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# A NOTE ON $\mathbb{A}$-BILINEAR MAPS

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Abstract. In this note we study various $\mathbb{A}$-bilinear maps and we prove among others that under certain conditions, orthosymmetric $\mathbb{A}$-bilinear maps are either symmetric or skew-symmetric.

## 0. Introduction

We consider $\mathbb{A}$-bilinear maps on an $\mathbb{A}$-module $E$, where $\mathbb{A}$ is a unital commutative algebra over a field $\mathbb{K}$ (of characteristic different from 2) without zero-divisors.

In Section 2, we prove that each $\mathbb{A}$-bilinear form $\varphi : E \times E \to \mathbb{A}$ is uniquely decomposed as a sum of a symmetric and a skew-symmetric $\mathbb{A}$-bilinear form (Proposition 2.1). In Section 3, we characterize the non-degenerate $\mathbb{A}$-bilinear forms (Lemma 3.1, Corollary 3.1) and in the case of a free $\mathbb{A}$-module of finite rank the $\mathbb{A}$-polars and $\mathbb{A}$-bipolars of $\mathbb{A}$-submodules and prove some properties of them analogous to the classical ones (Theorem 3.1). In the final Section 4, using the preceding results we show under certain conditions that an orthosymmetric $\mathbb{A}$-bilinear form $\varphi : E \times E \to \mathbb{A}$ is either symmetric or skew-symmetric (Theorem 4.1). As a consequence of this result, we obtain that the set of all orthosymmetric $\mathbb{A}$-bilinear forms is not an $\mathbb{A}$-submodule of the $\mathbb{A}$-module of all $\mathbb{A}$-bilinear forms (Corollary 4.1). In Section 5, we give some applications of Theorem 4.1 which refer to matrix representations, over the algebra $\mathbb{A}$, of various orthosymmetric $\mathbb{A}$-forms.

## 1. A Note on $\mathbb{A}$-bilinear maps

**1. Preliminaries**. In the sequel $\mathbb{A}$ denotes a unital commutative algebra without divisors of zero, over a field $\mathbb{K}$ having characteristic different from 2.

The unit of $\mathbb{A}$ will be denoted by 1.

Let $E$ be an $\mathbb{A}$-module. (See [3]).

For an $\mathbb{A}$-bilinear map $\varphi : E \times E \to \mathbb{A}$, in the sequel we shall use the term $\mathbb{A}$-*form*.

We consider the following sets:

$$\mathcal{B}(E) := \{\varphi : E \times E \to \mathbb{A}, \quad \mathbb{A}\text{-bilinear}\}. \tag{1.1}$$

$$\mathcal{B}_s(E) := \{\varphi \in \mathcal{B}(E) : \ \varphi \text{ symmetric; namely } \varphi(x, y) = \varphi(y, x), \ x, y \in E\} \tag{1.2}$$

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$$\mathcal{B}_a(E) := \{\varphi \in \mathcal{B}(E) : \ \varphi \text{ skew-symmetric; namely } \varphi(x,y) = -\varphi(y,x), \ x,y \in E\} \tag{1.3}$$

$$\begin{aligned}\mathcal{B}_0(E) := \{&\varphi \in \mathcal{B}(E) : \ \varphi \text{ orthosymmetric; namely} \\ &\varphi(x,y) = 0 \Leftrightarrow \varphi(y,x) = 0, \ x,y \in E\}\end{aligned} \tag{1.4}$$

We define in $\mathcal{B}(E)$ the pointwise operations:

$$\begin{aligned}&\text{(i) } (\varphi + \sigma)(x,y) := \varphi(x,y) + \sigma(x,y) \ \text{ and} \\ &\text{(ii) } (\lambda\varphi)(x,y) := \lambda\varphi(x,y) \ \ \forall\, x,y \in E, \ \ \varphi,\sigma \in \mathcal{B}(E), \ \ \lambda \in \mathbb{A}.\end{aligned} \tag{1.5}$$

Evidently $\mathcal{B}(E)$ under the above two operations becomes an $\mathbb{A}$-module.

Yet, if $\varphi, \sigma$ are symmetric, resp. skew-symmetric, the same is true for $\varphi + \sigma$, $\lambda\varphi$, $\lambda \in \mathbb{A}$.

Thus $\mathcal{B}_s(E), \mathcal{B}_a(E)$ are $\mathbb{A}$-submodules of $\mathcal{B}(E)$. But $\mathcal{B}_s(E) \subseteq \mathcal{B}_0(E)$ and $\mathcal{B}_a(E) \subseteq \mathcal{B}_0(E)$ are only subsets of $\mathcal{B}(E)$. (See Chapter V in [1]).

## 2. Decomposition of the $\mathbb{A}$-module $\mathcal{B}(E)$

**Proposition 2.1.** *Let $(E, \varphi)$ be a pair consisting of an $\mathbb{A}$-module $E$ and an $\mathbb{A}$-form $\varphi$ on $E$ (i.e., $\varphi \in \mathcal{B}(E)$). Then:*

$$\mathcal{B}(E) = \mathcal{B}_s(E) \oplus \mathcal{B}_a(E).$$

*(See § (5.3) in* [1]*).*

*Proof.* (i) One can easily prove that:

$$\mathcal{B}_s(E) + \mathcal{B}_a(E) \subseteq \mathcal{B}(E).$$

(ii) Now for $\varphi \in \mathcal{B}(E)$ we put:

$$\begin{aligned}\varphi_1(x,y) &:= \frac{1}{2}(\varphi(x,y) + \varphi(y,x)), \quad x,y \in E, \\ \varphi_2(x,y) &:= \frac{1}{2}(\varphi(x,y) - \varphi(y,x)), \quad x,y \in E.\end{aligned}$$

Then easily we have that $\varphi_1 \in \mathcal{B}_s(E)$ and $\varphi_2 \in \mathcal{B}_a(E)$, with $\varphi = \varphi_1 + \varphi_2$.

Thus $\mathcal{B}(E) \subseteq \mathcal{B}_s(E) + \mathcal{B}_a(E)$, consequently $\mathcal{B}(E) = \mathcal{B}_s(E) + \mathcal{B}_a(E)$.

(iii) Let $\varphi \in \mathcal{B}_s(E) \cap \mathcal{B}_a(E)$.

Then $\varphi(x,y) = \varphi(y,x)$ and $\varphi(x,y) = -\varphi(y,x)$, $x,y \in E$.

Therefore $\varphi(x,y) = 0$, $\forall\, x,y \in E$ and so $\varphi = 0$.

This completes the proof. □

## 3. The dual $\mathbb{A}$-module $E^*$

Let $E$ be an $\mathbb{A}$-module. We define

$$E^* := Hom_{\mathbb{A}}(E, \mathbb{A}) \ \ \text{(see [3])}. \tag{3.1}$$

Then $E^*$ is an $\mathbb{A}$-module. Let now $E$ be a free $\mathbb{A}$-module of finite rank $n \in \mathbb{N}$.

We call dimension of $E$ and we denote by $\dim_{\mathbb{A}} E$ the rank of $E$. In this case, $E^*$ is the free $\mathbb{A}$-module of all $\mathbb{A}$-linear forms on $E$, of finite rank $n \in \mathbb{N}$, so that $E^* \cong \mathbb{A}^n$.

Let $B = \{x_1, x_2, \ldots, x_n\}$ be a base of $E$ and $\mathcal{B}^* = \{\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n\}$ the dual base of $\mathcal{B}$ in $E^*$, such that:

$$\overline{x}_i(x_j) = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \quad i, j = 1, 2, \ldots, n. \tag{3.2}$$

Then $E \cong \mathbb{A}^n \cong E^*$. (See § 4.6 in [1]).

Let now $E$ be an arbitrary $\mathbb{A}$-module and $\varphi \in \mathcal{B}(E)$. For a fixed $x \in E$, one may define the map:

$$\varphi_x : E \to \mathbb{A} : \varphi_x(y) := \varphi(x, y), \quad y \in E. \tag{3.3}$$

and similarly for a fixed $y \in E$, one defines

$${}_y\varphi : E \to \mathbb{A} : {}_y\varphi(x) := \varphi(x, y), \quad x \in E \tag{3.4}$$

Clearly these maps $\varphi_x$, ${}_y\varphi$ are in $E^*$.

Also we may define the maps:

$$\widehat{\varphi} : E \to E^* : \widehat{\varphi}(x) := \varphi_x \in E^* \quad \forall\, x \in E \tag{3.5}$$

such that:

$$\widehat{\varphi}(x)(y) = \varphi_x(y) = \varphi(x, y) \quad \forall\, y \in E \quad \text{and}$$

$$\overset{\vee}{\varphi} : E \to E^* : \overset{\vee}{\varphi}(y) := {}_y\varphi \in E^*, \quad \forall\, y \in E \tag{3.6}$$

such that:

$$\overset{\vee}{\varphi}(y)(x) = {}_y\varphi(x) = \varphi(x, y) \quad \forall\, x \in E.$$

These maps $\widehat{\varphi}$ and and $\overset{\vee}{\varphi}$ are $\mathbb{A}$-linear maps corresponding to the $\mathbb{A}$-form $\varphi$.

Let $(E, \varphi)$ be a pair consisting of an $\mathbb{A}$-module $E$ and an $\mathbb{A}$-form $\varphi : E \times E \to \mathbb{A}$. For any subset $\emptyset \neq M \subseteq E$ we consider the following $\mathbb{A}$-submodules, with respect to $\varphi$, defined as follows:

$$\begin{aligned} M^{\perp\varphi} \equiv M^{\perp} :=& \{x \in E : \varphi(x, y) = \varphi_x(y) \equiv \widehat{\varphi}(x)(y) = 0, \quad \forall\, y \in M\} \\ &= \{x \in E : x \perp M\} = \{x \in E : M \subseteq Ker\varphi_x\} \quad \text{and} \end{aligned} \tag{3.7}$$

$$\begin{aligned} M^{\top\varphi} \equiv M^{\top} :=& \{y \in E : \varphi(x, y) = {}_y\varphi(x) \equiv \overset{\vee}{\varphi}(y)(x) = 0, \quad \forall\, x \in M\} \\ &= \{y \in E : y \perp M\} = \{y \in E : M \subseteq Ker\, {}_y\varphi\} \end{aligned} \tag{3.8}$$

(see § 5.2 in [1]).

**Remark 3.1.** *If the $\mathbb{A}$-form $\varphi$, is orthosymmetric then, clearly $M^{\perp} = M^{\top}$.*

**Definition 3.1.** *Let $(E, \varphi)$ be a pair consisting of an $\mathbb{A}$-module $E$ and an $\mathbb{A}$-form $\varphi : E \times E \to \mathbb{A}$. Then $\varphi$ is called* **non-degenerate** *whenever the following condition holds true:*

*If $x \in E$ such that $\varphi(x, y) = 0$, for every $y \in E$, then $x = 0$, as well.*

*Equivalently one obtains:*

*For any $x \neq 0$ in $E$, there exists $y \in E$ (necessarily nonzero) such that $\varphi(x, y) \neq 0$.*

**Lemma 3.1.** *Let $(E, \varphi)$ be a pair consisting of an $\mathbb{A}$-module $E$ and an $\mathbb{A}$-form $\varphi$ on $E$. Then $\varphi$ is non-degenerate if and only if $E^{\perp} = \{0\}$.*

*Proof.* Let $\varphi$ be a non degenerate $\mathbb{A}$-form of $E$. Then for every $a \in E^{\perp} \subseteq E$, with $a \neq 0$, there exists $x \in E$, with $x \neq 0$ such that $\varphi(a, x) \neq 0$, But $a \in E^{\perp}$ implies $\varphi(a, y) = 0$, for all $y \in E$. Thus $\varphi(a, x) = 0$, therefore $x = 0$, a contradiction.

Let now $E^{\perp} = \{0\}$ and $\varphi(x, y) = 0$, for all $y \in E$. But then $x = 0$, therefore $\varphi$ is non-degenerate. □

**Corollary 3.1.** *Let $(E, \varphi)$ be a pair consisting of an $\mathbb{A}$-module $E$ and an $\mathbb{A}$-form $\varphi$ on $E$. Then we have:*

*(i) $E^{\perp} = Ker\widehat{\varphi}$*

*(ii) The $\mathbb{A}$-form $\varphi$, is non degenerate if and only if $\widehat{\varphi}$ is injective.*

*Proof.* (i) Let $x \in E^{\perp}$ then $\varphi(x, y) = 0$, for all $y \in E$; namely $\varphi(x, y) = \varphi_x(y) = \widehat{\varphi}(x)(y) = 0$, $\forall\, y \in E$, therefore $\widehat{\varphi}(x) = 0$, i.e., $x \in Ker\widehat{\varphi}$. Hence $E^{\perp} \subseteq Ker\widehat{\varphi}$.

Let $x \in Ker\widehat{\varphi}$. Then $\widehat{\varphi}(x)(y) = \varphi_x(y) = \varphi(x, y) = 0$, for all $y \in E$, therefore $x \in E^{\perp}$ (see (3.7)).

This completes the proof. □

(ii) Let $\widehat{\varphi}$ be injective. Then $Ker\widehat{\varphi} = E^{\perp} = \{0\}$ so that from Lemma 3.1 $\varphi$ is non-degenerate.

**Conversely:** Let $\varphi$ be non-degenerate and $x \in Ker\widehat{\varphi}$ such that $\widehat{\varphi}(x) = 0$. Then $\forall\, y \in E$ $\widehat{\varphi}(x)(y) = \varphi_x(y) = \varphi(x, y) = 0$, so that $x \in E^{\perp} = \{0\}$ (see Lemma 3.1). Therefore $x = 0$ and $\widehat{\varphi}$ is $1 - 1$.

**Definition 3.2.** *Let $(E, \varphi)$ be a pair consisting of an $\mathbb{A}$-module $E$ and an $\mathbb{A}$-form $\varphi$ on $E$. Let $\emptyset \neq M \subseteq E$. We define the set:*

$$M^0 := \{f \in E^* : f(x) = 0, \quad \forall\, x \in M\} \subseteq E^*.$$

*We call $M^0$ an* **$\mathbb{A}$-polar** *set of $M$ in $E^*$. It is easy to see that $M^0$ is an* **$\mathbb{A}$-submodule** *of $E^*$.*

**Definition 3.3.** *Let $(E, \varphi)$ be a pair consisting of an $\mathbb{A}$-module $E$ and an $\mathbb{A}$-form $\varphi$ on $E$. Let $\emptyset \neq N \subseteq E^*$. We define the set:*

$$N^0 := \{x \in E : f(x) = 0 \;\; \forall\, f \in N\} \subseteq E.$$

*We call $N^0$ an* **$\mathbb{A}$-polar** *set of $N$ in $E$. It is easy to see that $N^0$ is an* **$\mathbb{A}$-submodule** *of $E$.*

**Remark 3.2.** *The sets $M^{00} = (M^0)^0 \subseteq E$ and $N^{00} = (N^0)^0 \subseteq E^*$ are called* **$\mathbb{A}$-bipolars** *of $M$, $N$ respectively in the $\mathbb{A}$-modules $E$, $E^*$.*

*We also consider the sets:*

$$\begin{aligned} &(i)\quad (\widehat{\varphi}(M))^0 := \{x \in E : \widehat{\varphi}(x)(y) \equiv \varphi_x(y) = \varphi(x,y) = 0, \quad \forall\, y \in M\} \\ &(ii)\quad (\overset{\vee}{\varphi}(M))^0 := \{y \in E : \overset{\vee}{\varphi}(y)(x) \equiv {}_y\varphi(x) = \varphi(x,y) = 0, \quad \forall\, x \in M\}, \end{aligned} \tag{3.9}$$

*where $\varphi$ is an $\mathbb{A}$-form and $\emptyset \neq M \subseteq E$.*

**Proposition 3.1.** *Let $(E, \varphi)$ be a pair consisting of an $\mathbb{A}$-module $E$ and an $\mathbb{A}$-form $\varphi$ on $E$. Let also $\emptyset \neq M \subseteq E$. Then we have:*

$$M^{\perp} = (\widehat{\varphi}(M))^0.$$

*Proof.*

$$\begin{aligned} M^{\perp} :&= \{x \in E : \varphi(x,y) = 0, \quad \forall\, y \in M\} \\ &= \{x \in E : \varphi_x(y) = 0, \quad \forall\, y \in M\} \\ &= \{x \in E : \widehat{\varphi}(x)(y) = 0, \quad \forall\, y \in M\} = (\widehat{\varphi}(M))^0. \end{aligned}$$

In the same manner we have that

$$M^{\top} = (\overset{\vee}{\varphi}(M))^0.$$

□

**Remark 3.3.** *If $\varphi$ is orthosymmetric then*

$$(\widehat{\varphi}(M))^0 = (\overset{\vee}{\varphi}(M))^0.$$

**Proposition 3.2.** *Let $(E, \varphi)$ be a pair consisting of an $\mathbb{A}$-module $E$ and an $\mathbb{A}$-form $\varphi$ on $E$. Let $\emptyset \neq M \subseteq E$. Then we have:*
*(i) $M \subseteq (M^{\perp})^{\top}$ and (ii) $M \subseteq (M^{\top})^{\perp}$.*

*Proof.* By (3.7), for $y \in M$, we have $\varphi(x,y) = 0$, $\forall\, x \in M^{\perp}$. So $y \in (M^{\perp})^{\top}$. Therefore $M \subseteq (M^{\perp})^{\top}$.

In the same manner $M \subseteq (M^{\top})^{\perp}$. □

**Proposition 3.3.** *Let $(E, \varphi)$ be a pair consisting of an $\mathbb{A}$-module $E$ and an $\mathbb{A}$-form $\varphi$ on $E$. Then $E = (E^{\perp})^{\top}$ and $E = (E^{\top})^{\perp}$.*

*Proof.* From Proposition 3.2, $E \subseteq (E^{\perp})^{\top}$. But also $(E^{\perp})^{\top} \subseteq E$, so that $E = (E^{\perp})^{\top}$.

In the same manner $E = (E^{\top})^{\perp}$. □

**Corollary 3.2.** *Under the supositions of Proposition 3.3, if $\varphi$ is $\mathbb{A}$-orthosymmetric, then we have $E^{\perp\perp} = E$ and $E^{\top\top} = E$.*

**Theorem 3.1.** *Let $E$ be a free $\mathbb{A}$-module of finite rank $n \in \mathbb{N}$ (i.e. $\dim_{\mathbb{A}} E = n$).*

*Also let $M, N$ be $\mathbb{A}$-submodules of $E$, then we have:*

(i) $\dim_{\mathbb{A}} E = \dim_{\mathbb{A}} M + \dim_{\mathbb{A}} M^0$

(ii) $M^{00} = M$

(iii) $M \subseteq N \Leftrightarrow N^0 \subseteq M^0$

(iv) $(M+N)^0 = M^0 \cap N^0$

(v) $(M \cap N)^0 = M^0 + N^0$ *(see § 4.6 in* [1]*).*

*Proof.* The proof of this theorem goes exactly as in the classical case. For completeness' sake we include it here too.

(i) $E \cong \mathbb{A}^n$ and let $\mathcal{B} = \{x_1, x_2, \ldots, x_n\}$ be a base of $E$. We consider the $\mathbb{A}$-submodule $M$ of $E$ with $M = \langle x_1, x_2, \ldots, x_k \rangle$, $k \leq n$, where

$$\mathcal{B}_M = \{x_1, x_2, \ldots, x_k\} \text{ is a base of } M \text{ (i.e., } M \cong \mathbb{A}^k\text{).}$$
$$\mathcal{B}^* = \{\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n\} \text{ is the dual base of } \mathcal{B} \text{ (see Definition 3.2).}$$

Then $\forall\, f \in E^*$,

$$f = \lambda_1 \overline{x}_1 + \lambda_2 \overline{x}_2 + \cdots + \lambda_n \overline{x}_n, \quad \lambda_i \in \mathbb{A}, \quad i = 1, 2, \ldots, n.$$

If $f \in M^0$ (see Definition 3.2), then

$$f(x_1) = f(x_2) = \cdots = f(x_k) = 0,$$

so that

$$f(x_i) = (\lambda_1 \overline{x}_1 + \lambda_2 \overline{x}_2 + \cdots + \lambda_n \overline{x}_n)(x_i) = \lambda_i \overline{x}_i(x_i) = \lambda_i = 0, \quad i = 1, 2, \ldots, k.$$

Thus $f \in M^0$ if and only if $\lambda_1 = \lambda_2 = \cdots = \lambda_k = 0$, so we may write $f = \lambda_{k+1}\overline{x}_{k+1} + \lambda_{k+2}\overline{x}_{k+2} + \cdots + \lambda_n \overline{x}_n$. This means that the set $\{\overline{x}_{k+1}, \overline{x}_{k+2}, \ldots, \overline{x}_n\}$ generates $M^0$ and since these elements are $\mathbb{A}$-linearly independent we understand that they constitute a base of $M^0$. So $\dim_{\mathbb{A}} M^0 = n - k$.

But $\dim_{\mathbb{A}} M = k$, so that

$$\dim_{\mathbb{A}} E = \dim_{\mathbb{A}} M + \dim_A M^0 = k + n - k = n.$$

(ii) For every $x \in M$ we have that $f(x) = 0$ for all $f \in M^0$, so $x \in M^{00}$, therefore $M \subseteq M^{00}$.

**Conversely:** From (i) $\dim_{\mathbb{A}} M^0 = \dim_A E - \dim_{\mathbb{A}} M$. So for the $\mathbb{A}$-submodule $M^0$ it is valid that

$$\dim_{\mathbb{A}} M^{00} = \dim_{\mathbb{A}} E^\star - \dim_{\mathbb{A}} M^0 = \dim_{\mathbb{A}} E^* - (\dim_{\mathbb{A}} E - \dim_{\mathbb{A}} M) = \dim_{\mathbb{A}} M.$$

So $M^{00} = M$.

(iii) Let $M \subseteq N$ and $f \in N^0$. From Definition 3.3 we have that, $\forall\, x \in N$, $f(x) = 0$. So $\forall\, x \in M \subseteq N$, $f(x) = 0$, therefore $f \in M^0$. Hence $N^0 \subseteq M^0$.

Let $N^0 \subseteq M^0$. Then reasoning as above we have $M^{00} \subseteq N^{00}$ and from (ii) $M \subseteq N$.

(iv) Let $f \in (M+N)^0$. Then $x \in M+N$, $f(x) = 0$ and if $x = x_1 + x_2$, $x_1 \in M$, $x_2 \in N$, we have

$$0 = f(x) = f(x_1 + x_2) = f(x_1) + f(x_2).$$

So $f \in M^0$ and $f \in N^0$, that is $f \in M^0 \cap N^0$, therefore $(M+N)^0 \subseteq M^0 \cap N^0$.

Let $f \in M^0 \cap N^0$ that is $f \in M^0$ and $f \in N^0$.

From Definition 3.3 $\forall x_1 \in M$, $f(x_1) = 0$ and $\forall x_2 \in N$, $f(x_2) = 0$, so $0 = f(x_1) + f(x_2) = f(x_1 + x_2)$ for all $x_1 + x_2 = x \in M+N$. So $f \in (M+N)^0$, therefore $M^0 \cap N^0 \subseteq (M+N)^0$. Thus (iv) is proved.

(v) From (ii) and (iv) we have that

$$(M^0 + N^0)^0 = M^{00} \cap N^{00} = M \cap N \quad \text{and}$$

$$(M \cap N)^0 = (M^0 + N^0)^{00} = M^0 + N^0.$$

□

## 4. Orthosymmetric $\mathbb{A}$-forms

**Lemma 4.1.** *Let $E$, $F$ be two free $\mathbb{A}$-modules having finite rank and $f, g : E \to F$ two $\mathbb{A}$-isomorphisms (hence $\dim_{\mathbb{A}} E = \dim_{\mathbb{A}} F = n$).*

*We suppose that $f(M) = g(M)$, for all $\mathbb{A}$-submodules $M$ of $E$. Then there is $\lambda \in \mathbb{A}$, such that $g = \lambda f$.*

*Proof.* (i) Let $\dim_{\mathbb{A}} E = 1$. Then $E = \langle a \rangle$, $a \neq 0$, that is $E \cong \mathbb{A}$ and since by hypothesis $f(M) = g(M) \Leftrightarrow f(\langle a \rangle) = g(\langle a \rangle)$, we have $g(a) \in f(\langle a \rangle)$. So there is $\lambda_a \in \mathbb{A}$, such that $g(a) = \lambda_a f(a)$. For $0 \neq b \in \langle a \rangle$ we have $b = \mu a$, $\mu \in \mathbb{A} \backslash \{0\}$ and $g(b) = \lambda_b f(b)$ with $\lambda_b \in \mathbb{A}$ and $g(b) = g(\mu a) = \mu g(a) = \mu \lambda_a f(a) = \lambda_a \mu f(a) = \lambda_a f(\mu a) = \lambda_a f(b)$. Hence $\lambda_b f(b) = \lambda_a f(b)$ with $f(b) \neq 0$, since $f$ is injective. Since moreover the algebra $\mathbb{A}$ has no zero-divisors, we conclude that $\lambda_b = \lambda_a = \lambda \in \mathbb{A}$ and so $g = \lambda f$.

(ii) Let $\dim_{\mathbb{A}} E \geq 2$. Then there exist at least two $\mathbb{A}$-linear independent elements $a, b \in E$ such that $g(a+b) = \lambda_{a+b} f(a+b)$ with $a+b \in E$, $\lambda_{a+b} \in \mathbb{A}$ and

$$g(a+b) = g(a) + g(b) = \lambda_a f(a) + \lambda_b f(b), \quad \lambda_a, \lambda_b \in \mathbb{A}$$

therefore $\lambda_{a+b} f(a+b) = \lambda_a f(a) + \lambda_b f(b)$ so

$$\lambda_{a+b} f(a) + \lambda_{a+b} f(b) = \lambda_a f(a) + \lambda_b f(b) \quad \text{and}$$

$$(\lambda_{a+b} - \lambda_a) f(a) + (\lambda_{a+b} - \lambda_b) f(b) = 0.$$

But $f(a), f(b)$ are linear independent, so that we have $\lambda_{a+b} = \lambda_a = \lambda_b = \lambda \in \mathbb{A}$, that is $g(a) = \lambda f(a)$ and $g(b) = \lambda f(b)$.

Let now $c \neq 0$, $c \in E$, being $\mathbb{A}$-linear independent either with $a$ or $b$. Then $\lambda_c = \lambda_a = \lambda_b = \lambda \in \mathbb{A}$.

Therefore $\lambda$ is independent of any element $\omega \in E$.

Indeed, let $\mathcal{B} = \{x_1, x_2, \ldots, x_n\}$ be a base of $E$. Then for $w \in E$ we have:
$\omega = \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n$ with $\lambda_i \in \mathbb{A}$, $i = 1, 2, \ldots, n$ and
$g(\omega) = g(\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n) = \lambda_1 g(x_1) + \lambda_2 g(x_2) + \cdots + \lambda_n g(x_n) =$
$\lambda_1 \lambda f(x_1) + \lambda_2 \lambda f(x_2) + \cdots + \lambda_n \lambda f(x_n) = \lambda\lambda_1 f(x_1) + \lambda\lambda_2 f(x_2) + \cdots + \lambda\lambda_n f(x_n) =$
$\lambda(f(\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n)) = \lambda f(\omega)$.

Therefore $g = \lambda f$, $\lambda \in \mathbb{A}$. □

**Lemma 4.2.** *Let $E$, $F$ be two free $\mathbb{A}$-modules of the same finite rank. Also we suppose that $\mathbb{A}$ is PID (Principal ideal domain).*

*Let $f, g : E \to F$ be two $\mathbb{A}$-linear maps with $f(M) = g(M)$ for all $\mathbb{A}$-submodules $M$ of $E$, and $Kerf = Kerg$. Then there exist $\lambda \in \mathbb{A}$ such that $g = \lambda f$.*

*Proof.* Note that since $\mathbb{A}$ is a PID algebra we have that the $\mathbb{A}$-modules $E/Kerf$ and $E/Kerg$ are free of finite rank.

The following maps:

$$\widetilde{f} : E/Kerf \longrightarrow Imf : \ \widetilde{f}(x + Kerf) := f(x), \ x \in E$$
$$\widetilde{g} : E/Kerg \longrightarrow Img : \ \widetilde{g}(x + Kerg) := g(x), \ x \in E$$

are well defined $\mathbb{A}$-isomorphisms.

Let $N$ be an $\mathbb{A}$-submodule of $E/Kerf$.

Consider the natural $\mathbb{A}$-homomorphism: $\pi : E \longrightarrow E/Kerf$, where
$\pi(x) := x + Kerf$, $\forall\, x \in E$.

The set $\pi^{-1}(N) := \{x \in E : \pi(x) \in N\} \subseteq E$ is clearly an $\mathbb{A}$-submodule of $E$, while $\pi^{-1}(N)/Kerf$ is an $\mathbb{A}$-submodule of $E/Kerf$.

In general, for every $\mathbb{A}$-submodule $M$ of $E$, the quotient $M/Kerf$ is an $\mathbb{A}$-submodule of $E/Kerf$. It is now easily seen that $N \cong \pi^{-1}(N)/Kerf$.

Let $a \in \widetilde{f}\big(\pi^{-1}(N)/Kerf\big)$. Then there is $z \in \pi^{-1}(N)/Kerf$ such that $\widetilde{f}(z) = a$, $z = x + Kerf$, $x \in \pi^{-1}(N)$, so that $a = \widetilde{f}(z) = \widetilde{f}(x + Kerf) = f(x) \in f(\pi^{-1}(N))$.

But by assumption we have $f(\pi^{-1}(N)) = g(\pi^{-1}(N))$ for all $\mathbb{A}$-submodules of $E$ and also $Kerf = Kerg$. So for $x \in \pi^{-1}(N)$, $f(x) \in f(\pi^{-1}(N))$ and
$f(x) \in g(\pi^{-1}(N))$ and so there exists $x' \in \pi^{-1}(N)$ such that $f(x) = g(x') = a$.

Thus $\widetilde{f}(x+Kerf) = \widetilde{g}(x'+Kerf) = \widetilde{g}(x'+Kerg) = a$, i.e., $a \in \widetilde{g}\big(\pi^{-1}(N)/Kerg\big)$. Therefore $\widetilde{f}\big(\pi^{-1}(N)/Kerf\big) \subseteq \widetilde{g}\big(\pi^{-1}(N)/Kerg\big)$.

Reasoning in the same way, we obtain

$$\widetilde{g}\big(\pi^{-1}(N)/Kerg\big) \subseteq \widetilde{f}\big(\pi^{-1}(N)/Kerf\big), \quad \text{consequently}$$
$$\widetilde{f}\big(\pi^{-1}(N)/Kerf\big) = \widetilde{g}\big(\pi^{-1}(N)/Kerg\big).$$

So we have proved that the $\mathbb{A}$-isomorphisms $\widetilde{f}, \widetilde{g}$ fulfill the assumptions of Lemma 4.1. So $\exists\, \lambda \in \mathbb{A}$ such that $\widetilde{g} = \lambda \widetilde{f}$ which implies $g = \lambda f$. □

**Theorem 4.1.** *Let $(E, \varphi)$ be a pair consisting of a free $\mathbb{A}$-module $E$ of finite rank and an $\mathbb{A}$-form $\varphi$ on $E$. Suppose that $\mathbb{A}$ is PID and $\varphi$ is an orthosymmetric $\mathbb{A}$-form. Then $\varphi$ is either symmetric or skew-symmetric (see § (5.3) in* [1]*).*

*Proof.* Let $M$ be an $\mathbb{A}$-submodule of $E$. By Proposition 3.1 $M^{\perp} = (\widehat{\varphi}(M))^0$ and $M^{\top} = (\overset{\vee}{\varphi}(M))^0$ (for $\widehat{\varphi}$, $\overset{\vee}{\varphi}$ see (3.5) and (3.6)).

Since $\varphi$ is orthosymmetric we have $M^{\perp} = M^{\top}$, therefore $(\widehat{\varphi}(M))^0 = (\overset{\vee}{\varphi}(M))^0$ and $(\widehat{\varphi}(M))^{00} = (\overset{\vee}{\varphi}(M))^{00}$. Moreover by Theorem 3.1 $\widehat{\varphi}(M) = \overset{\vee}{\varphi}(M)$ and

$$\begin{aligned} Ker\widehat{\varphi} &= \{x \in E :\ \widehat{\varphi}(x) = \varphi_x = 0\} = \{x \in E :\ \varphi_x(y) = 0\ \forall\, y \in E\} \\ &= \{x \in E :\ \varphi(x, y) = 0,\ \forall\, y \in E\} \\ &= \{x \in E :\ \varphi(y, x) = 0,\ \forall\, y \in E\} = \{x \in E\ {}_x\varphi(y) = 0\ \forall\, y \in E\} \\ &= \{x \in E :\ \overset{\vee}{\varphi}(x)(y) = 0,\ \forall\, y \in E\} = \{x \in E\ \overset{\vee}{\varphi}(x) = 0\} = Ker\overset{\vee}{\varphi}. \end{aligned}$$

That is $Ker\widehat{\varphi} = Ker\overset{\vee}{\varphi}$ and $\widehat{\varphi}(M) = \overset{\vee}{\varphi}(M)$, therefore by Lemma 4.2 there exists $\lambda \in \mathbb{A}$ such that $\overset{\vee}{\varphi} = \lambda\widehat{\varphi}$.

Hence

$$\varphi(x, y) = {}_y\varphi(x) = \overset{\vee}{\varphi}(y)(x) = \lambda\widehat{\varphi}(y)(x) = \lambda\varphi_y(x) = \lambda\varphi(y, x), \quad \text{for all} \quad x, y \in E.$$

In a similar way we obtain

$$\varphi(y, x) = \lambda\varphi(x, y), \quad x, y \in E, \quad \text{so}$$
$$\varphi(x, y) = \lambda^2\varphi(x, y) \quad \text{for all} \quad x, y \in E \quad \text{and} \quad \lambda \in \mathbb{A}.$$

Therefore $\lambda^2 = 1$, the unit of $\mathbb{A}$, since $\mathbb{A}$ has no zero-divisors.

If $\lambda = 1$ we have $\varphi(x, y) = \varphi(y, x)$, $x, y \in E$, that is $\varphi$ is symmetric.

If $\lambda = -1$ we have $\varphi(x, y) = -\varphi(y, x)$, $x, y \in E$ that is $\varphi$ is skew-symmetric. □

As a result we have the following.

An immediate consequence of Theorem 4.1 is the following.

**Corollary 4.1.** *Let $(E, \varphi)$ be a pair consisting of an free $\mathbb{A}$-module $E$ having finite rank and an $\mathbb{A}$-form $\varphi$ on $E$. Let also $\mathbb{A}$ be PID. Then we have:*

$$\mathcal{B}_0(E) = \mathcal{B}_s(E) \cup \mathcal{B}_a(E).$$

## 5. Matrix representation

Let $(E, \varphi)$ be a pair consisting of a free $\mathbb{A}$-module $E$ of finite rank $n \in \mathbb{N}$ and an $\mathbb{A}$-form $\varphi$ on $E$, and let

$$\mathcal{B} = \{x_1, x_2, \ldots, x_n\} \text{ be a base of } E;$$

then one defines the matrix of $\varphi$ with respect to the base $\mathcal{B}$, as follows:

$$(\varphi : \mathcal{B}) = \begin{pmatrix} \varphi(x_1, x_1) & \varphi(x_1, x_2) \ldots \varphi(x_1, x_n) \\ \varphi(x_2, x_1) & \varphi(x_2, x_2) \ldots \varphi(x_2, x_n) \\ \vdots & \\ \varphi(x_n, x_1) & \varphi(x_n, x_2) \ldots \varphi(x_n, x_n) \end{pmatrix} \in \mathbb{A}^{n \times n}. \tag{5.1}$$

If the $\mathbb{A}$-form $\varphi : E \times E \to \mathbb{A}$ is orthosymmetric and $\mathbb{A}$ a unital, commutative, PID algebra, without divisors of zero, then from Theorem 4.1, $\varphi$ is either symmetric or skew-symmetric. In each case, the matrix of $\varphi$ is simplified.

1) (i) If $\varphi$ is a symmetric $\mathbb{A}$-form and non-degenerated, namely $\varphi(x, y) = \varphi(y, x)$, for every $x, y \in E$, then $E$ admits the following ***orthogonal decomposition***:

$$E = \mathbb{A}[x_1] \oplus \mathbb{A}[x_2] \oplus \cdots \oplus \mathbb{A}[x_n], \tag{5.2}$$

where $\mathbb{A}[x_i]$'s are the $\mathbb{A}$-submodules generated by $x_i$'s, for every $i = 1, 2, \ldots, n$.

Since $\varphi(x_i, x_i) = 1$ and $\dim \mathbb{A}[x_i] = 1$, for every $i = 1, 2, \ldots, n$, we call the $\mathbb{A}$-submodules ***hyperbolic*** $\mathbb{A}$***-lines*** and the base $\mathcal{B}$ of $E$, ***hyperbolic base***, (see (5.23) in [2]).

Then the matrix of $\varphi$ over $\mathbb{A}$ with respect to $\mathcal{B}$ is the following:

$$(\varphi : \mathcal{B}) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & 1 \end{pmatrix}. \tag{5.3}$$

In case we have $\varphi(x_i, x_i) = 1$, for some $i$'s, and for the rest of $i$'s $\varphi(x_i, x_i) = -1$, then the preceding matrix will have in the diagonal $i$-times $1 \in \mathbb{A}$ and $(n-i)$-times $-1 \in \mathbb{A}$.

(ii) If $\varphi$ is a degenerate symmetric $\mathbb{A}$-form then $Ker\widehat{\varphi} \neq \{0\}$ and $Ker\widehat{\varphi} = E^{\perp}$, (see Lemma 3.1 and Corollary 3.1).

The orthogonal decomposition of the $\mathbb{A}$-module $E$, in this case, is:

$$\begin{aligned} E &= \mathbb{A}[x_1] \oplus \mathbb{A}[x_2] \oplus \cdots \oplus \mathbb{A}[x_r] \oplus Ker\widehat{\varphi} \\ &= \mathbb{A}[x_1] \oplus \mathbb{A}[x_2] \oplus \cdots \oplus \mathbb{A}[x_r] \oplus E^{\perp}, \end{aligned} \tag{5.4}$$

(see (5.24) in [2]).

Then the matrix of $\varphi$ over $\mathbb{A}$, with respect to the hyperbolic base $\{x_1, x_2, \ldots, x_r\}$ and the base $\{x_{r+1}, \ldots, x_n\}$ of $Ker\widehat{\varphi}$ takes the form

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & \cdots & 0 \\ \vdots & & & & & \\ 0 & 0 & \cdots & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & & & & \\ 0 & 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix}, \tag{5.5}$$

where again, in case we have $\varphi(x_i, x_i) = 1$, for some $i$'s, and for the rest of $i$'s $\varphi(x_i, x_i) = -1$, 1 appears $i$-times in the diagonal and $-1, (n-i)$-times, $i = 1, 2, \ldots, r$.

2) (i) If $\varphi$ is a non-degenerate skew-symmetric $\mathbb{A}$-form, namely $\varphi(x, y) = -\varphi(y, x)$, for all $x, y \in E$, then $E$ admits the following ***orthogonal decomposition***:

$$E = L_1 \oplus L_2 \oplus \cdots \oplus L_s \tag{5.6}$$

where $2s = n = \dim E$, $L_i = \mathbb{A}[x_i, y_i]$, $i = 1, 2, \ldots, s$, are the $\mathbb{A}$-subsmodules of $E$ generated by the basic vectors $x_i, y_i$ of $E$, such that $\varphi(x_i, y_i) = 1$, $i = 1, 2, \ldots, s$.

Since $\dim L_i = 2$ and $\varphi(x_i, y_i) = 1$, we call $L_i$, $i = 1, 2, \ldots, s$ ***hyperbolic Langrangian $\mathbb{A}$-planes***. (See (3.2) and (3.3) in [2]).

Then the matrix over $\mathbb{A}$ of $\varphi$ with respect to the base $\mathcal{B} = \{x_1, y_1, x_2, y_2, \ldots, x_s, y_s\}$ is of the form:

$$(\varphi : \mathcal{B}) = \begin{pmatrix} 0 & 1 & & & & & & \\ -1 & 0 & & & & \mathrm{O} & & \\ & & 0 & 1 & & & & \\ & & -1 & 0 & & & & \\ & & & & \ddots & & & \\ & & & & & \ddots & & \\ & \mathrm{O} & & & & & \ddots & \\ & & & & & & 0 & 1 \\ & & & & & & -1 & 0 \end{pmatrix} \tag{5.7}$$

(see (4.7) in [2]).

(ii) If $\varphi$ is a degenerate, skew-symmetric $\mathbb{A}$-form, then $Ker\widehat{\varphi} \neq \{0\}$ and $Ker\widehat{\varphi} = E^{\perp}$ (see Lemma 3.1 and Corollary 3.1).

The orthogonal decomposition of the $\mathbb{A}$-module $E$ is then:

$$E = L_1 \oplus L_2 \oplus \cdots \oplus L_r \oplus Ker\widehat{\varphi} = L_1 \oplus L_2 \oplus \cdots \oplus L_r \oplus E^{\perp} \tag{5.8}$$

with $k = 2r$, where $k = \dim E/Ker\widehat{\varphi}$ (see (3.23) and (3.26) in [2]).

In this case the matrix over $\mathbb{A}$ of $\varphi$ with respect to the base

$$\mathcal{B} = \{x_1, y_1, x_2, y_2, \ldots, x_r, y_r, x_{r+1} y_{r+1}, \ldots, x_s, y_s\}$$

has the form

$$(\varphi : \mathcal{B}) = \begin{pmatrix} 0 & 1 & & & & \\ -1 & 0 & & & \mathrm{O} & \\ & & \ddots & & & \\ & & & \ddots & & \\ & & & & 0 & 1 \\ & \mathrm{O} & & & -1 & 0 \\ & & & & & \mathrm{O} \end{pmatrix} \tag{5.9}$$

(see (4.8) in [2]).

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