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FIXED AND COMMON FIXED POINT FOR MAPPINGS SATISFYING SOME NONLINEAR CONTRACTIONS IN b-METRIC SPACES

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Abstract. In this paper, we introduce the concept of a c-comparison function with base s and utilized our new notion to establish some common fixed point theorems for nonlinear contractions in a complete b-metric space with base s in the sense of Czerwik [Czerwik S., Nonlinear set-valued contraction mappings in b-metric spaces, Atti Sem. Mat. Univ. Modena, 1998, 46, 263-276].

1. Introduction and preliminaries

Czerwik [1] introduced the notion of b-metric spaces as a generalization of metric spaces. Then after, many authors established many fixed point theorems in b-metric spaces (see [3]-[24]). Aydi et al. [5] established a nice fixed point theorem for set-valid quasi contraction in b-metric space. Also, Vahid Parvaneh et al. [17] studied some tripled coincidence point in ordered metric spaces. While Roshan et al. [19]studied some common fixed points of almost generalized $(\psi, \phi)_s$-contractive mappings in ordered b-metric spaces. Moreover, Shi et al. [21] studied some nice fixed point theorems in cone b-metric spaces and Mustafa et al. [14] studied some coupled coincidence point results in generalized b-metric spaces.

Now, we present the definition of b-metric spaces.

Definition 1.1. [1, 2] *Let X be an nonempty set. A real-valued function $d\colon X \times X \to [0, +\infty)$ is called a b-metric on X with constant $s \geq 1$ if the next axioms are fulfilled:*

1) $d(x, y) = 0$ if $x = y$,
2) $d(x, y) = d(y, x)$ for all $x, y \in X$,
3) $d(x, y) \leq s(d(x, a) + d(a, y))$ for each $x, y, a \in X$.
The pair (X, d) is called a b-metric space.

Note that every metric space is a b-metric space with base 1. Some authors established some b-metric spaces which is not metric spaces.

Following Boriceanu et al. [7], we have the following definitions and remarks:

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Definition 1.2. [7] *Let (X, d) be a b-metric space. Then a sequence (x_n) in X is called:*

(1) *convergent if and only if there exists $x \in X$ such that $d(x_n, x) \to 0$ as $n \to +\infty$. In this case, we write $\lim_{n\to+\infty} x_n = x$.*
(2) *Cauchy if and only if $d(x_n, x_m) \to 0$ as $m, n \to +\infty$.*

Definition 1.3. [7] *The b-metric space (X, d) is complete if every Cauchy sequence in X converges.*

The following remark is crucial in our subsequent arguments.

Remark. [7] *In a b-metric space (X, d) the following assertions hold:*

(1) *a convergent sequence has a unique limit;*
(2) *each convergent sequence is Cauchy;*
(3) *in general, a b-metric is not continuous.*

Definition 1.4. [7] *Let (X, d) and (X', d') be b-metric spaces with constant s and s' respectively. Then the mapping $T : X \to X'$ is said to be continuous if whenever (x_n) is a sequence in X converges to some $x \in X$ with respect to d, then (Tx_n) converges to Tx with respect to d'.*

2. Main Results

We start our work by introducing the following definitions.

Definition 2.1. *Let s be a constant with $s \geq 1$. A map $\phi\colon [0, +\infty) \to [0, +\infty)$ is called a c-comparison function with base s if ϕ satisfies the following:*

(i) *ϕ is monotone nondecreasing,*
(ii) *$\sum_{n=0}^{\infty} s^n \phi^n(st)$ converges for all $t \geq 0$,*

Remark. *If ϕ is a c-comparison function with base s, then $\phi(t) < t$ for all $t > 0$.*

Theorem 2.1. *Let (X, d) be a complete b-metric space with constant $s \geq 1$ and $T, S\colon X \to X$ be two mappings. suppose that there exist a c-comparison function ϕ with base s such that*

$$d(Tx, Sy) \leq \frac{1}{s}\phi\left(\max\left\{ sd(x, y), sd(x, Tx), sd(y, Sy), \frac{1}{2}(d(x, Sy) + d(Tx, y)) \right\}\right) \tag{2.1}$$

holds for all $x, y \in X$. If ϕ is a continuous function, then T and S have a unique common fixed point.

Proof. Since X is nonempty, we start with $x_0 \in X$ and define a sequence (x_n) in X inductively by putting $x_{2n+1} = Tx_{2n}$ and $x_{2n+2} = Sx_{2n+1}$. Suppose that $x_n = x_{n+1}$ for some $n \in \mathbb{N}$. Without loss of generality, we assume that $n = 2k$ for

some $k \in \mathbb{N}$. By (2.1), we have

$$\begin{aligned}
& sd(x_{2k+1}, x_{2k+2}) \\
= \; & sd(Tx_{2k}, Sx_{2k+1}) \\
\leq \; & \phi\Bigg(\max\Bigg\{ sd(x_{2k}, x_{2k+1}), sd(x_{2k}, Tx_{2k}), sd(x_{2k+1}, Sx_{2k+1}), \\
& \frac{1}{2}(d(x_{2k}, Sx_{2k+1}) + d(Tx_{2k}, x_{2k+1}))\Bigg\}\Bigg) \\
= \; & \phi\Bigg(\max\Bigg\{ sd(x_{2k+1}, x_{2k+2}), \frac{1}{2}d(x_{2k}, x_{2k+2})\Bigg\}\Bigg) \\
\leq \; & \phi\Bigg(\max\Bigg\{ sd(x_{2k+1}, x_{2k+2}), \frac{s}{2}(d(x_{2k}, x_{2k+1}) + d(x_{2k+1}, x_{2k+2}))\Bigg\}\Bigg) \\
\leq \; & \phi(sd(x_{2k+1}, x_{2k+2})).
\end{aligned}$$

Using the fact that $\phi(t) < t$ for all $t > 0$, we conclude that $sd(x_{2k+1}, x_{2k+2}) = 0$ and hence $x_{2k} = x_{2k+1} = x_{2k+2}$. Therefore $x_{2k} = Tx_{2k} = Sx_{2k}$. Therefore x_k is a fixed point of T and S. Thus, we may assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. For $n \in \mathbb{N}$, we study the following cases.

Case 1: n is even. Here $n = 2t$ for some $t \in \mathbb{N}$. By (2.1), we have

$$\begin{aligned}
& sd(x_{2t}, x_{2t+1}) \\
= \; & sd(x_{2t+1}, x_{2t}) \\
= \; & sd(Tx_{2t}, Sx_{2t-1}) \\
\leq \; & \phi(\max\{sd(x_{2t}, x_{2t-1}), sd(x_{2t}, Tx_{2t}), sd(x_{2t-1}, Sx_{2t-1}), \\
& \frac{1}{2}(d(x_{2t}, Sx_{2t-1}) + d(Tx_{2t}, x_{2t-1}))\}) \\
= \; & \phi\Bigg(\max\Bigg\{ sd(x_{2t}, x_{2t-1}), sd(x_{2t}, x_{2t+1}), \frac{1}{2}d(x_{2t+1}, x_{2t-1})\Bigg\}\Bigg) \\
\leq \; & \phi\Bigg(\max\Bigg\{ sd(x_{2t}, x_{2t-1}), sd(x_{2t}, x_{2t+1}), \frac{s}{2}(d(x_{2t-1}, x_{2t}) + d(x_{2t}, x_{2t+1}))\Bigg\}\Bigg) \\
= \; & \phi(\max\{sd(x_{2t}, x_{2t-1}), sd(x_{2t}, x_{2t+1})\})
\end{aligned}$$

If $\max\{sd(x_{2t}, x_{2t-1}), sd(x_{2t}, x_{2t+1})\} = sd(x_{2t}, x_{2t+1})$, then

$$sd(x_{2t}, x_{2t+1}) \leq \phi(sd(x_{2t}, x_{2t+1})) < sd(x_{2t}, x_{2t+1})$$

which is a contradiction. Thus

$$\max\{sd(x_{2t}, x_{2t-1}), sd(x_{2t}, x_{2t+1})\} = sd(x_{2t}, x_{2t-1}).$$

Therefore

$$sd(x_{2t}, x_{2t+1}) \leq \phi(sd(x_{2t}, x_{2t-1})). \tag{2.2}$$

Case 2: n is odd. Here $n = 2t + 1$ for some $t \in \mathbb{N} \cup \{0\}$. Using similar arguments as those given in Case 1 , we can show that

$$sd(x_{2t+1}, x_{2t+2}) \leq \phi(sd(x_{2t}, x_{2t+1})). \tag{2.3}$$

Combining (2.2) and (2.3) together, we arrive to

$$sd(x_n, x_{n+1}) \leq \phi(sd(x_{n-1}, x_n)). \tag{2.4}$$

From (2.4), we have

$$\begin{aligned} sd(x_n, x_{n+1}) &\leq \phi(sd(x_{n-1}, x_n)) \\ &\leq \phi^2(sd(x_{n-2}, x_{n-1})) \\ &\leq \phi^3(sd(x_{n-3}, x_{n-2})) \\ &\vdots \\ &\leq \phi^n(sd(x_0, x_1)). \end{aligned} \tag{2.5}$$

Given $n, m \in \mathbb{N}$ with $m > n$. By the definition of b-metric and (2.5), we have

$$\begin{aligned} & d(x_n, x_m) \\ \leq\ & sd(x_n, x_{n+1}) + sd(x_{n+1}, x_m) \\ \leq\ & sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + s^2 d(x_{n+2}, x_m) \\ \vdots\ & \\ \leq\ & sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + s^3 d(x_{n+2}, x_{n+3}) + \cdots + s^{m-n} d(x_{m-1}, x_m) \\ \leq\ & \phi^n(sd(x_0, x_1)) + s\phi^{n+1}(sd(x_0, x_1)) + s^2\phi^{n+2}(sd(x_0, x_1)) + \cdots + s^{m-n-1}\phi^{m-1}(sd(x_0, x_1). \end{aligned}$$

Using the fact that $s \geq 1$, we get

$$d(x_n, x_m) \leq \sum_{j=n}^{+\infty} s^j \phi^j (sd(x_0, x_1)).$$

Since ϕ is (c)-comparison with base s, we have

$$\lim_{n,m \to +\infty} d(x_n, x_m) = 0. \tag{2.6}$$

Thus (x_n) is Cauchy in the complete b-metric space (X, d). So there exists $u \in X$ such that $x_n \to u$ as $n \to +\infty$.

Since

$$d(x_{2n+1}, x_{2n}) \leq s(d(x_{2n+1}, u) + d(u, x_{2n}))$$

we have $\lim_{n \to +\infty} d(x_{2n+1}, x_{2n}) = 0$. Now assume that $d(u, Su) \neq 0$. Using (2.1), we have

$$\begin{aligned} & sd(x_{2n+1}, Su) \\ =\ & sd(Tx_{2n}, Su) \\ =\ & \phi\Bigg(\max\left\{ sd(x_{2n}, u), sd(x_{2n}, x_{2n+1}), sd(u, Su), \frac{1}{2}(d(x_{2n+1}, u) + d(x_{2n}, Su)) \right\}\Bigg) \\ \leq\ & \phi\Bigg(\max\Bigg\{ sd(x_{2n}, u), sd(x_{2n}, x_{2n+1}), sd(u, x_{2n+1}) + sd(x_{2n+1}, Su), \\ & \frac{1}{2}(d(x_{2n+1}, u) + sd(x_{2n}, x_{2n+1}) + sd(x_{2n+1}, Su)) \Bigg\}\Bigg). \end{aligned}$$

Taking the limit sup and use the continuity of ϕ, we get the following

$$\limsup_{n \to +\infty} sd(x_{2n+1}, Su) \leq \phi(\limsup_{n \to +\infty} sd(x_{2n+1}, Su)).$$

By the properties of ϕ, we get

$$\limsup_{n \to +\infty} d(x_{2n+1}, Su) = 0$$

and hence

$$\lim_{n\to+\infty} d(x_{2n+1}, Su) = .0$$

On letting $n \to +\infty$ in

$$d(u, Su) \leq sd(u, x_{2n+1}) + sd(x_{2n+1}, Su),$$

we get $d(u, Su) = 0$ and hence $Su = u$. From (2.1), we have

$$\begin{aligned} sd(Tu, u) &= sd(Tu, Su) \\ &\leq \phi(\max\{sd(u, Tu), sd(u, Su), \frac{1}{2}(d(Tu, u) + d(u, Su))\}) \\ &= \phi(sd(u, Tu)). \end{aligned}$$

By using the properties of ϕ, we get $sd(u, Tu) = 0$ and hence $u = Tu$. Thus u is a common fixed point of T and S.

□

By taking $S = T$ in Theorem 2.1, we have the following results.

Corollary 2.2. *Let (X, d) be a complete b-metric space with constant $s \geq 1$ and $T\colon X \to X$ be a mapping. Suppose that there exist a c-comparison function ϕ with base s such that*

$$d(Tx, Ty) \leq \frac{1}{s}\phi\left(\max\left\{sd(x, y), sd(x, Tx), sd(y, Ty), \frac{1}{2}(d(x, Ty) + d(Tx, y))\right\}\right)$$

holds for all $x, y \in X$. If ϕ is continuous, then T has a unique fixed point.

Corollary 2.3. *Let (X, d) be a complete b-metric space with constant $s \geq 1$ and $T, S\colon X \to X$ be two mappings. Suppose that there exist $a > 0$ such that*

$$d(Tx, Sy) \leq \frac{1}{s+a}\max\left\{d(x, y), d(x, Tx), d(y, Sy), \frac{1}{2s}(d(x, Sy) + d(Tx, y))\right\}$$

holds for all $x, y \in X$. Then T and S have a unique common fixed point.

Proof. Define $\phi : [0, +\infty) \to [0, +\infty)$ by $\phi(t) = \frac{1}{s+a}t$, then ϕ is a continuous c-comparison function with constant s. Note that ϕ satisfies the following inequality

$$d(Tx, Sy) \leq \frac{1}{s}\phi\left(\max\left\{sd(x, y), sd(x, Tx), sd(y, Sy), \frac{1}{2}(d(x, Sy) + d(Tx, y))\right\}\right)$$

for all $x, y \in X$. So that T and S satisfy all the hypotheses of Theorem 2.1. Thus T and S have a unique common fixed point. □

Corollary 2.4. *Let (X, d) be a complete b-metric space with constant $s \geq 1$ and $T, S\colon X \to X$ be two mappings. Suppose the following inequality*

$$d(Tx, Sy) \leq \frac{1}{s^2}\max\left\{d(x, y), d(x, Tx), d(y, Sy), \frac{1}{2s}(d(x, Sy) + d(Tx, y))\right\}$$

holds for all $x, y \in X$. Then T and S have a unique common fixed point.

Proof. Take $a = s^2 - s$. Then $a > 0$ and $s^2 = a + s$. Hence the result follows from Corollary 2.3. □

If we take $S = T$ in Corollary 2.3, we get the following result.

Corollary 2.5. *Let (X, d) be a complete b-metric space with constant $s \geq 1$ and $T, S\colon X \to X$ be two mappings. Suppose that there exist $a > 0$ such that*

$$d(Tx, Ty) \leq \frac{1}{s+a} \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2s}(d(x,Ty) + d(Tx,y))\right\}$$

holds for all $x, y \in X$. Then T has a unique fixed point.

The continuity of ϕ in Theorem 2.1 can be dropped. For this instance, we will assume that the mapping T or S is continuous.

Theorem 2.6. *Let (X, d) be a complete b-metric space with constant $s \geq 1$ and $T, S\colon X \to X$ be two mappings. Suppose that there exist a c-comparison function ϕ with base s such that*

$$d(Tx, Sy) \leq \frac{1}{s}\phi\left(\max\left\{sd(x,y), sd(x,Tx), d(y,Sy), \frac{1}{2}(d(x,Sy) + d(Tx,y))\right\}\right) \tag{2.7}$$

holds for all $x, y \in X$. If T or S is continuous, then T and S have a unique common fixed point.

Proof. Using the same method in the proof of Theorem 2.1 we construct a Cauchy sequence (x_n) in the b-metric space (X, d). Since X is complete, then there is $u \in X$ such that $x_n \to u$ as $n \to +\infty$. Now, we prove that $Tu = Su = u$. It is an easy matter to realize that $Tx_{2n} = x_{2n+1} \to u$ and $Sx_{2n+1} = x_{2n+2} \to u$ as $n \to +\infty$. Now, we prove that $Su = u$ and $Tu = u$. Without loss of generality, we may assume that$S$ is continuous. Since $x_{2n+1} \to u$ and S is continuous, we get that $x_{2n+2} = Sx_{2n+1} \to Su$. By uniqueness of limit, we conclude that $Su = u$. To prove that $Tu = u$, suppose to the contrary. By (2.7), we have

$$\begin{aligned} sd(Tu, u) &= sd(Tu, Su) \\ &\leq \phi(\max\{sd(u,Tu), sd(u,Su), \frac{1}{2}(d(Tu,u) + d(u,Su))\}) \\ &= \phi(sd(u,Tu)). \end{aligned}$$

Since $\phi(sd(u,Tu)) < sd(u,Tu)$, we arrive to a contradiction. Thus u is a common fixed point of T and S. To prove the uniqueness of a common fixed point of T and S, we assume that T and S have two different common fixed point, say u and v. By (2.7), we have

$$\begin{aligned} d(u, v) &= d(Tu, Sv) \\ &\leq \frac{1}{s}\phi\left(\max\{sd(u,v), sd(u,Tu), sd(v,Sv), \frac{1}{2}(d(u,Sv) + d(Tu,v))\}\right) \\ &\leq \frac{1}{s}\phi(sd(u,v)). \end{aligned}$$

Using the property of ϕ, we get $d(u, v) = 0$ and hence $u = v$.

□

By taking $S = T$ in Theorem 2.6, we have the following results.

Corollary 2.7. *Let (X,d) be a complete b-metric space with constant s and $T\colon X \to X$ be a mapping. suppose that there exist a c-comparison function ϕ with base s such that*

$$d(Tx,Ty) \leq \frac{1}{s}\phi\Big(\max\Big\{sd(x,y), sd(x,Tx), d(y,Ty), \frac{1}{2}(d(x,Ty)+d(Tx,y))\Big\}\Big)$$

holds for all $x,y \in X$. If T is continuous, then T has a unique fixed point.

Corollary 2.8. *Let (X,d) be a complete b-metric space with constant $s \geq 1$ and $T,S\colon X \to X$ be two mappings. Suppose that there exist a c-comparison function ϕ with base s such that*

$$d(Tx,Sy) \leq \frac{1}{s}\phi\Big(\max\Big\{d(x,y), d(x,Tx), d(y,Sy), \frac{1}{2}(d(x,Sy)+d(Tx,y))\Big\}\Big)$$

holds for all $x,y \in X$. If ϕ is a continuous function, then T and S have a unique common fixed point.

Proof. Follows from Theorem 2.1 and the fact that

$$\begin{aligned} & \frac{1}{s}\phi\Big(\max\Big\{d(x,y), d(x,Tx), d(y,Sy), \frac{1}{2}(d(x,Sy)+d(Tx,y))\Big\}\Big) \\ \leq\ & \frac{1}{s}\phi\Big(\max\Big\{sd(x,y), sd(x,Tx), sd(y,Sy), \frac{1}{2}(d(x,Sy)+d(Tx,y))\Big\}\Big). \end{aligned}$$

□

Corollary 2.9. *Let (X,d) be a complete b-metric space with constant $s \geq 1$ and $T,S\colon X \to X$ be two mappings. Suppose that there exist a c-comparison function ϕ with base s such that*

$$d(Tx,Sy) \leq \frac{1}{s}\phi\Big(\max\Big\{d(x,y), d(x,Tx), d(y,Sy), \frac{1}{2}(d(x,Sy)+d(Tx,y))\Big\}\Big)$$

holds for all $x,y \in X$. If T or S is continuous, then T and S have a unique common fixed point.

Proof. Follows from Theorem 2.6 and the fact that

$$\begin{aligned} & \frac{1}{s}\phi\Big(\max\Big\{d(x,y), d(x,Tx), d(y,Sy), \frac{1}{2}(d(x,Sy)+d(Tx,y))\Big\}\Big) \\ \leq\ & \frac{1}{s}\phi\Big(\max\Big\{sd(x,y), sd(x,Tx), sd(y,Sy), \frac{1}{2}(d(x,Sy)+d(Tx,y))\Big\}\Big). \end{aligned}$$

□

Taking $S = T$ in Corollary 2.8 and Corollary 2.9, we get the following results.

Corollary 2.10. *Let (X,d) be a complete b-metric space with constant $s \geq 1$ and $T,S\colon X \to X$ be two mappings. Suppose that there exist a c-comparison function ϕ with base s such that*

$$d(Tx,Ty) \leq \frac{1}{s}\phi\Big(\max\Big\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2}(d(x,Ty)+d(Tx,y))\Big\}\Big)$$

holds for all $x, y \in X$. If ϕ is a continuous function, then T has a unique common fixed point.

Corollary 2.11. *Let (X, d) be a complete b-metric space with constant $s \geq 1$ and $T\colon X \to X$ be two mappings. Suppose that there exist a c-comparison function ϕ with base s such that*

$$d(Tx, Ty) \leq \frac{1}{s}\phi\left(\max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2}(d(x,Ty) + d(Tx,y))\right\}\right)$$

holds for all $x, y \in X$. If T is continuous mapping, then T has a unique common fixed point.

3. EXAMPLES

In this section, we introduce some examples to show the validity of our main results.

Example 3.1. *Let $X = [0, +\infty)$. Consider the complete b-metric space $d : X \times X \to [0, +\infty)$, $d(x, y) = (x - y)^2$ with constant $s = 2$. Define the mappings $T, S : X \to X$ by $Tx = \frac{1}{3}x$ and $Sx = \frac{1}{6}x$. Also define $\phi : [0, +\infty) \to [0, +\infty)$ by $\phi(t) = \frac{1}{4}x$. Then*

(1) *ϕ is a continuous c-comparison function.*
(2) *T, S and ϕ satisfy the following inequality:*

$$d(Tx, Sy) \leq \frac{1}{s}\phi\left(\max\left\{sd(x,y), sd(x,Tx), sd(y,Sy), \frac{1}{2}(d(x,Sy) + d(Tx,y))\right\}\right),$$

for all $x, y \in X$.

Proof. It is clear that ϕ is a nondecreasing continuous function. Now, given $t \in [0, +\infty)$. Then, it is an easy matter to show that

$$\phi^n(st) = \phi^n(2t) = \frac{1}{4^n}(2t).$$

Thus

$$\sum_{n=0}^{\infty} s^n \phi^n(st) = \sum_{n=0}^{\infty} \frac{2^n}{4^n}(2t) = 2t \sum_{n=0}^{\infty} \frac{1}{2^n} < +\infty.$$

So ϕ is a continuous c-comparison function. This prove (1).

To prove (2), let $x, y \in X$. Then

$$d(Tx, Sy) = d\left(\frac{1}{3}x, \frac{1}{6}y\right) = \left(\frac{1}{3}x - \frac{1}{6}y\right)^2 = \frac{1}{9}\left(x - \frac{1}{2}y\right)^2.$$

Consider the following cases:

Case I: $x = \frac{1}{2}y$. Here, we have

$$d(Tx, Sy) = 0 \leq \frac{1}{s}\phi\left(\max\left\{sd(x,y), sd(x,Tx), sd(y,Sy), \frac{1}{2}(d(x,Sy) + d(Tx,y))\right\}\right).$$

Case II: $x > \frac{1}{2}y$. Here, we have

$$\begin{aligned}
d(Tx, Sy) &= \frac{1}{9}\left(x - \frac{1}{2}y\right)^2 \leq \frac{x^2}{6} \\
&= \frac{1}{2}(2)\left(\frac{2}{3}x\right)^2\left(\frac{1}{4}\right) \\
&= \frac{1}{2}\phi\left(2(x - \frac{1}{3}x)^2\right) \\
&= \frac{1}{2}\phi\left(2d\left(x, \frac{1}{3}x\right)\right) \\
&= \frac{1}{s}\phi(sd(x, Tx)) \\
&\leq \frac{1}{s}\phi\left(\max\left\{sd(x,y), sd(x,Tx), sd(y,Sy), \frac{1}{2}(d(x,Sy) + d(Tx,y))\right\}\right).
\end{aligned}$$

Case III: $x < \frac{1}{2}y$. Here, we have

$$\begin{aligned}
d(Tx, Sy) &= \frac{1}{9}\left(x - \frac{1}{2}y\right)^2 \leq \frac{y^2}{36} \\
&\leq \left(\frac{25}{36}\right)\left(\frac{y^2}{4}\right) \\
&= \frac{1}{2}\phi\left(2\left(\frac{25}{36}\right)y^2\right) \\
&= \frac{1}{2}\phi\left(2(y - \frac{1}{6}y)^2\right) \\
&= \frac{1}{2}\phi\left(2d\left(y, \frac{1}{6}y\right)\right) \\
&= \frac{1}{s}\phi(sd(y, Sx)) \\
&\leq \frac{1}{s}\phi\left(\max\left\{sd(x,y), sd(x,Tx), sd(y,Sy), \frac{1}{2}(d(x,Sy) + d(Tx,y))\right\}\right).
\end{aligned}$$

Note that T, S and ϕ satisfy all the hypotheses of Theorem 2.1. So T and S have a unique common fixed point. □

Example 3.2. *Let* $X = \{1, 2, 3, \ldots, 100\}$. *Define* $d : X \times X \to [0, +\infty)$ *by*

$$d(x,y) = \begin{cases} 0, & \text{if } x = y \\ \left|\frac{1}{x} - \frac{1}{y}\right|, & \text{if } x \text{ and } y \text{ are even with } x \neq y \\ 5, & \text{if } x \text{ and } y \text{ are odd with } x \neq y \\ 2, & \text{other wise.} \end{cases}$$

Also, define the two mappings $T, S : X \to X$ *by*

$$Tx = \begin{cases} 2, & \text{if } x \text{ is odd;} \\ 4, & \text{if } x \text{ is even} \end{cases}$$

and

$$Sx = \begin{cases} 6, & \text{if } x \text{ is odd;} \\ 4, & \text{if } x \text{ is even} \end{cases}$$

Also, define $\phi : [0, +\infty) \to [0, +\infty)$ *by*

$$\phi(t) = \begin{cases} \frac{1}{2}t, & \text{if } t \geq 1 \\ 0, & \text{if } t < 1. \end{cases}$$

Then:

(1) (X, d) *is a complete b-metric space with* $s = \frac{5}{4}$.
(2) ϕ *is a noncontinuous c-comparison function.*
(3) T *is continuous.*
(4) T, S *and* ϕ *satisfy the following inequality:*

$$d(Tx, Sy) \leq \frac{1}{s}\phi\left(\max\left\{ sd(x, y), sd(x, Tx), sd(y, Sy), \frac{1}{2}(d(x, Sy) + d(Tx, y)) \right\}\right),$$

for all $x, y \in X$.

Proof. With a simple technique, one can easily prove (1). Note that ϕ is nondecreasing. Now, given $t \in [0, +\infty)$. Then

$$\phi^n(st) = \phi^n\left(\frac{5}{4}t\right) \leq \frac{5t}{(2^n)(4)}.$$

Hence

$$\sum_{n=0}^{\infty} s^n \phi^n(st) = \sum_{n=0}^{\infty} \frac{5^n}{4^n}\phi^n\left(\frac{5}{4}t\right) \leq \frac{5t}{4}\sum_{n=0}^{\infty}\left(\frac{5}{8}\right)^n < +\infty.$$

Thus ϕ is a c-comparison function. Also, Since $1 - \frac{1}{n} \to 1$, and $(\phi(1 - \frac{1}{n}))$ doesn't converge to $\phi(1)$, we conclude that ϕ is not continuous. This prove (2). To prove (3), let (x_n) be a sequence in X such that $x_n \to x$ for some $x \in X$. Then $x_n = x$ for all but finitely many. So $Tx_n = Tx$ for all but finitely many. So $d(Tx_n, Tx) = 0$ for all but finitely many. Hence $Tx_n \to Tx$. So T is continuous.

To prove (4), given $x, y \in X$. We study the following cases:

Case I: x is odd and y is odd. Here, we have $Tx = 2$ and $Sy = 6$. Thus

$$d(Tx, Sy) = d(2, 6) = \frac{1}{3} \leq 2.5 = \left(\frac{4}{5}\right)\left(\frac{25}{4}\right)\left(\frac{1}{2}\right)$$

$$= \frac{4}{5}\phi\left(\frac{25}{4}\right) = \frac{4}{5}\phi\left(\frac{5}{4}d(x, y)\right) = \frac{1}{s}\phi(sd(x, y)).$$

Thus

$$d(Tx, Sy) \leq \frac{1}{s}\phi\left(\max\left\{ sd(x, y), sd(x, Tx), sd(y, Sy), \frac{1}{2}(d(x, Sy) + d(Tx, y)) \right\}\right).$$

Case II: x is odd and y is even. Here, we have $Tx = 2$ and $Sy = 4$. Thus

$$d(Tx, Sy) = d(2, 4) = \frac{1}{4} \leq 1 = \left(\frac{4}{5}\right)\left(\frac{10}{4}\right)\left(\frac{1}{2}\right)$$

$$= \frac{4}{5}\phi\left(\frac{10}{4}\right) = \frac{4}{5}\phi\left(\frac{5}{4}d(x, y)\right) = \frac{1}{s}\phi(sd(x, y)).$$

Thus

$$d(Tx, Sy) \leq \frac{1}{s}\phi\left(\max\left\{ sd(x,y), sd(x,Tx), sd(y,Sy), \frac{1}{2}(d(x,Sy) + d(Tx,y)) \right\}\right).$$

Case III: x is even and y is odd. Here, we have $Tx = 4$ and $Sy = 6$. Thus

$$d(Tx, Sy) = d(4,6) = \frac{1}{12} \leq 1 = \left(\frac{4}{5}\right)\left(\frac{10}{4}\right)\left(\frac{1}{2}\right)$$

$$= \frac{4}{5}\phi\left(\frac{10}{4}\right) = \frac{4}{5}\phi\left(\frac{5}{4}d(x,y)\right) = \frac{1}{s}\phi(sd(x,y)).$$

Thus

$$d(Tx, Sy) \leq \frac{1}{s}\phi\left(\max\left\{ sd(x,y), sd(x,Tx), sd(y,Sy), \frac{1}{2}(d(x,Sy) + d(Tx,y)) \right\}\right).$$

Case IV: x and y are even. Here $Tx = Sy = 4$. Thus $d(Tx, Sy) = 0$. Therefore

$$d(Tx, Sy) \leq \frac{1}{s}\phi\left(\max\left\{ sd(x,y), sd(x,Tx), sd(y,Sy), \frac{1}{2}(d(x,Sy) + d(Tx,y)) \right\}\right).$$

So T, S and ϕ satisfy all the hypotheses of Theorem 2.6. So T and S have a unique common fixed point. □

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