A NOTE ON $A$-BILINEAR MAPS

A. CONTE-THRASYVOULIDOU

Abstract. In this note we study various $A$-bilinear maps and we prove among others that under certain conditions, orthosymmetric $A$-bilinear maps are either symmetric or skew-symmetric.

0. Introduction

We consider $A$-bilinear maps on an $A$-module $E$, where $A$ is a unital commutative algebra over a field $K$ (of characteristic different from 2) without zero-divisors.

In Section 2 we prove that each $A$-bilinear form $\varphi : E \times E \to A$ is uniquely decomposed as a sum of a symmetric and a skew-symmetric $A$-bilinear form (Proposition 2.1). In Section 3 we characterize the non-degenerate $A$-bilinear forms (Lemma 3.1, Corollary 3.1) and in the case of a free $A$-module of finite rank the $A$-polars and $A$-bipolars of $A$-submodules and prove some properties of them analogous to the classical ones (Theorem 3.1). In the final Section 4 using the preceding results we show under certain conditions that an orthosymmetric $A$-bilinear form $\varphi : E \times E \to A$ is either symmetric or skew-symmetric (Theorem 4.1). As a consequence of this result, we obtain that the set of all orthosymmetric $A$-bilinear forms is not an $A$-submodule of the $A$-module of all $A$-bilinear forms (Corollary 4.1). In Section 5 we give some applications of Theorem 4.1 which refer to matrix representations, over the algebra $A$, of various orthosymmetric $A$-forms.

1. A Note on $A$-bilinear maps

1. Preliminaries. In the sequel $A$ denotes a unital commutative algebra without divisors of zero, over a field $K$ having characteristic different from 2.

The unit of $A$ will be denoted by 1.

Let $E$ be an $A$-module. (See [3]).

For an $A$-bilinear map $\varphi : E \times E \to A$, in the sequel we shall use the term $A$-form.

We consider the following sets:

$$B(E):=\{\varphi : E \times E \to A, \ A\text{-bilinear}\}. \quad (1.1)$$

$$B_s(E):=\{\varphi \in B(E) : \varphi \text{ symmetric; namely } \varphi(x,y)=\varphi(y,x), \ x, y \in E\} \quad (1.2)$$

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Let \( \varphi, \sigma \) be symmetric, resp. skew-symmetric, the same is true for \( \varphi + \sigma, \lambda \varphi, \lambda \in \mathbb{A} \).

Thus \( B_s(E), B_a(E) \) are \( \mathbb{A} \)-submodules of \( B(E) \). But \( B_s(E) \subseteq B_0(E) \) and \( B_a(E) \subseteq B_0(E) \) are only subsets of \( B(E) \). (See Chapter V in [1]).

2. Decomposition of the \( \mathbb{A} \)-module \( B(E) \)

Proposition 2.1. Let \((E, \varphi)\) be a pair consisting of an \( \mathbb{A} \)-module \( E \) and an \( \mathbb{A} \)-form \( \varphi \) on \( E \) \((i.e., \varphi \in B(E))\). Then:

\[
B(E) = B_s(E) \oplus B_a(E).
\]

(See § (5.3) in [1].)

Proof. (i) One can easily prove that:

\[
B_s(E) + B_a(E) \subseteq B(E).
\]

(ii) Now for \( \varphi \in B(E) \) we put:

\[
\varphi_1(x, y) := \frac{1}{2}(\varphi(x, y) + \varphi(y, x)), \quad x, y \in E,
\]

\[
\varphi_2(x, y) := \frac{1}{2}(\varphi(x, y) - \varphi(y, x)), \quad x, y \in E.
\]

Then easily we have that \( \varphi_1 \in B_s(E) \) and \( \varphi_2 \in B_a(E) \), with \( \varphi = \varphi_1 + \varphi_2 \).

Thus \( B(E) \subseteq B_s(E) + B_a(E) \), consequently \( B(E) = B_s(E) + B_a(E) \).

(iii) Let \( \varphi \in B_s(E) \cap B_a(E) \).

Then \( \varphi(x, y) = \varphi(y, x) \) and \( \varphi(x, y) = -\varphi(y, x), x, y \in E \).

Therefore \( \varphi(x, y) = 0, \forall x, y \in E \) and so \( \varphi = 0 \).

This completes the proof. \( \square \)

3. The dual \( \mathbb{A} \)-module \( E^* \)

Let \( E \) be an \( \mathbb{A} \)-module. We define

\[
E^* := \text{Hom}_{\mathbb{A}}(E, \mathbb{A}) \quad \text{(see [3])}.
\]

Then \( E^* \) is an \( \mathbb{A} \)-module. Let now \( E \) be a free \( \mathbb{A} \)-module of finite rank \( n \in \mathbb{N} \).
We call dimension of $E$ and we denote by $\dim_A E$ the rank of $E$. In this case, $E^*$ is the free $A$-module of all $A$-linear forms on $E$, of finite rank $n \in \mathbb{N}$, so that $E^* \cong A^n$.

Let $B = \{x_1, x_2, \ldots, x_n\}$ be a base of $E$ and $B^* = \{x_1, x_2, \ldots, x_n\}$ the dual base of $B$ in $E^*$, such that:

$$\pi_i(x_j) = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}, \quad i, j = 1, 2, \ldots, n. \quad (3.2)$$

Then $E \cong A^n \cong E^*$. (See § 4.6 in [1]).

Let now $E$ be an arbitrary $A$-module and $\varphi \in \mathcal{B}(E)$. For a fixed $x \in E$, one may define the map:

$$\varphi_x : E \to A : \varphi_x(y) := \varphi(x, y), \quad y \in E. \quad (3.3)$$

and similarly for a fixed $y \in E$, one defines

$$y\varphi : E \to A : y\varphi(x) := \varphi(x, y), \quad x \in E \quad (3.4)$$

Clearly these maps $\varphi_x, y\varphi$ are in $E^*$.

Also we may define the maps:

$$\hat{\varphi} : E \to E^* : \hat{\varphi}(x) := \varphi_x \in E^* \quad \forall x \in E \quad (3.5)$$

such that:

$$\hat{\varphi}(x)(y) = \varphi_x(y) = \varphi(x, y) \quad \forall y \in E \quad \text{and}$$

$$\check{\varphi} : E \to E^* : \check{\varphi}(y) := y\varphi \in E^*, \quad \forall y \in E \quad (3.6)$$

such that:

$$\check{\varphi}(y)(x) = y\varphi(x) = \varphi(x, y) \quad \forall x \in E.$$

These maps $\hat{\varphi}$ and $\check{\varphi}$ are $A$-linear maps corresponding to the $A$-form $\varphi$.

Let $(E, \varphi)$ be a pair consisting of an $A$-module $E$ and an $A$-form $\varphi : E \times E \to A$. For any subset $\emptyset \neq M \subseteq E$ we consider the following $A$-submodules, with respect to $\varphi$, defined as follows:

$$M^{\perp \varphi} := \{x \in E : \varphi(x, y) = \varphi_y(x) \equiv \hat{\varphi}(x)(y) = 0, \quad \forall y \in M\}$$

$$= \{x \in E : x \perp M\} = \{x \in E : M \subseteq \ker \varphi_x\} \quad \text{and} \quad (3.7)$$

$$M^{\top \varphi} := \{y \in E : \varphi(x, y) = y\varphi(x) \equiv \check{\varphi}(y)(x) = 0, \quad \forall x \in M\}$$

$$= \{y \in E : y \perp M\} = \{y \in E : M \subseteq \ker y\varphi\} \quad (3.8)$$

(see § 5.2 in [1]).

**Remark 3.1.** If the $A$-form $\varphi$, is orthosymmetric then, clearly $M^{\perp} = M^{\top}$.

**Definition 3.1.** Let $(E, \varphi)$ be a pair consisting of an $A$-module $E$ and an $A$-form $\varphi : E \times E \to A$. Then $\varphi$ is called **non-degenerate** whenever the following condition holds true:

If $x \in E$ such that $\varphi(x, y) = 0$, for every $y \in E$, then $x = 0$, as well.

Equivalently one obtains:

For any $x \neq 0$ in $E$, there exists $y \in E$ (necessarily nonzero) such that $\varphi(x, y) \neq 0$. 

Lemma 3.1. Let \((E, \varphi)\) be a pair consisting of an \(\mathcal{A}\)-module \(E\) and an \(\mathcal{A}\)-form \(\varphi\) on \(E\). Then \(\varphi\) is non-degenerate if and only if \(E^\perp = \{0\}\).

Proof. Let \(\varphi\) be a non-degenerate \(\mathcal{A}\)-form of \(E\). Then for every \(a \in E^\perp \subseteq E\), with \(a \neq 0\), there exists \(x \in E\) with \(x \neq 0\) such that \(\varphi(a, x) \neq 0\). But \(a \in E^\perp\) implies \(\varphi(a, y) = 0\), for all \(y \in E\). Thus \(\varphi(a, x) = 0\), therefore \(x = 0\), a contradiction.

Let now \(E^\perp = \{0\}\) and \(\varphi(x, y) = 0\), for all \(y \in E\). But then \(x = 0\), therefore \(\varphi\) is non-degenerate. \(\square\)

Corollary 3.1. Let \((E, \varphi)\) be a pair consisting of an \(\mathcal{A}\)-module \(E\) and an \(\mathcal{A}\)-form \(\varphi\) on \(E\). Then we have:

(i) \(E^\perp = \text{Ker} \hat{\varphi}\)

(ii) The \(\mathcal{A}\)-form \(\varphi\), is non degenerate if and only if \(\hat{\varphi}\) is injective.

Proof. (i) Let \(x \in E^\perp\) then \(\varphi(x, y) = 0\), for all \(y \in E\); namely \(\varphi(x, y) = \varphi_x(y) = \hat{\varphi}(x)(y) = 0\), \(\forall y \in E\), therefore \(\hat{\varphi}(x) = 0\), i.e., \(x \in \text{Ker} \hat{\varphi}\). Hence \(E^\perp \subseteq \text{Ker} \hat{\varphi}\).

Let \(x \in \text{Ker} \hat{\varphi}\). Then \(\hat{\varphi}(x)(y) = \varphi_x(y) = \varphi(x, y) = 0\), for all \(y \in E\), therefore \(x \in E^\perp\) (see (3.7)).

This completes the proof. \(\square\)

(ii) Let \(\hat{\varphi}\) be injective. Then \(\text{Ker} \hat{\varphi} = E^\perp = \{0\}\) so that from Lemma 3.1 \(\varphi\) is non-degenerate.

Conversely: Let \(\varphi\) be non-degenerate and \(x \in \text{Ker} \hat{\varphi}\) such that \(\hat{\varphi}(x) = 0\). Then \(\forall y \in E\) \(\hat{\varphi}(x)(y) = \varphi_x(y) = \varphi(x, y) = 0\), so that \(x \in E^\perp = \{0\}\) (see Lemma 3.1). Therefore \(x = 0\) and \(\hat{\varphi}\) is 1-1.

Definition 3.2. Let \((E, \varphi)\) be a pair consisting of an \(\mathcal{A}\)-module \(E\) and an \(\mathcal{A}\)-form \(\varphi\) on \(E\). Let \(\emptyset \neq M \subseteq E\). We define the set:

\[M^0 := \{f \in E^* : f(x) = 0, \ \forall x \in M\} \subseteq E^*.\]

We call \(M^0\) an \(\mathcal{A}\)-polar set of \(M\) in \(E^*\). It is easy to see that \(M^0\) is an \(\mathcal{A}\)-submodule of \(E^*\).

Definition 3.3. Let \((E, \varphi)\) be a pair consisting of an \(\mathcal{A}\)-module \(E\) and an \(\mathcal{A}\)-form \(\varphi\) on \(E\). Let \(\emptyset \neq N \subseteq E^*\). We define the set:

\[N^0 := \{x \in E : f(x) = 0 \ \forall f \in N\} \subseteq E.\]

We call \(N^0\) an \(\mathcal{A}\)-polar set of \(N\) in \(E\). It is easy to see that \(N^0\) is an \(\mathcal{A}\)-submodule of \(E\).

Remark 3.2. The sets \(M^{00} = (M^0)^0 \subseteq E\) and \(N^{00} = (N^0)^0 \subseteq E^*\) are called \(\mathcal{A}\)-bipolars of \(M, N\) respectively in the \(\mathcal{A}\)-modules \(E, E^*\).
We also consider the sets:

\[(i) \quad (\hat{\varphi}(M))^0 := \{ x \in E : \hat{\varphi}(x)(y) \equiv \varphi(x) = \varphi(x,y) = 0, \forall y \in M \} \]

\[(ii) \quad (\check{\varphi}(M))^0 := \{ y \in E : \check{\varphi}(y)(x) \equiv y\varphi(x) = \varphi(x,y) = 0, \forall x \in M \}, \]

where \( \varphi \) is an \( \mathbb{A} \)-form and \( \emptyset \neq M \subseteq E \).

**Proposition 3.1.** Let \((E, \varphi)\) be a pair consisting of an \( \mathbb{A} \)-module \( E \) and an \( \mathbb{A} \)-form \( \varphi \) on \( E \). Let also \( \emptyset \neq M \subseteq E \). Then we have:

\[ M^\perp = (\hat{\varphi}(M))^0. \]

**Proof.**

\[
M^\perp := \{ x \in E : \varphi(x,y) = 0, \forall y \in M \} = \{ x \in E : \varphi_x(y) = 0, \forall y \in M \} = \{ x \in E : \hat{\varphi}(x)(y) = 0, \forall y \in M \} = (\hat{\varphi}(M))^0.
\]

In the same manner we have that

\[ M^\perp = (\check{\varphi}(M))^0. \]

**Remark 3.3.** If \( \varphi \) is orthosymmetric then

\[ (\hat{\varphi}(M))^0 = (\check{\varphi}(M))^0. \]

**Proposition 3.2.** Let \((E, \varphi)\) be a pair consisting of an \( \mathbb{A} \)-module \( E \) and an \( \mathbb{A} \)-form \( \varphi \) on \( E \). Let \( \emptyset \neq M \subseteq E \). Then we have:

(i) \( M \subseteq (M^\perp)^\perp \) and (ii) \( M \subseteq (M^\perp)^\perp \).

**Proof.** By (3.7), for \( y \in M \), we have \( \varphi(x,y) = 0, \forall x \in M^\perp \). So \( y \in (M^\perp)^\perp \). Therefore \( M \subseteq (M^\perp)^\perp \).

In the same manner \( M \subseteq (M^\perp)^\perp \). \( \square \)

**Proposition 3.3.** Let \((E, \varphi)\) be a pair consisting of an \( \mathbb{A} \)-module \( E \) and an \( \mathbb{A} \)-form \( \varphi \) on \( E \). Then \( E = (E^\perp)^\perp \).

**Proof.** From Proposition 3.2, \( E \subseteq (E^\perp)^\perp \). But also \( (E^\perp)^\perp \subseteq E \), so that \( E = (E^\perp)^\perp \).

In the same manner \( E = (E^\perp)^\perp \). \( \square \)

**Corollary 3.2.** Under the supositions of Proposition 3.3, if \( \varphi \) is \( \mathbb{A} \)-orthosymmetric, then we have \( E^{\perp\perp} = E \) and \( E^{\perp\perp} = E \).

**Theorem 3.1.** Let \( E \) be a free \( \mathbb{A} \)-module of finite rank \( n \in \mathbb{N} \) (i.e. \( \dim_\mathbb{A} E = n \)).

Also let \( M, N \) be \( \mathbb{A} \)-submodules of \( E \), then we have:
(i) $\dim_{\mathbb{A}} E = \dim_{\mathbb{A}} M + \dim_{\mathbb{A}} M^0$
(ii) $M^{00} = M$
(iii) $M \subseteq N \iff N^0 \subseteq M^0$
(iv) $(M + N)^0 = M^0 \cap N^0$
(v) $(M \cap N)^0 = M^0 + N^0$ (see §4.6 in [1]).

**Proof.** The proof of this theorem goes exactly as in the classical case. For completeness’ sake we include it here too.

(i) $E \cong \mathbb{A}^n$ and let $B = \{x_1, x_2, \ldots, x_n\}$ be a base of $E$. We consider the $\mathbb{A}$-submodule $M$ of $E$ with $M = \langle x_1, x_2, \ldots, x_k \rangle$, $k \leq n$, where

$B_M = \{x_1, x_2, \ldots, x_k\}$ is a base of $M$ (i.e., $M \cong \mathbb{A}^k$).

$B^* = \{\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n\}$ is the dual base of $B$ (see Definition 3.2).

Then $\forall f \in E^*$, $f = \lambda_1 \overline{x}_1 + \lambda_2 \overline{x}_2 + \cdots + \lambda_n \overline{x}_n$, $\lambda_i \in \mathbb{A}$, $i = 1, 2, \ldots, n$.

If $f \in M^0$ (see Definition 3.2), then

$f(x_1) = f(x_2) = \cdots = f(x_k) = 0$,

so that

$f(x_i) = (\lambda_1 \overline{x}_1 + \lambda_2 \overline{x}_2 + \cdots + \lambda_n \overline{x}_n)(x_i) = \lambda_i \overline{x}_i(x_i) = \lambda_i = 0$, $i = 1, 2, \ldots, k$.

Thus $f \in M^0$ if and only if $\lambda_1 = \lambda_2 = \cdots = \lambda_k = 0$, so we may write $f = \lambda_{k+1} \overline{x}_{k+1} + \lambda_{k+2} \overline{x}_{k+2} + \cdots + \lambda_n \overline{x}_n$. This means that the set $\{\overline{x}_{k+1}, \overline{x}_{k+2}, \ldots, \overline{x}_n\}$ generates $M^0$ and since these elements are $\mathbb{A}$-linearly independent we understand that they constitute a base of $M^0$. So $\dim_{\mathbb{A}} M^0 = n - k$.

But $\dim_{\mathbb{A}} M = k$, so that

$\dim_{\mathbb{A}} E = \dim_{\mathbb{A}} M + \dim_{\mathbb{A}} M^0 = k + n - k = n$.

(ii) For every $x \in M$ we have that $f(x) = 0$ for all $f \in M^0$, so $x \in M^{00}$, therefore $M \subseteq M^{00}$.

**Conversely:** From (i) $\dim_{\mathbb{A}} M^0 = \dim_{\mathbb{A}} E - \dim_{\mathbb{A}} M$. So for the $\mathbb{A}$-submodule $M^0$ it is valid that

$\dim_{\mathbb{A}} M^{00} = \dim_{\mathbb{A}} E^* - \dim_{\mathbb{A}} M^0 = \dim_{\mathbb{A}} E^* - (\dim_{\mathbb{A}} E - \dim_{\mathbb{A}} M) = \dim_{\mathbb{A}} M$.

So $M^{00} = M$.

(iii) Let $M \subseteq N$ and $f \in N^0$. From Definition 3.3 we have that, $\forall x \in N$, $f(x) = 0$. So $\forall x \in M \subseteq N$, $f(x) = 0$, therefore $f \in M^0$. Hence $N^0 \subseteq M^0$. 
Let $N^0 \subseteq M^0$. Then reasoning as above we have $M^{00} \subseteq N^{00}$ and from (ii) $M \subseteq N$.

(iv) Let $f \in (M + N)^0$. Then $x \in M + N$, $f(x) = 0$ and if $x = x_1 + x_2$, $x_1 \in M$, $x_2 \in N$, we have
\[ 0 = f(x) = f(x_1 + x_2) = f(x_1) + f(x_2). \]
So $f \in M^0$ and $f \in N^0$, that is $f \in M^0 \cap N^0$, therefore $(M + N)^0 \subseteq M^0 \cap N^0$.

Let $f \in M^0 \cap N^0$ that is $f \in M^0$ and $f \in N^0$.

From Definition 3.3 all $x \in M$, $f(x_1) = 0$ and $\forall x_2 \in N$, $f(x_2) = 0$, so $0 = f(x_1) + f(x_2) = f(x_1 + x_2)$ for all $x_1 + x_2 = x \in M + N$. So $f \in (M + N)^0$, therefore $M^0 \cap N^0 \subseteq (M + N)^0$. Thus (iv) is proved.

(v) From (ii) and (iv) we have that
\[
(M^0 + N^0)^0 = M^{00} \cap N^{00} = M \cap N \text{ and }
(M \cap N)^0 = (M^0 + N^0)^{00} = M^0 + N^0.
\]

\[ \square \]

4. Orthosymmetric $\mathbb{A}$-forms

Lemma 4.1. Let $E, F$ be two free $\mathbb{A}$-modules having finite rank and $f, g : E \to F$ two $\mathbb{A}$-isomorphisms (hence $\text{dim}_\mathbb{A} E = \text{dim}_\mathbb{A} F = n$).

We suppose that $f(M) = g(M)$, for all $\mathbb{A}$-submodules $M$ of $E$. Then there is \( \lambda \in \mathbb{A} \), such that $g = \lambda f$.

Proof. (i) Let $\text{dim}_\mathbb{A} E = 1$. Then $E = \langle a \rangle$, $a \neq 0$, that is $E \cong \mathbb{A}$ and since by hypothesis $f(M) = g(M)$ if $f(\langle a \rangle) = g(\langle a \rangle)$, we have $g(a) \in f(\langle a \rangle)$. So there is $\lambda_a \in \mathbb{A}$, such that $g(a) = \lambda_a f(a)$. For $0 \neq b \in \langle a \rangle$ we have $b = \mu a$, $\mu \in \mathbb{A} \setminus \{0\}$ and $g(b) = \lambda_b f(b)$ with $\lambda_b \in \mathbb{A}$ and $g(b) = g(\mu a) = \mu g(a) = \mu \lambda_a f(a) = \lambda_a f(\mu a) = \lambda_a f(b)$. Hence $\lambda_b f(b) = \lambda_a f(b)$ with $f(b) \neq 0$, since $f$ is injective. Since moreover the algebra $\mathbb{A}$ has no zero-divisors, we conclude that $\lambda_b = \lambda_a = \lambda \in \mathbb{A}$ and so $g = \lambda f$.

(ii) Let $\text{dim}_\mathbb{A} E \geq 2$. Then there exist at least two $\mathbb{A}$-linear independent elements $a, b \in E$ such that $g(a + b) = \lambda_{a+b} f(a + b)$ with $a + b \in E$, $\lambda_{a+b} \in \mathbb{A}$ and
\[ g(a + b) = g(a) + g(b) = \lambda_a f(a) + \lambda_b f(b), \quad \lambda_a, \lambda_b \in \mathbb{A} \]
therefore $\lambda_{a+b} f(a + b) = \lambda_a f(a) + \lambda_b f(b)$ so
\[ \lambda_{a+b} f(a) + \lambda_{a+b} f(b) = \lambda_a f(a) + \lambda_b f(b) \]
and
\[ (\lambda_{a+b} - \lambda_a) f(a) + (\lambda_{a+b} - \lambda_b) f(b) = 0. \]
But $f(a), f(b)$ are linear independent, so that we have $\lambda_{a+b} = \lambda_a = \lambda_b = \lambda \in \mathbb{A}$, that is $g(a) = \lambda f(a)$ and $g(b) = \lambda f(b)$. 

Let now $c \neq 0$, $c \in E$, being $\mathbb{A}$-linear independent either with $a$ or $b$. Then
$$\lambda_c = \lambda_a = \lambda_b = \lambda \in \mathbb{A}.$$ 
Therefore $\lambda$ is independent of any element $\omega \in E$.
Indeed, let $B = \{x_1, x_2, \ldots, x_n\}$ be a base of $E$. Then for $\omega \in E$ we have:
$$\omega = \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n \text{ with } \lambda_i \in \mathbb{A}, \ i = 1, 2, \ldots, n$$
and
$$g(\omega) = g(\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n) = \lambda_1 g(x_1) + \lambda_2 g(x_2) + \cdots + \lambda_n g(x_n) = \lambda_1 \lambda f(x_1) + \lambda_2 \lambda f(x_2) + \cdots + \lambda_n \lambda f(x_n) = \lambda(\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n) = \lambda f(\omega).$$
Therefore $g = \lambda f, \lambda \in \mathbb{A}$.

\textbf{Lemma 4.2.} Let $E, F$ be two free $\mathbb{A}$-modules of the same finite rank. Also we suppose that $\mathbb{A}$ is PID (Principal ideal domain).

Let $f, g : E \to F$ be two $\mathbb{A}$-linear maps with $f(M) = g(M)$ for all $\mathbb{A}$-submodules $M$ of $E$, and $\text{Ker} f = \text{Kerg}$. Then there exist $\lambda \in \mathbb{A}$ such that $g = \lambda f$.

\textbf{Proof.} Note that since $\mathbb{A}$ is a PID algebra we have that the $\mathbb{A}$-modules $E/\text{Ker} f$ and $E/\text{Kerg}$ are free of finite rank.

The following maps:

$$\tilde{f} : E/\text{Ker} f \to \text{Img} : \tilde{f}(x + \text{Ker} f) := f(x), \ x \in E$$
$$\tilde{g} : E/\text{Kerg} \to \text{Img} : \tilde{g}(x + \text{Kerg}) := g(x), \ x \in E$$

are well defined $\mathbb{A}$-isomorphisms.

Let $N$ be an $\mathbb{A}$-submodule of $E/\text{Ker} f$.

Consider the natural $\mathbb{A}$-homomorphism: $\pi : E \to E/\text{Ker} f$, where $\pi(x) := x + \text{Ker} f, \forall x \in E$.

The set $\pi^{-1}(N) := \{x \in E : \pi(x) \in N\} \subseteq E$ is clearly an $\mathbb{A}$-submodule of $E$, while $\pi^{-1}(N)/\text{Ker} f$ is an $\mathbb{A}$-submodule of $E/\text{Ker} f$.

In general, for every $\mathbb{A}$-submodule $M$ of $E$, the quotient $M/\text{Ker} f$ is an $\mathbb{A}$-submodule of $E/\text{Ker} f$. It is now easily seen that $N \cong \pi^{-1}(N)/\text{Ker} f$.

Let $a \in \tilde{f}(\pi^{-1}(N)/\text{Ker} f)$. Then there is $z \in \pi^{-1}(N)/\text{Ker} f$ such that $\tilde{f}(z) = a$, $z = x + \text{Ker} f, x \in \pi^{-1}(N)$, so that $a = \tilde{f}(z) = \tilde{f}(x + \text{Ker} f) = f(x) \in f(\pi^{-1}(N))$.

But by assumption we have $f(\pi^{-1}(N)) = g(\pi^{-1}(N))$ for all $\mathbb{A}$-submodules of $E$ and also $\text{Ker} f = \text{Kerg}$. So for $x \in \pi^{-1}(N), f(x) \in f(\pi^{-1}(N))$ and $f(x) \in g(\pi^{-1}(N))$ and so there exists $x' \in \pi^{-1}(N)$ such that $f(x) = g(x') = a$.

Thus $\tilde{f}(x + \text{Ker} f) = g(z + \text{Ker} f) = g(x + \text{Kerg}) = a$, i.e., $a \in \tilde{g}(\pi^{-1}(N)/\text{Kerg})$.
Therefore $\tilde{f}(\pi^{-1}(N)/\text{Ker} f) \subseteq \tilde{g}(\pi^{-1}(N)/\text{Kerg})$. 

\textbf{□}
Reasoning in the same way, we obtain
\[ g(\pi^{-1}(N)/\text{Ker}g) \subseteq f(\pi^{-1}(N)/\text{Ker}f), \] consequently
\[ f(\pi^{-1}(N)/\text{Ker}f) = g(\pi^{-1}(N)/\text{Ker}g). \]
So we have proved that the \( \mathbb{A} \)-isomorphisms \( f, g \) fulfill the assumptions of Lemma 4.1. So \( \exists \lambda \in \mathbb{A} \) such that \( g = \lambda f \) which implies \( g = \lambda f \).

**Theorem 4.1.** Let \((E, \varphi)\) be a pair consisting of a free \( \mathbb{A} \)-module \( E \) of finite rank and an \( \mathbb{A} \)-form \( \varphi \) on \( E \). Suppose that \( \mathbb{A} \) is PID and \( \varphi \) is an orthosymmetric \( \mathbb{A} \)-form. Then \( \varphi \) is either symmetric or skew-symmetric (see § (5.3) in [1]).

**Proof.** Let \( M \) be an \( \mathbb{A} \)-submodule of \( E \). By Proposition 3.1 \( M^\perp = (\hat{\varphi}(M))^0 \) and \( M^\top = (\hat{\varphi}(M))^0 \) (for \( \hat{\varphi}, \hat{\varphi} \) see (3.5) and (3.6)).

Since \( \varphi \) is orthosymmetric we have \( M^\perp = M^\top \), therefore \((\hat{\varphi}(M))^0 = (\hat{\varphi}(M))^0 \) and \((\hat{\varphi}(M))^00 = (\hat{\varphi}(M))^00 \). Moreover by Theorem 3.1 \( \hat{\varphi}(M) = \hat{\varphi}(M) \) and
\[
\text{Ker} \hat{\varphi} = \{ x \in E : \hat{\varphi}(x) = \varphi_x = 0 \} = \{ x \in E : \varphi_x(y) = 0 \ \forall \ y \in E \}
= \{ x \in E : \varphi(y, x) = 0, \ \forall \ y \in E \} = \{ x \in E : x \varphi(y) = 0 \ \forall \ y \in E \}
= \{ x \in E : \hat{\varphi}(x)(y) = 0, \ \forall \ y \in E \} = \{ x \in E : \hat{\varphi}(x) = 0 \} = \text{Ker} \hat{\varphi}.

That is \( \text{Ker} \hat{\varphi} = \text{Ker} \hat{\varphi} \) and \( \hat{\varphi}(M) = \hat{\varphi}(M) \), therefore by Lemma 4.2 there exists \( \lambda \in \mathbb{A} \) such that \( \hat{\varphi} = \lambda \hat{\varphi} \).

Hence
\[ \varphi(x, y) = y \varphi(x) = \hat{\varphi}(y)(x) = \lambda \hat{\varphi}(y)(x) = \lambda \varphi_y(x) = \lambda \varphi(y, x), \] for all \( x, y \in E \).

In a similar way we obtain
\[ \varphi(y, x) = \lambda \varphi(x, y), \] \( x, y \in E \), so
\[ \varphi(x, y) = \lambda^2 \varphi(x, y) \] for all \( x, y \in E \) and \( \lambda \in \mathbb{A} \).

Therefore \( \lambda^2 = 1 \), the unit of \( \mathbb{A} \), since \( \mathbb{A} \) has no zero-divisors.

If \( \lambda = 1 \) we have \( \varphi(x, y) = \varphi(y, x), \) \( x, y \in E \), that is \( \varphi \) is symmetric.

If \( \lambda = -1 \) we have \( \varphi(x, y) = -\varphi(y, x), \) \( x, y \in E \) that is \( \varphi \) is skew-symmetric. \( \square \)

As a result we have the following.
An immediate consequence of Theorem 4.1 is the following.

**Corollary 4.1.** Let \((E, \varphi)\) be a pair consisting of an free \( \mathbb{A} \)-module \( E \) having finite rank and an \( \mathbb{A} \)-form \( \varphi \) on \( E \). Let also \( \mathbb{A} \) be PID. Then we have:
\[ \mathcal{B}_0(E) = \mathcal{B}_s(E) \cup \mathcal{B}_a(E). \]
Matrix representation

Let \((E, \varphi)\) be a pair consisting of a free \(A\)-module \(E\) of finite rank \(n \in \mathbb{N}\) and an \(A\)-form \(\varphi\) on \(E\), and let
\[
B = \{x_1, x_2, \ldots, x_n\}
\]
be a base of \(E\); then one defines the matrix of \(\varphi\) with respect to the base \(B\), as follows:
\[
(\varphi : B) = \begin{pmatrix}
\varphi(x_1, x_1) & \varphi(x_1, x_2) & \cdots & \varphi(x_1, x_n) \\
\varphi(x_2, x_1) & \varphi(x_2, x_2) & \cdots & \varphi(x_2, x_n) \\
\vdots & \vdots & \ddots & \vdots \\
\varphi(x_n, x_1) & \varphi(x_n, x_2) & \cdots & \varphi(x_n, x_n)
\end{pmatrix} \in A^{n \times n}.
\tag{5.1}
\]

If the \(A\)-form \(\varphi : E \times E \to A\) is orthosymmetric and \(A\) a unital, commutative, PID algebra, without divisors of zero, then from Theorem 4.1, \(\varphi\) is either symmetric or skew-symmetric. In each case, the matrix of \(\varphi\) is simplified.

1) (i) If \(\varphi\) is a symmetric \(A\)-form and non-degenerated, namely \(\varphi(x, y) = \varphi(y, x)\), for every \(x, y \in E\), then \(E\) admits the following orthogonal decomposition:
\[
E = A[x_1] \oplus A[x_2] \oplus \cdots \oplus A[x_n],
\tag{5.2}
\]
where \(A[x_i]\)'s are the \(A\)-submodules generated by \(x_i\)'s, for every \(i = 1, 2, \ldots, n\).

Since \(\varphi(x_i, x_i) = 1\) and \(\dim A[x_i] = 1\), for every \(i = 1, 2, \ldots, n\), we call the \(A\)-submodules hyperbolic \(A\)-lines and the base \(B\) of \(E\), hyperbolic base, (see (5.23) in [2]).

Then the matrix of \(\varphi\) over \(A\) with respect to \(B\) is the following:
\[
(\varphi : B) = \begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{pmatrix}.
\tag{5.3}
\]

In case we have \(\varphi(x_i, x_i) = 1\), for some \(i\)'s, and for the rest of \(i\)'s \(\varphi(x_i, x_i) = -1\), then the preceding matrix will have in the diagonal \(i\)-times \(1 \in A\) and \((n-i)\)-times \(-1 \in A\).

(ii) If \(\varphi\) is a degenerate symmetric \(A\)-form then \(\text{Ker}\widehat{\varphi} \neq \{0\}\) and \(\text{Ker}\widehat{\varphi} = E^\perp\), (see Lemma 3.1 and Corollary 3.1).

The orthogonal decomposition of the \(A\)-module \(E\), in this case, is:
\[
E = A[x_1] \oplus A[x_2] \oplus \cdots \oplus A[x_r] \oplus \text{Ker}\widehat{\varphi}
= A[x_1] \oplus A[x_2] \oplus \cdots \oplus A[x_r] \oplus E^\perp,
\tag{5.4}
\]
(see (5.24) in [2]).

Then the matrix of \(\varphi\) over \(A\), with respect to the hyperbolic base \(\{x_1, x_2, \ldots, x_r\}\) and the base \(\{x_{r+1}, \ldots, x_n\}\) of \(\text{Ker}\widehat{\varphi}\) takes the form
\[
\begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 1 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1 & 0
\end{pmatrix},
\tag{5.5}
\]
where again, in case we have \( \varphi(x_i, x_i) = 1 \), for some \( i \)'s, and for the rest of \( i \)'s \( \varphi(x_i, x_i) = -1 \), 1 appears \( i \)-times in the diagonal and \( -1, (n-i) \)-times, \( i = 1, 2, \ldots, r \).

2) (i) If \( \varphi \) is a non-degenerate skew-symmetric \( A \)-form, namely \( \varphi(x, y) = -\varphi(y, x) \), for all \( x, y \in E \), then \( E \) admits the following orthogonal decomposition:

\[
E = L_1 \oplus L_2 \oplus \cdots \oplus L_s
\]

(5.6)

where \( 2s = n = \dim E \), \( L_i = A[x_i, y_i], \quad i = 1, 2, \ldots, s \), are the \( A \)-submodules of \( E \) generated by the basic vectors \( x_i, y_i \) of \( E \), such that \( \varphi(x_i, y_i) = 1, \quad i = 1, 2, \ldots, s \).

Since \( \dim L_i = 2 \) and \( \varphi(x_i, y_i) = 1 \), we call \( L_i, \quad i = 1, 2, \ldots, s \) hyperbolic Langrangian \( A \)-planes. (See (3.2) and (3.3) in [2].)

Then the matrix over \( A \) of \( \varphi \) with respect to the base \( B = \{ x_1, y_1, x_2, y_2, \ldots, x_s, y_s \} \) is of the form:

\[
(\varphi : B) = 
\begin{pmatrix}
  0 & 1 & 0 & 0 & \cdots & 0
  -1 & 0 & 0 & 0 & \cdots & 0
  0 & 1 & 0 & 0 & \cdots & 0
  -1 & 0 & 0 & 0 & \cdots & 0
  \vdots & \vdots & \vdots & \vdots & \ddots & \vdots
  \vdots & \vdots & \vdots & \vdots & \ddots & \vdots
  0 & 1 & 0 & 0 & \cdots & 0
  -1 & 0 & 0 & 0 & \cdots & 0
\end{pmatrix}
\]

(5.7)

(see (4.7) in [2]).

(ii) If \( \varphi \) is a degenerate, skew-symmetric \( A \)-form, then \( \text{Ker} \hat{\varphi} \neq \{0\} \) and \( \text{Ker} \hat{\varphi} = E^\perp \) (see Lemma 3.1 and Corollary 3.1).

The orthogonal decomposition of the \( A \)-module \( E \) is then:

\[
E = L_1 \oplus L_2 \oplus \cdots \oplus L_r \oplus \text{Ker} \hat{\varphi} = L_1 \oplus L_2 \oplus \cdots \oplus L_r \oplus E^\perp
\]

(5.8)

with \( k = 2r \), where \( k = \dim E / \text{Ker} \hat{\varphi} \) (see (3.23) and (3.26) in [2]).

In this case the matrix over \( A \) of \( \varphi \) with respect to the base \( B = \{ x_1, y_1, x_2, y_2, \ldots, x_r, y_r, x_{r+1} y_{r+1}, \ldots, x_s, y_s \} \)

has the form

\[
(\varphi : B) = 
\begin{pmatrix}
  0 & 1 & 0 & 0 & \cdots & 0
  -1 & 0 & 0 & 0 & \cdots & 0
  0 & 1 & 0 & 0 & \cdots & 0
  -1 & 0 & 0 & 0 & \cdots & 0
  \vdots & \vdots & \vdots & \vdots & \ddots & \vdots
  \vdots & \vdots & \vdots & \vdots & \ddots & \vdots
  0 & 1 & 0 & 0 & \cdots & 0
  -1 & 0 & 0 & 0 & \cdots & 0
\end{pmatrix}
\]

(5.9)

(see (4.8) in [2]).

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A. CONTE-THRASYVOULIDOU,

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ATHENS,, PANEPISTIMIOUPOLIS, ATHENS 15784, GREECE

E-mail address: aconte@math.uoa.gr