{aligned}$$

Letting $n \to \infty$, it follows

$$\lim_{n\to\infty} p(x_{r_n}, x_{q_n}) = \lim_{n\to\infty} p(x_{r_n - 1}, x_{q_n}) = r + \epsilon. \tag{2.12}$$

Also,

$$p(x_{r_n}, x_{q_n}) \leq p(x_{r_n}, x_{q_n - 1}) + p(x_{q_n - 1}, x_{q_n}) - p(x_{q_n - 1}, x_{q_n - 1}),$$

$$p(x_{r_n}, x_{q_n - 1}) \leq p(x_{r_n}, x_{q_n}) + p(x_{q_n}, x_{q_n - 1}) - p(x_{q_n}, x_{q_n}).$$

Considering the limit leads us to

$$\lim_{n\to\infty} p(x_{r_n}, x_{q_n - 1}) = r + \epsilon. \tag{2.13}$$

Since μ is a non-decreasing function and T is weakly Chatterjea contractive, it follows

$$\begin{aligned}\mu(\epsilon + r) \quad &\leq \mu(p(x_{r_n}, x_{q_n})) = \mu(p(Tx_{r_n - 1}, Tx_{q_n - 1})) \\ &\leq \mu\left(\frac{p(x_{q_n - 1}, x_{r_n}) + p(x_{r_n - 1}, x_{q_n})}{2}\right) \\ &-\psi(p(x_{q_n - 1}, x_{r_n}) - p(x_{r_n}, x_{r_n}), p(x_{r_n - 1}, x_{q_n}) - p(x_{q_n}, x_{q_n})).\end{aligned}$$

Letting $n \to \infty$ and using (2.12) and (2.13), we obtain

$$\mu(\epsilon + r) \leq \mu(r + \epsilon) - \psi(\epsilon, \epsilon) \Rightarrow \psi(\epsilon, \epsilon) = 0 \Rightarrow \epsilon = 0,$$

which is contradicting with $\epsilon > 0$.

Therefore,

$$\lim_{m,n\to\infty} p(x_n, x_m) = \lim_{n\to\infty} p(x_n, x_n) = \lim_{n\to\infty} p(x_n, x_{n+1}) = r \tag{2.14}$$

and $\{x_n\}$ is a Cauchy sequence in a complete partial metric space. In conclusion, there exists $u \in X$ such that

$$\lim_{m,n\to\infty} p(x_n, x_m) = \lim_{n\to\infty} p(x_n, u) = p(u, u). \tag{2.15}$$

Moreover, combining (2.14) and (2.15) leads to $p(u, u) = r$.

STEP 3. In the following, we shall prove that u is a fixed point of T. Again, we start with T being a weakly Chatterjea-type contractive mapping:

$$\begin{aligned} \mu(p(x_{n+1}, Tu)) &\leq \mu\left(\frac{p(u, x_{n+1}) + p(x_n, Tu)}{2}\right) \\ &\quad -\psi\left(p(u, x_{n+1}) - p(x_{n+1}, x_{n+1}), p(x_n, Tu) - p(Tu, Tu)\right) \\ &\leq \mu\left(\frac{p(u, x_{n+1}) + p(x_n, Tu)}{2}\right). \end{aligned} \tag{2.16}$$

Considering the limit and based on the monotonicity of μ, it follows

$$p(u, Tu) \leq \frac{p(u, u) + p(u, Tu)}{2},$$

that is $p(u, Tu) \leq p(u, u)$. On the other hand, property (p_2) of partial metrics assures us about the converse inequality $p(u, Tu) \geq p(u, u)$. Hence $p(u, Tu) = p(u, u)$. Moreover, substituting these in relation (2.16), after considering the limit, leads to

$$\psi(0, p(u, Tu) - p(Tu, Tu)) = 0,$$

resulting that $p(u, Tu) - p(Tu, Tu) = 0$. Therefore, $p(u, Tu) = p(u, u) = p(Tu, Tu)$ and, using property (p_1) it follows $u = Tu$, i.e. u is a fixed point for T.

STEP 4. Finally, let us prove the uniqueness of the fixed point. Suppose there exist two fixed points u and v for mapping T. Using again the properties of weakly Chatterjea contractive mappings, we find

$$\begin{aligned} \mu(p(Tu, Tv)) &\leq \mu\left(\frac{p(v, Tu) + p(u, Tv)}{2}\right) \\ &\quad -\psi\left(p(v, Tu) - p(Tu, Tu), p(u, Tv) - p(Tv, Tv)\right), \end{aligned}$$

that is

$$\mu(p(u, v)) \leq \mu(p(u, v)) - \psi\left(p(u, v) - p(u, u), p(u, v) - p(v, v)\right).$$

We obtain $\psi\left(p(u, v) - p(u, u), p(u, v) - p(v, v)\right) = 0$, so $p(u, v) = p(u, u) = p(v, v) = 0$, and get $u = v$. □ □

3. Best proximity point theorems on partial metric spaces

In the previous section we analyzed the fixed point properties of some classes of contractive mappings, that is determine a fixed point of a selfmapping $T\colon X \to X$. A natural question arises: *what happens if T is a nonselfmapping*? From this point of view the interest is focused on studying the existence of best proximity points, that is, if A and B are nonempty subsets in a complete metric space (X, d), and $T\colon A \to B$ a nonselfmapping, find a point $u \in A$, so that $d(u, Tu) = d(A, B)$. Using the generalized contractions we have defined, we state and prove best proximity point theorems in partial metric context. Let us start by introducing the notions used to investigate best proximity point theorems.

Consider the sets

$$A_0 = \{x \in A \mid p(x, y) = p(A, B) \text{ for some } y \in B\},$$
$$B_0 = \{y \in B \mid p(x, y) = p(A, B) \text{ for some } x \in A\}.$$

The following result will be needed in the sequel.

Lemma 3.1 ([14]). *The subset B_0 is closed with respect to the induced metric topology $\tau(d_p)$.*

In 2013, a very useful tool in the study of best proximity point theory was introduced in [15].

Definition 3.2 ([15])**.** *Let (A, B) be a pair of nonempty subsets of a partial metric space (X, p), such that $A_0 \neq \emptyset$. The pair (A, B) is said to have the weak P-property if and only if, for any $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$,*

$$\begin{cases} p(x_1, y_1) = p(A, B), \\ p(x_2, y_2) = p(A, B) \end{cases} \Rightarrow p(x_1, x_2) \leq p(y_1, y_2).$$

In [14], the weakly Kannan contractive mappings with respect to the pair (A, B) were defined as follows, and a corresponding best proximity point property was obtained. More precisely, a mapping $T\colon A \to B$ was said to be weakly Kannan provided that

$$p(Tx, Ty) \leq \frac{\alpha(x, y)}{2} \left[p(x, Tx) + p(y, Ty) - 2p(A, B)\right],$$

where, for all $0 < a < b$, the function $\alpha\colon A \times A \to [0, 1)$ satisfies the condition

$$\sup\{\alpha(x, y) | a \leq p(x, y) \leq b\} < 1.$$

The result of Zhang and Su [14] is the following.

Theorem 3.3. *(A, B) is a pair of subsets of the partial metric space (X, p), which are nonempty and closed, and so that A_0 is nonempty. If $T\colon A \to B$ is a continuous weakly Kannan contration, $TA_0 \subset B_0$, and (A, B) is endowed with the P property, then T has a unique best proximity point $x^* \in A_0$.*

The results we are going to state and prove in the sequel are more general than those of Zhang and Su.

3.1. **Weakly Kannan-type I mappings on proper subsets of partial metric spaces.**

Definition 3.4. *Let (A, B) be a pair of nonempty subsets of a partial metric space (X, p). A mapping $T\colon A \to B$ is called generalized weakly Kannan-type I contraction associated to the pair (A, B) if*

$$\mu(p(Tx, Ty)) \leq \mu\left(\frac{p(x, Tx) + p(y, Ty) - 2p(A, B) + p(x, x) + p(y, y)}{2}\right)$$
$$-\psi(p(x, Tx) - p(A, B), p(y, Ty) - p(A, B)),$$

for all x, $y \in A$.

Theorem 3.5. *Let (A, B) be a pair of nonempty subsets of a complete partial metric space (X, p) such that $A_0 \neq \emptyset$ and let $T\colon A \to B$ be a weakly Kannan-type I contraction. Suppose $TA_0 \subseteq B_0$ and the pair (A, B) has the weak P-property. Then T has a unique best proximity point $u \in A_0$ and, for each point $x_0 \in A_0$, the iteration sequence $\{x_n\}$ defined by*

$$p(x_{n+1}, y_n) = p(A, B),\ y_n = Tx_n,\ n = 0, 1, 2, \ldots$$

converges to u with respect to d_p.

Proof. For start, let us prove the existence of the two sequences. Consider $x_0 \in A_0$. Since $TA_0 \subset B_0$, it follows that $y_0 = Tx_0 \in B_0$, and there exists $x_1 \in A_0$ such that $p(x_1, y_0) = p(A, B)$. By continuing this procedure, we obtain the sequences $\{x_n\}$ and $\{y_n\}$ with the above-mentioned properties.

STEP 1. The definition of the sequence $\{x_n\}$, together with the weak P-property of pair (A, B) lead to the following relations:

(i) $p(x_n, y_{n-1}) = p(A, B),\ \forall n = 1, 2, \ldots;$

(ii) $p(x_n, x_{n+1}) \le p(y_{n-1}, y_n),\ \forall n = 1, 2, \ldots;$

(iii) $p(x_n, x_n) \le p(y_{n-1}, y_{n-1}),\ \forall n = 1, 2, \ldots;$

Since T is a Kannan-type contraction, from (i)-(iii) and (p_4) it follows

$$\begin{aligned}
&\mu(p(y_n, y_{n+1})) = \mu(p(Tx_n, Tx_{n+1})) \\
\le\ &\mu\left(\frac{p(x_n, y_n) + p(x_{n+1}, y_{n+1}) - 2p(A, B) + p(x_n, x_n) + p(x_{n+1}, x_{n+1})}{2}\right) \\
&-\psi(p(x_n, y_n) - p(A, B), p(x_{n+1}, y_{n+1}) - p(A, B)) \\
\le\ &\mu\left(\frac{p(x_n, y_n) + p(x_{n+1}, y_{n+1}) - 2p(A, B) + p(y_{n-1}, y_{n-1}) + p(y_n, y_n)}{2}\right) \\
&-\psi(p(x_n, y_n) - p(A, B), p(x_{n+1}, y_{n+1}) - p(A, B)) \\
\le\ &\mu\left(\frac{p(y_{n-1}, y_n) + p(y_n, y_{n+1})}{2}\right) \\
&-\psi(p(x_n, y_n) - p(A, B), p(x_{n+1}, y_{n+1}) - p(A, B)) \\
\le\ &\mu\left(\frac{p(y_{n-1}, y_n) + p(y_n, y_{n+1})}{2}\right).
\end{aligned} \tag{3.1}$$

Then

$$p(y_n, y_{n+1}) \le \frac{p(y_{n-1}, y_n) + p(y_n, y_{n+1})}{2}, \text{ which is } p(y_n, y_{n+1}) \le p(y_{n-1}, y_n),$$

that is, $\{p(y_n, y_{n+1})\}$ is a monotone decreasing sequence of non-negative real numbers, therefore it is convergent. Let $r \in \mathbb{R}^+$ denote its limit.

Also, the hypotheses (2.1) on ψ, together with inequalities (3.1), lead to

$$\lim_{n\to\infty} p(x_n, y_n) = p(A, B). \tag{3.2}$$

On the other hand, $p(x_n, x_n) \le p(y_{n-1}, y_{n-1}) \le p(y_{n-1}, y_n)$, so relations (3.1) imply

$$\begin{aligned}
&\mu(p(y_n, y_{n+1})) = \mu(p(Tx_n, Tx_{n+1})) \\
\le\ &\mu\left(\frac{p(x_n, y_n) + p(x_{n+1}, y_{n+1}) - 2p(A, B) + p(y_{n-1}, y_{n-1}) + p(x_{n+1}, x_{n+1})}{2}\right) \\
&-\psi(p(x_n, y_n) - p(A, B), p(x_{n+1}, y_{n+1}) - p(A, B)) \\
\le\ &\mu\left(\frac{p(x_{n+1}, y_{n+1}) - p(A, B) + p(y_{n-1}, y_n) + p(x_{n+1}, x_{n+1})}{2}\right) \\
\le\ &\mu\left(\frac{p(y_{n-1}, y_n) + p(y_n, y_{n+1})}{2}\right)
\end{aligned}$$

Taking the limit when $n \to \infty$, and having in mind relation (3.2), we obtain $\lim_{n\to\infty} p(x_n, x_n) = r$.

STEP 2. In the following, we shall prove that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences. Moreover, we shall prove that

$$\lim_{m,n\to\infty} p(x_m, x_n) = \lim_{m,n\to\infty} p(y_m, y_n) = r.$$

Assume the contrary. Thus, there exists $\epsilon > 0$ such that, for all $n \in N$, we can find $r_n > q_n \geq n$ with either $p(x_{r_n}, x_{q_n}) \leq -\epsilon + r$, or $p(x_{r_n}, x_{q_n}) \geq \epsilon + r$. Let us eliminate, each at a time those two possibilities.

First, suppose $p(x_{r_n}, x_{q_n}) \leq -\epsilon + r$. Property (p_2) of partial metrics leads to $p(x_{r_n}, x_{r_n}) \leq -\epsilon + r$ and, by letting $n \to \infty$, we obtain $r \leq -\epsilon + r$, that is $\epsilon \leq 0$, which is contradiction with $\epsilon > 0$.

Let us evaluate next the second possibility, that is $p(x_{r_n}, x_{q_n}) \geq \epsilon + r$. Since μ is a non-decreasing function and T is weakly Kannan contractive, it follows

$$\begin{aligned}
&\mu(\epsilon + r) \leq \mu(p(x_{r_n}, x_{q_n})) \leq \mu(p(y_{r_n-1}, y_{q_n-1})) = \mu(p(Tx_{r_n-1}, Tx_{q_n-1}))\\
&\leq \mu\left(\frac{p(x_{r_n-1}, y_{r_n-1}) + p(x_{q_n-1}, y_{q_n-1}) - 2p(A,B) + p(x_{r_n-1}, x_{r_n-1}) + p(x_{q_n-1}, x_{q_n-1})}{2}\right)\\
&-\psi(p(x_{r_n-1}, y_{r_n-1}) - p(A,B), p(x_{q_n-1}, y_{q_n-1}) - p(A,B)).
\end{aligned}$$

Letting $n \to \infty$ and based on relation (3.2), we obtain $\mu(\epsilon + r) \leq \mu(r)$, and, due to the non-decreasing behavior of μ, it follows $\epsilon \leq 0$, which is contradiction with $\epsilon > 0$. Therefore, $\{x_n\}$ is a Cauchy sequence and, consequence of (X, p) being a complete partial metric space, there exists $u \in X$ such that $\lim_{n\to\infty} x_n = u$. Moreover, the sequence $\{x_n\}$, by definition, is included in A_0, therefore $u \in \overline{A_0}$.

Similar arguments as above are used in order to prove that $\{y_n\}$ is a Cauchy sequence. More precisely, the first possibility stands precisely as above. Let us evaluate next the second possibility, that is $p(y_{r_n}, y_{q_n}) \geq \epsilon + r$. Since μ is a non-decreasing function and T is weakly Kannan contractive, it follows

$$\begin{aligned}
&\mu(\epsilon + r) \leq \mu(p(y_{r_n}, y_{q_n})) = \mu(p(Tx_{r_n}, Tx_{q_n}))\\
&\leq \mu\left(\frac{p(x_{r_n}, y_{r_n}) + p(x_{q_n}, y_{q_n}) - 2p(A,B) + p(x_{r_n}, x_{r_n}) + p(x_{q_n}, x_{q_n})}{2}\right)\\
&-\psi(p(x_{r_n}, y_{r_n}) - p(A,B), p(x_{q_n}, y_{q_n}) - p(A,B)).
\end{aligned}$$

Letting $n \to \infty$, we obtain $\mu(\epsilon+r) \leq \mu(r)$, and, due to the non-decreasing behavior of μ, it follows $\epsilon \leq 0$, which is contradiction with $\epsilon > 0$. Therefore, $\{y_n\}$ is a Cauchy sequence in B_0. Since, according to Lemma 3.1, B_0 is a closed subset of a complete partial metric space, there exists $v \in B_0$ such that $\lim_{n\to\infty} y_n = v$.

In conclusion, we have obtained

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} Tx_n = v \in B_0;$$

$$\lim_{n\to\infty} x_n = u \in \overline{A_0};$$

$$p(u, v) = p(A, B) \Rightarrow u \in A_0;\ p(u, u) = p(v, v) = r.$$

STEP 3. In order to complete the proof, let us show that $v = Tu$. We start with T being a weakly Kannan-type contraction. Then

$$\begin{aligned}
&\mu(p(Tu, y_n)) = \mu(p(Tu, Tx_n))\\
&\leq \mu\left(\frac{p(u, Tu) + p(x_n, y_n) - 2p(A,B) + p(u,u) + p(x_n, x_n)}{2}\right)\\
&-\psi(p(u, Tu) - p(A,B), p(x_n, y_n) - p(A,B)).
\end{aligned}$$

Letting $n \to \infty$ and using the results obtained in the previous step, the continuity property with respect to μ, Lemma 1.2 on p, and the monotonicity of μ , we obtain

$$\begin{aligned} \mu(p(Tu,v)) &\leq \mu\left(\frac{p(u,Tu)-p(A,B)+p(u,u)+p(u,u)}{2}\right) \\ &\quad -\psi(p(u,Tu)-p(A,B),0) \\ &\leq \mu\left(\frac{p(v,Tu)+p(u,u)}{2}\right)-\psi(p(u,Tu)-p(A,B),0), \quad (3.3) \end{aligned}$$

which implies

$$p(Tu,v) \leq \frac{p(v,Tu)+p(u,u)}{2},$$

that is $p(Tu,v) \leq p(u,u) = p(v,v)$. On the other hand, $p(Tu,v) \geq p(v,v)$, therefore

$$p(Tu,v) = p(v,v). \tag{3.4}$$

Replacing in (3.3) leads to $\psi(p(u,Tu)-p(A,B),0) = 0$, or $p(u,Tu) = p(A,B)$. Finally, since (A,B) has the weak P-property, it follows $p(u,u) \leq p(Tu,Tu)$ generating the following inequalities:

$$p(v,v) = p(u,u) \leq p(Tu,Tu) \leq p(Tu,v) = p(v,v). \tag{3.5}$$

Combining (3.4) and (3.5) and based on property (p_1) it follows $v = Tu$ and $p(u,Tu) = p(A,B)$.

Step 4. Next, we shall prove the uniqueness of the best proximity point. Suppose $x, y \in A_0$ satisfy $p(x,Tx) = p(y,Ty) = p(A,B)$. The weak P-property ensures us that

$$p(x,y) \leq p(Tx,Ty),\ p(x,x) \leq p(Tx,Tx),\ p(y,y) \leq p(Ty,Ty).$$

On the other hand,

$$\begin{aligned} &\mu(p(x,y)) \leq \mu(p(Tx,Ty)) \\ &\leq \mu\left(\frac{p(x,Tx)+p(y,Ty)-2p(A,B)+p(x,x)+p(y,y)}{2}\right) \\ &-\psi(p(x,Tx)-p(A,B),p(y,Ty)-p(A,B)) \\ &= \mu\left(\frac{p(x,x)+p(y,y)}{2}\right)-\psi(0,0) \\ &\leq \mu\left(\frac{p(x,x)+p(y,y)}{2}\right). \end{aligned}$$

Since μ is non-decreasing, $p(x,y) \leq \dfrac{p(x,x)+p(y,y)}{2} \Rightarrow d_p(x,y) \leq 0$, so $x = y$. □ □

Considering the case when $A = B$, we obtain the following corollary.

Corollary 3.6. *Let (X,p) be a complete partial metric space and $T\colon X \to X$ be a mapping which satisfy the inequality*

$$\begin{aligned} \mu(p(Tx,Ty)) &\leq \mu\left(\frac{p(x,Tx)+p(y,Ty)-2p(X,X)+p(x,x)+p(y,y)}{2}\right) \\ &- \psi(p(x,Tx)-p(X,X),p(y,Ty)-p(X,X)), \end{aligned}$$

for all x, $y \in A$. Then T has a unique fixed point u in X and, for each initial point $x \in X$, the Picard sequence of iterates $\{T^n x\}_{n \in N}$ converges to u, with respect to d_p.

One can easily see that the corollary is a more general statement than that of Theorem 2.2.

3.2. **Weakly Chatterjea-type I mappings on proper subsets of partial metric spaces.**

Definition 3.7. *Let (A, B) be a pair of nonempty subsets of a partial metric space (X, p). A mapping $T\colon A \to B$ is called generalized weakly Chatterjea-type I contraction associated to pair (A, B) if*

$$\mu(p(Tx, Ty)) \leq \mu\left(\frac{p(y, Tx) + p(x, Ty) - 2p(A, B) + p(x, x) + p(y, y)}{2}\right)$$
$$-\psi(p(y, Tx) - p(A, B), p(x, Ty) - p(A, B)),$$

for all x, $y \in A$.

Theorem 3.8. *Let (A, B) be a pair of nonempty subsets of a complete partial metric space (X, p) such that $A_0 \neq \emptyset$ and let $T\colon A \to B$ be a weakly Chatterjea-type I contraction. Suppose $TA_0 \subseteq B_0$ and the pair (A, B) has the weak P-property. Then T has a unique best proximity point $u \in A_0$ and, for each point $x_0 \in A$, the iteration sequence $\{x_n\}$ defined by*

$$p(x_{n+1}, y_n) = p(A, B),\ y_n = Tx_n,\ n = 0, 1, 2, \dots$$

converges to u, with respect to d_p.

Proof. Step 1. In the sequel, we are going to use properties (i)-(iii) from the proof of Theorem 3.5. Since T is a Chatterjea-type contraction and based on the properties (i)-(iii) and (p_4), we have

$$\begin{aligned}
&\mu(p(y_n, y_{n+1})) = \mu(p(Tx_n, Tx_{n+1}))\\
\leq\ &\mu\left(\frac{p(x_{n+1}, y_n) + p(x_n, y_{n+1}) - 2p(A, B) + p(x_n, x_n) + p(x_{n+1}, x_{n+1})}{2}\right)\\
&-\psi(p(x_{n+1}, y_n) - p(A, B), p(x_n, y_{n+1}) - p(A, B))\\
\leq\ &\mu\left(\frac{p(x_n, y_{n+1}) - p(A, B) + p(y_{n-1}, y_{n-1}) + p(y_n, y_n)}{2}\right)\\
&-\psi(0, p(x_n, y_{n+1}) - p(A, B))\\
\leq\ &\mu\left(\frac{p(y_{n-1}, y_n) + p(y_n, y_{n+1})}{2}\right) - \psi(0, p(x_n, y_{n+1}) - p(A, B))\\
\leq\ &\mu\left(\frac{p(y_{n-1}, y_n) + p(y_n, y_{n+1})}{2}\right).
\end{aligned} \tag{3.6}$$

Since μ is a non-decreasing function, we find

$$p(y_n, y_{n+1}) \leq \frac{p(y_{n-1}, y_n) + p(y_n, y_{n+1})}{2}. \tag{3.7}$$

The immediate consequence consists in having the monotone decreasing sequence $\{p(y_n, y_{n+1})\}$, of non-negative real numbers, that is, a convergent sequence. Let r denote its limit. Returning to (3.6), it follows

$$\lim_{n\to\infty} p(x_n, y_{n+1}) = p(A, B). \tag{3.8}$$

Going back to relations (3.6), it follows that

$$\begin{aligned} p(y_n, y_{n+1}) &\leq \frac{p(x_n, y_{n+1}) - p(A, B) + p(y_n, y_{n-1}) + p(y_n, y_n)}{2} \\ &\leq \frac{p(x_n, y_{n+1}) - p(A, B) + 2p(y_n, y_{n-1})}{2} \end{aligned}$$

so, by taking the limit, we get $\lim_{n\to\infty} p(y_n, y_n) = r$. Applying the same method, it follows $\lim_{n\to\infty} p(x_n, x_n) = r$.

Also, using (p_2) and (ii), (iii), we obtain

$$p(x_n, x_n) \leq p(x_n, x_{n+1}) \leq p(y_{n-1}, y_n),$$

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = r. \tag{3.9}$$

Moreover, based on the properties of the partial metric, the following set of inequalities holds:

$$\begin{aligned} p(x_n, y_n) &\leq p(x_n, x_{n+1}) + p(x_{n+1}, y_n) - p(x_{n+1}, x_{n+1}); \\ p(x_n, y_{n+1}) &\leq p(x_n, y_n) + p(y_n, y_{n+1}) - p(y_n, y_n). \end{aligned}$$

Letting $n \to \infty$, leads to

$$\lim_{n\to\infty} p(x_n, y_n) = p(A, B).$$

Step 2. In the following, we shall prove that

$$\lim_{m,n\to\infty} p(x_m, x_n) = \lim_{m,n\to\infty} p(y_m, y_n) = r,$$

meaning that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences. Assume the contrary. Thus, there exists $\epsilon > 0$ such that, for all $n \in N$, we can find $r_n > q_n \geq n$ with either $p(x_{r_n}, x_{q_n}) \leq -\epsilon + r$, or $p(x_{r_n}, x_{q_n}) \geq \epsilon + r$. The first possibility may be eliminated with same arguments as in the proof of Theorem 3.5.

Suppose next that $p(x_{r_n}, x_{q_n}) \geq \epsilon + r$. Let r_n denote the smallest integer such that $r_n > q_n \geq n$ and $p(x_{r_n}, x_{q_n}) \geq \epsilon + r$. Then either 1) $r_n - 1 = q_n$, or 2) $p(x_{r_n-1}, x_{q_n}) < \epsilon + r$.

Case 1) may be approached same way as in Theorem 3.5.

Case 2) Suppose $p(x_{r_n-1}, x_{q_n}) < \epsilon + r$. Then

$$\begin{aligned} \epsilon + r &\leq p(x_{r_n}, x_{q_n}) \leq p(x_{r_n}, x_{r_n-1}) + p(x_{r_n-1}, x_{q_n}) - p(x_{r_n-1}, x_{r_n-1}) \\ &< \epsilon + r + p(x_{r_n}, x_{r_n-1}) - p(x_{r_n-1}, x_{r_n-1}). \end{aligned}$$

Letting $n \to \infty$ leads to

$$\lim_{n\to\infty} p(x_{r_n}, x_{q_n}) = \lim_{n\to\infty} p(x_{r_n-1}, x_{q_n}) = r + \epsilon. \tag{3.10}$$

Also,

$$p(x_{r_n}, x_{q_n}) \leq p(x_{r_n}, x_{q_n-1}) + p(x_{q_n}, x_{q_n-1}) - p(x_{q_n-1}, x_{q_n-1}),$$

$$p(x_{r_n}, x_{q_n-1}) \leq p(x_{r_n}, x_{q_n}) + p(x_{q_n}, x_{q_n-1}) - p(x_{q_n}, x_{q_n}).$$

Letting $n \to \infty$ leads to

$$\lim_{n\to\infty} p(x_{r_n}, x_{q_n-1}) = r + \epsilon. \tag{3.11}$$

Moreover,

$$\begin{aligned}
p(x_{r_n-1}, y_{q_n-1}) &\leq p(x_{r_n-1}, x_{q_n}) + p(x_{q_n}, y_{q_n-1}) - p(x_{q_n}, x_{q_n}) \\
&= p(x_{r_n-1}, x_{q_n}) + p(A, B) - p(x_{q_n}, x_{q_n}) \\
p(x_{q_n-1}, y_{r_n-1}) &\leq p(x_{q_n-1}, x_{r_n}) + p(x_{r_n}, y_{r_n-1}) - p(x_{r_n}, x_{r_n}) \\
&= p(x_{q_n-1}, x_{r_n}) + p(A, B) - p(x_{r_n}, x_{r_n}).
\end{aligned}$$

Since μ is a non-decreasing function and T is weakly Chatterjea contractive, it follows

$$\begin{aligned}
&\mu(\epsilon + r) \leq \mu(p(x_{r_n}, x_{q_n})) \leq \mu(p(y_{r_n-1}, y_{q_n-1})) = \mu(p(Tx_{r_n-1}, Tx_{q_n-1})) \\
\leq\ & \mu\left(\frac{p(x_{q_n-1}, y_{r_n-1}) + p(x_{r_n-1}, y_{q_n-1}) - 2p(A, B) + p(x_{r_n-1}, x_{r_n-1}) + p(x_{q_n-1}, x_{q_n-1})}{2}\right) \\
&-\psi(p(x_{q_n-1}, y_{r_n-1}) - p(A, B), p(x_{r_n-1}, y_{q_n-1}) - p(A, B)) \\
\leq\ & \mu\left(\frac{p(x_{q_n-1}, x_{r_n}) + p(x_{r_n-1}, x_{q_n}) - p(x_{r_n}, x_{r_n}) - p(x_{q_n}, x_{q_n})}{2}\right. \\
&\left.+\frac{p(x_{r_n-1}, x_{r_n-1}) + p(x_{q_n-1}, x_{q_n-1})}{2}\right) \\
&-\psi(p(x_{q_n-1}, y_{r_n-1}) - p(A, B), p(x_{r_n-1}, y_{q_n-1}) - p(A, B)). \qquad (3.12)
\end{aligned}$$

Letting $n \to \infty$, we obtain, based on the properties of ψ,

$$\lim_{n\to\infty} p(x_{r_n-1}, y_{q_n-1}) = \lim_{n\to\infty} p(x_{q_n-1}, y_{r_n-1}) = p(A, B).$$

Substituting these in (3.12), leads to $\mu(\epsilon+r) \leq \mu(r)$, and, due to the non-decreasing behavior of μ, it follows $\epsilon \leq 0$, which is contradiction with $\epsilon > 0$. Therefore, x_n is a Cauchy sequence and, consequence of (X, p) being a complete partial metric space, there exists $u \in X$ such that $\lim\limits_{n\to\infty} x_n = u$. Moreover, the sequence $\{x_n\}$, by definition, is included in A_0, therefore $u \in \overline{A_0}$.

Similar arguments as above are used in order to prove that $\{y_n\}$ is a Cauchy sequence. Same as for $\{x_n\}$ the most complex assumption is to consider $p(y_{r_n}, y_{q_n}) \geq \epsilon + r$ and $p(y_{r_n-1}, y_{q_n}) < \epsilon + r$. Then, since relations (3.7), (3.8) and (3.9) emphasize a similar behavior for $\{x_n\}$ and $\{y_n\}$, it immediate follows that analogue relations to (3.10) and (3.11) stay valid for sequence $\{y_n\}$. More precisely, $\lim\limits_{n\to\infty} p(y_{r_n}, y_{q_n}) = \lim\limits_{n\to\infty} p(y_{r_n-1}, y_{q_n}) = \lim\limits_{n\to\infty} p(y_{r_n}, y_{q_n-1}) = r + \epsilon$.

Moreover,

$$\begin{aligned}
p(x_{r_n}, y_{q_n}) &\leq p(x_{r_n}, y_{r_n}) + p(y_{r_n}, y_{q_n}) - p(y_{r_n}, y_{r_n}) \\
p(x_{q_n}, y_{r_n}) &\leq p(x_{q_n}, y_{q_n}) + p(y_{q_n}, y_{r_n}) - p(y_{q_n}, y_{q_n}). \qquad (3.13)
\end{aligned}$$

Since T is weakly Chatterjea contractive, it follows, based on relation (3.13):

$$\begin{aligned}
&\mu(\epsilon+r)\leq \mu(p(y_{r_n},y_{q_n}))=\mu(p(Tx_{r_n},Tx_{q_n}))\\
\leq\ &\mu\left(\frac{p(x_{q_n},y_{r_n})+p(x_{r_n},y_{q_n})-2p(A,B)+p(x_{r_n},x_{r_n})+p(x_{q_n},x_{q_n})}{2}\right)\\
&-\psi(p(x_{q_n},y_{r_n})-p(A,B),p(x_{r_n},y_{q_n})-p(A,B))\\
\leq\ &\mu\left(\frac{p(x_{q_n},y_{q_n})+p(x_{r_n},y_{r_n})+2p(y_{q_n},y_{r_n})-2p(A,B)}{2}\right.\\
&+\left.\frac{-p(y_{q_n},y_{q_n})-p(y_{r_n},y_{r_n})+p(x_{r_n},x_{r_n})+p(x_{q_n},x_{q_n})}{2}\right)\\
&-\psi(p(x_{q_n},y_{r_n})-p(A,B),p(x_{r_n},y_{q_n})-p(A,B)). \qquad (3.14)
\end{aligned}$$

Letting $n\to\infty$, we obtain

$$\lim_{n\to\infty}p(x_{r_n},y_{q_n})=\lim_{n\to\infty}p(x_{q_n},y_{r_n})=p(A,B).$$

Substituting these in (3.14), leads to $\mu(\epsilon+r)\leq\mu(r)$, and, due to the non-decreasing behavior of μ, it follows $\epsilon\leq 0$, which is contradiction with $\epsilon>0$. Therefore, y_n is also a Cauchy sequence in B_0 and, since B_0 is a closed subset of a complete partial metric space, there exists $v\in B_0$ such that $\lim\limits_{n\to\infty}y_n=v$. In conclusion, we have obtained

$$\lim_{n\to\infty}y_n=\lim_{n\to\infty}Tx_n=v\in B_0;$$
$$\lim_{n\to\infty}x_n=u;$$
$$p(u,v)=p(A,B)\Rightarrow u\in A_0;\ p(u,u)=p(v,v)=r.$$

Step 3. In order to complete the proof, we need to show that $v=Tu$. Let us start again with T being a weakly Chatterjea-type contraction. Then

$$\begin{aligned}
&\mu(p(Tu,y_n))=\mu(p(Tu,Tx_n))\\
&\leq\mu\left(\frac{p(x_n,Tu)+p(u,y_n)-2p(A,B)+p(u,u)+p(x_n,x_n)}{2}\right)\\
&-\psi(p(x_n,Tu)-p(A,B),p(x,y_n)-p(A,B)).
\end{aligned}$$

Letting $n\to\infty$, we obtain

$$\begin{aligned}
\mu(p(Tu,v))\quad&\leq\mu\left(\frac{p(u,Tu)-p(A,B)+p(u,u)+p(u,u)}{2}\right)\\
&-\psi(p(u,Tu)-p(A,B),0),
\end{aligned}$$

leading to $v=Tu$.

Step 4. Next, we shall prove the uniqueness of the best proximity point. Suppose $x,y\in A_0$ satisfy $p(x,Tx)=p(y,Ty)=p(A,B)$. The weak P-property ensures us that

$$p(x,y)\leq p(Tx,Ty),\ p(x,x)\leq p(Tx,Tx),\ p(y,y)\leq p(Ty,Ty).$$

On the other hand,

$$\begin{aligned}
\mu(p(x,y))\quad&\leq\quad\mu(p(Tx,Ty))\\
&\leq\quad\mu\left(\frac{p(y,Tx)+p(x,Ty)-2p(A,B)+p(x,x)+p(y,y)}{2}\right)\\
&\quad\ -\psi(p(y,Tx)-p(A,B),p(x,Ty)-p(A,B)).
\end{aligned}$$

Moreover, $p(x,Ty) \leq p(x,y)+p(y,Ty)-p(y,y)$ and $p(y,Tx) \leq p(x,y)+p(x,Tx)-p(x,x)$, which leads to

$$\psi(p(y,Tx)-p(A,B), p(x,Ty)-p(A,B)) = 0,$$

so

$$p(x,Ty) = p(y,Tx) = p(A,B).$$

Using the monotonicity of μ we obtain $p(x,y) \leq \dfrac{p(x,x)+p(y,y)}{2}$, hence $d_p(x,y) \leq 0$, or $x = y$. □ □

Taking now the case when $A = B$, we obtain the following corollary.

Corollary 3.9. *Let (X,p) be a complete partial metric space and $T\colon X \to X$ a mapping which fulfills*

$$\begin{aligned}\mu(p(Tx,Ty)) \quad &\leq \mu\left(\frac{p(y,Tx)+p(x,Ty)-2p(X,X)+p(x,x)+p(y,y)}{2}\right)\\ &-\psi(p(y,Tx)-p(X,X), p(x,Ty)-p(X,X)).\end{aligned}$$

Then T has a unique fixed point u in X and, for each $x \in X$, the Picard sequence of iterates $\{T^n x\}_{n\in\mathbb{N}}$ converges to u, with respect to d_p.

The statement is more general than the one we proved in Theorem 2.4.

We provide now an example to prove the usability of our results.

Example 3.10. Consider $X = \mathbb{R}^+$ and the partial metric $p_{\max}\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$, $p_{\max} = \max\{x,y\}$. Let $A = [2,\infty)$ and $B = [1,\infty)$, and $T\colon A \to B$, $Tx = \frac{x+1}{2}$. Then (A,B) has the weak P-property and $p(A,B) = 2$.

We shall prove next that T is both a weakly Kannan-type I and a weakly Chatterjea-type I contraction, provided that $\mu\colon [0,\infty) \to [0,\infty)$, $\mu(x) = x$ and $\psi\colon [0,\infty)^2 \to [0,\infty)$, $\psi(x,y) = \dfrac{x+y}{2}$. Indeed, we have

$$\mu(p(Tx,Ty)) = \frac{\max\{x,y\}+1}{2}$$

and, by denoting

$$\begin{aligned}\alpha \quad &= \mu\left(\frac{p(x,Tx)+p(y,Ty)-2p(A,B)+p(x,x)+p(y,y)}{2}\right)\\ &-\psi(p(x,Tx)-p(A,B), p(y,Ty)-p(A,B))\end{aligned}$$

and

$$\begin{aligned}\beta \quad &= \mu\left(\frac{p(y,Tx)+p(x,Ty)-2p(A,B)+p(x,x)+p(y,y)}{2}\right)\\ &-\psi(p(y,Tx)-p(A,B), p(x,Ty)-p(A,B))\end{aligned},$$

we find

$$\alpha = \beta = \frac{x+y}{2}.$$

Finally, $\mu(p(Tx,Ty)) \leq \alpha$, which is $\max\{x,y\}+1 \leq x+y$, that is $1 \leq \min\{x,y\}$, which is a true assertion since $x,y \in [1,\infty)$. Therefore, T is both a weakly Kannan-type I and a weakly Chatterjea-type I contraction and, by applying Theorem 3.5 or Theorem 3.8, we conclude that there exists a unique best proximity point, namely $u = 1$. □

3.3. **Weakly Kannan-type II and Chatterjea-type II mappings on proper subsets with inverse weak P-property.**

Definition 3.11. *Let (A, B) be a pair of nonempty subsets of a partial metric space (X, p), such that $A_0 \neq \emptyset$. The pair (A, B) is said to have the inverse weak P-property if and only if, for any $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$,*

$$\begin{cases} p(x_1, y_1) = p(A, B) \\ p(x_2, y_2) = p(A, B) \end{cases} \Rightarrow p(x_1, x_2) \geq p(y_1, y_2).$$

Definition 3.12. *Let (A, B) be a pair of nonempty subsets of a partial metric space (X, p). A mapping $T\colon A \to B$ is called generalized weakly Kannan-type II contraction associated to pair (A, B) if*

$$\mu(p(Tx, Ty)) \leq \mu\left(\frac{p(x, Tx)+p(y, Ty)-2p(A, B)+p(Tx, Tx)+p(Ty, Ty)}{2}\right)$$
$$-\psi(p(x, Tx) - p(A, B), p(y, Ty) - p(A, B)),$$

for all x, $y \in A$.

Definition 3.13. *Let (A, B) be a pair of nonempty subsets of a partial metric space (X, p). A mapping $T\colon A \to B$ is called generalized weakly Chatterjea-type II contraction associated to pair (A, B) if*

$$\mu(p(Tx, Ty)) \leq \mu\left(\frac{p(y, Tx)+p(x, Ty)-2p(A, B)+p(Tx, Tx)+p(Ty, Ty)}{2}\right)$$
$$-\psi(p(y, Tx) - p(A, B), p(x, Ty) - p(A, B)),$$

for all x, $y \in A$.

Using similar arguments as in the proof of Theorem 3.5 and Theorem 3.8, with the differences arising from considering inverse P-property and type II contractions, the following outcomes may be proved.

Theorem 3.14. *Let (A, B) be a pair of nonempty subsets of a complete partial metric space (X, p) such that $A_0 \neq \emptyset$ and let $T\colon A \to B$ be a weakly Kannan-type II contraction. Suppose $T(A_0) \subseteq B_0$ and the pair (A, B) has the inverse weak P-property. Then T has a unique best proximity point $u \in A_0$ and, for each point $x_0 \in A$, the iteration sequence $\{x_n\}$ defined by*

$$p(x_{n+1}, y_n) = p(A, B),\ y_n = Tx_n,\ n = 0, 1, 2, \ldots$$

converges to u, with respect to d_p.

Theorem 3.15. *Let (A, B) be a pair of nonempty subsets of a complete partial metric space (X, p) such that $A_0 \neq \emptyset$ and let $T\colon A \to B$ be a weakly Chatterjea-type II contraction. Suppose $T(A_0) \subseteq B_0$ and the pair (A, B) has the inverse weak P-property. Then T has a unique best proximity point $u \in A_0$ and, for each point $x_0 \in A$, the iteration sequence $\{x_n\}$ defined by*

$$p(x_{n+1}, y_n) = p(A, B),\ y_n = Tx_n,\ n = 0, 1, 2, \ldots$$

converges to u, with respect to d_p.

Example 3.16. Consider again the partial metric $p_{\max}\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$, $p_{\max} = \max\{x, y\}$. Let $A = [2, \infty)$ and $B = [1, \infty)$, and $T\colon A \to B$, $Tx = \dfrac{x}{2}$. Then $p(A, B) = 2$ and the pair (A, B) has the inverse weak P-property.

We shall prove next that T is both a weakly Kannan-type II and a weakly Chatterjea-type II contraction, provided that $\mu\colon [0,\infty) \to [0,\infty)$, $\mu(x) = x$ and $\psi\colon [0,\infty)^2 \to [0,\infty)$, $\psi(x,y) = \frac{x+y}{4}$. For start, let us remark that ψ satisfies property (2.1) and $\mu(p(Tx,Ty)) = \dfrac{\max\{x,y\}}{2}$. On the other side, by computing

$$\begin{aligned}\alpha \quad &= \mu\left(\frac{p(x,Tx)+p(y,Ty)-2p(A,B)+p(Tx,Tx)+p(Ty,Ty)}{2}\right)\\ &-\psi(p(x,Tx)-p(A,B),p(y,Ty)-p(A,B))\end{aligned}$$

and

$$\begin{aligned}\beta \quad &= \mu\left(\frac{p(y,Tx)+p(x,Ty)-2p(A,B)+p(Tx,Tx)+p(Ty,Ty)}{2}\right)\\ &-\psi(p(y,Tx)-p(A,B),p(x,Ty)-p(A,B)),\end{aligned}$$

we find

$$\alpha = \frac{x+y-2}{2}$$

and

$$\beta = \begin{cases} \dfrac{3\max\{x,y\}}{8} + \dfrac{x+y}{4} - 1, & \text{if } \min\{x,y\} \le \dfrac{\max\{x,y\}}{2} \\ \dfrac{x+y-2}{2}, & \text{otherwise.} \end{cases}$$

Finally, let us prove that $\mu(p(Tx,Ty)) \le \alpha, \beta,\ \forall x,y \in [2,\infty)$.

Case I. $\dfrac{\max\{x,y\}}{2} \le \dfrac{x+y-2}{2}$, that is $\max\{x,y\} \le x+y-2$, which is $2 \le \min\{x,y\}$, a true assertion since $x,y \in [2,\infty)$.

Case II. Suppose $\min\{x,y\} \le \dfrac{\max\{x,y\}}{2}$. It follows $x+y \ge 3\min\{x,y\}$. Then

$$\frac{\max\{x,y\}}{2} \le \frac{3\max\{x,y\}}{8} + \frac{x+y}{4} - 1,$$

that is $8 \le x+y+\min\{x,y\}$, which holds true since $x+y+\min\{x,y\} \ge 4\min\{x,y\} \ge 8,\ \forall x,y \in [2,\infty)$. Therefore, T is both a weakly Kannan and a weakly Chatterjea-type II contraction, and since all the hypotheses in Theorem 3.14 and Theorem 3.15 are satisfied, it follows that there exists a unique best proximity point, namely $u = 2$.

Remark. To extend the results in the theorem of Zhang and Su, the definition of the weakly Kannan contractions, may be rephrased in the following weaker form:

$$p(Tx,Ty) \le \frac{\alpha(x,y)}{2}\left[p(x,Tx)+p(y,Ty)-2p(A,B)+p(Tx,Tx)+p(Ty,Ty)\right],$$

without affecting the proof of the corresponding main result.

4. Conclusion

In this paper, weak contractive conditions of the type Kannan and Chaterjea have been used in order to state and prove some fixed point theorems. The weak P-property has been an important tool to prove some best proximity point theorems in the setting of the partial metric space. Examples have been provided, in order to sustain the results obtained here.

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