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CONTROL PROBLEMS WITH KUHN-TUCKER AND FRITZ JOHN GENERALIZED INVEXITY

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Abstract. KT-ρ-(η, ξ, θ)-invexity and FJ-ρ-(η, ξ, θ)-invexity are defined on the functionals of a control problem and considered a fresh characterization result of these conditions. Also prove the KT-ρ-(η, ξ, θ)-invexity and FJ-ρ-(η, ξ, θ)-invexity are both necessary and sufficient in order to characterize the optimal solution set using Kuhn-Tucker and Fritz John points.

1. Introduction

Mond and Smart [16] proved that invexity of functional is necessary and sufficient for its critical points to be global minimum, which coincides with the original concept of invex function [11]. Arana *et al.* [2] provided a class of functionals, called KT-invex and showed that Kuhn-Tucker points being optimal solutions for the control problem. Again Arana *et al.* [3] extended this result for new weaker conditions on the involved functionals, that is FJ-invexity. Giorgi [10] described numerous results and remarks towards the necessary optimality conditions of Fritz John type for a nonlinear programming problem with inequality and equality constraints. Husain and Srivastav [13] obtained sufficient Fritz John optimality conditions for a control problem, where the objective functional was pseudo-convex and constraint functions were quasi-convex or semi-strictly quasi-convex. Also established dual to the control problem using Fritz John type optimality criteria instead of Karush-Kuhn-Tucker optimality criteria and prove that it did not require a regularity condition. Flores-Bazán [9] established an alternative-type version of Fritz John optimality conditions at points not necessarily optimal, which covers situations where no result appearing elsewhere was appearing. Also presented a variant of the KKT conditions. Slimani and Radjef [25] introduced a generalized Fritz John condition which was the necessary and sufficient conditions for a feasible point to be an optimal solution under weak invexity. Considering order relationship between two closed intervals in real numbers, Singh *et al.* [24] developed a theoretical as well as practical solution method for multiobjective programming problems with interval valued objective functions. They obtained Karush-Kuhn-Tucker optimality

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for this problem under assumptions of generalized convexity and differentiability of interval valued functions. Recently, Pandey and Mishra [17] proposed the concept of M-stationary point for the nonsmooth multiobjective semi-infinite mathematical programming problems with equilibrium constraints in terms of the Clarke subdifferentials. Again they showed that the M-stationary conditions were strong KKT type sufficient optimality conditions for the nonsmooth multiobjective semi-infinite mathematical programming problems with equilibrium constraints under generalized invexity assumptions. Emphasizing possible applications to applied sciences, Pitea *et al.* studied some classes of multiobjective optimization problems by means of several classes of generalized convexity in a geometric framework; please, see [1], and [20, 21, 22, 23].

In the present investigation, we extend the results of Arana *et al.* [2, 3] with more weaker conditions on the functionals involved in it, that is KT-ρ-(η, ξ, θ)-invexity and FJ-ρ-(η, ξ, θ)-invexity. These type of problems are applied to engineering problems, like the control design for autonomous vechiles or impulsive control problems [18, 19], optimal control of static elastoplasticity [4], electrical power production [6], economy [14], medicine [26], ecology [15], computer integrated manufacturing Robotics [8], wavelet analysis [12] etc.

2. Preliminaries

Consider the following control problem (CP):

$$\text{(CP)} \quad H(x,u) = \int_a^b f(t,x,u)dt$$

$$x(a) = \alpha, \quad x(b) = \beta, \tag{2.1}$$

$$g(t,x,u) \le 0, \tag{2.2}$$

$$h(t,x,u) = \dot{x}, \quad t \in I. \tag{2.3}$$

Here $I = [a,b]$ is a real interval, while $f\colon I \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, $g\colon I \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^k$ and $h\colon I \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ are continuously differential functions. The partial derivatives of f are denoted by

$$f_t = \frac{\partial f}{\partial t}, \quad f_x = \Big(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n}\Big), \quad f_u = \Big(\frac{\partial f}{\partial u_1}, \frac{\partial f}{\partial u_2}, \dots, \frac{\partial f}{\partial u_m}\Big).$$

Similarly, the partial derivatives of g and h can be denoted by $g_t,\ g_x,\ g_u$ and $h_t,\ h_x,\ h_u$. X is the space of piecewise smooth state functions $x\colon I \to \mathbb{R}^n$ such that $x(a) = \alpha$ and $x(b) = \beta$ which is equipped with the norm $\|x\| = \|x\|_\infty + \|Dx\|_\infty$. Y is the space of piecewise continuous control functions $u\colon I \to \mathbb{R}^m$, endowed with the uniform norm $\|\cdot\|$. The feasible solutions of the control problem (CP) lies in the invex set

$$K = \{x \in X,\ y \in Y : x(a) = \alpha,\ x(b) = \beta,\ g(t,x,u) \le 0,\ h(t,x,u) - \dot{x} = 0,\ t \in I\}.$$

Definition 2.1. *A point $(\bar{x}, \bar{u}) \in K$ is said to be Kuhn-Tucker point if there exist piecewise smooth functions $\lambda\colon I \to \mathbb{R}^k$ and $\mu\colon I \to \mathbb{R}^n$ satisfying the following conditions.*

$$f_x(t,\bar{x},\bar{u}) + \lambda(t)^T g_x(t,\bar{x},\bar{u}) + \mu(t)^T h_x(t,\bar{x},\bar{u}) + \dot{\mu}(t) = 0, \tag{2.4}$$

$$f_u(t,\bar{x},\bar{u}) + \lambda(t)^T g_u(t,\bar{x},\bar{u}) + \mu(t)^T h_u(t,\bar{x},\bar{u}) = 0, \tag{2.5}$$

$$\lambda(t)^T g(t, \bar{x}, \bar{u}) = 0, \tag{2.6}$$

$$\lambda(t) \geq 0, \tag{2.7}$$

for all $t \in I$, except the discontinuities.

If the constraints (2.1) and (2.2) are removed from (CP), then Kuhn-Tucker points reduce to a critical points of f.

Theorem 2.2 ([2]). *If $(\bar{x}, \bar{u}) \in K$ is an optimal solution for* (CP) *and $(\bar{x}, \bar{u})$ is normal, then $(\bar{x}, \bar{u})$ is Kuhn-Tucker point.*

3. Main results

The following definition is required to extend the results of Arana *et al.* [2].

Definition 3.1. (CP) *is said to be* KT-ρ-(η, ξ, θ)*-invex at the point $(\bar{x}, \bar{u}) \in K$ if for all $(x, u) \in K$, and for all $\lambda \colon I \to \mathbb{R}^k$, which satisfies $\lambda(t)^T g(t, \bar{x}, \bar{u}) = 0$, $\lambda(t) \geq 0$, and $\mu \colon I \to \mathbb{R}^n$ piecewise smooth functions, there exist differentiable vector functions $\eta(t, \bar{x}, x, \bar{u}, u, \lambda, \mu)$ and $\xi(t, \bar{x}, x, \bar{u}, u, \lambda, \mu)$; and a vector function $\theta(t, \bar{x}, x, \bar{u}, u, \lambda, \mu)$ with $\rho\|\theta(t, \bar{x}, x, \bar{u}, u, \lambda, \mu)\|^2 - \mu(t)\eta(t, \bar{x}, x, \bar{u}, u, \lambda, \mu)\big|_a^b \geq 0$, $\rho \in \mathbb{R}$ such that*

$$\begin{aligned}
&H(x,u) - H(\bar{x},\bar{u}) < 0 \quad \Rightarrow \\
&\int_a^b \big[\big\{f_x(t,\bar{x},\bar{u}) + \lambda(t)^T g_x(t,\bar{x},\bar{u}) + \mu(t)^T h_x(t,\bar{x},\bar{u})\big\}\eta(t,\bar{x},x,\bar{u},u,\lambda,\mu) \\
&-\mu(t)^T \dot{\eta}(t,\bar{x},x,\bar{u},u,\lambda,\mu) \\
&+\big\{f_u(t,\bar{x},\bar{u}) + \lambda(t)^T g_u(t,\bar{x},\bar{u}) + \mu(t)^T h_u(t,\bar{x},\bar{u})\big\}\xi(t,\bar{x},x,\bar{u},u,\lambda,\mu)\big]dt \\
&+\rho\|\theta(t,\bar{x},x,\bar{u},u,\lambda,\mu)\|^2 < 0.
\end{aligned}$$

(CP) *is said to be* KT-ρ-(η, ξ, θ)*-invex if it is true for all $(\bar{x}, \bar{u}) \in K$.*

Remark that:
1) The norm used in the above is standard 2-norm.
2) Theorem (2.2) tells that Kuhn-Tucker optimality conditions are necessary for a feasible point of (CP) to be an optimal solution. Now the converse part is studied under KT-ρ-(η, ξ, θ)-invexity assumptions.

Theorem 3.2. *If* (CP) *is KT-ρ-(η, ξ, θ)-invex, then all Kuhn-Tucker points are optimal solutions for* (CP).

Proof. Given that (CP) is KT-ρ-(η, ξ, θ)-invex, let $(\bar{x}, \bar{u})$ be a Kuhn-Tucker point, i.e. there exist $\lambda \colon I \to \mathbb{R}^k$ and $\mu \colon I \to \mathbb{R}^n$ satisfying (2.4)-(2.7).
Now

$$\begin{aligned}
&\int_a^b \big[\big\{f_x(t,\bar{x},\bar{u}) + \lambda(t)^T g_x(t,\bar{x},\bar{u}) + \mu(t)^T h_x(t,\bar{x},\bar{u})\big\}\eta(t,\bar{x},x,\bar{u},u,\lambda,\mu) \\
&-\mu(t)^T \dot{\eta}(t,\bar{x},x,\bar{u},u,\lambda,\mu) \\
&+\big\{f_u(t,\bar{x},\bar{u}) + \lambda(t)^T g_u(t,\bar{x},\bar{u}) + \mu(t)^T h_u(t,\bar{x},\bar{u})\big\}\xi(t,\bar{x},x,\bar{u},u,\lambda,\mu)\big]dt \\
&+\rho\|\theta(t,\bar{x},x,\bar{u},u,\lambda,\mu)\|^2
\end{aligned}$$

$$\begin{aligned}
&= \int_a^b \big[\big\{f_x(t,\bar{x},\bar{u}) + \lambda(t)^T g_x(t,\bar{x},\bar{u}) + \mu(t)^T h_x(t,\bar{x},\bar{u}) + \dot{\mu}(t)\big\}\eta(t,\bar{x},x,\bar{u},u,\lambda,\mu) \\
&\quad + \big\{f_u(t,\bar{x},\bar{u}) + \lambda(t)^T g_u(t,\bar{x},\bar{u}) + \mu(t)^T h_u(t,\bar{x},\bar{u})\big\}\xi(t,\bar{x},x,\bar{u},u,\lambda,\mu)\big]dt \\
&\quad + \rho\|\theta(t,\bar{x},x,\bar{u},u,\lambda,\mu)\|^2 - \mu(t)\eta(t,\bar{x},x,\bar{u},u,\lambda,\mu)\big|_a^b \ \big(\text{by integration by parts}\big) \\
&= \rho\|\theta(t,\bar{x},x,\bar{u},u,\lambda,\mu)\|^2 - \mu(t)\eta(t,\bar{x},x,\bar{u},u,\lambda,\mu)\big|_a^b . \ \big(\text{using (2.4) and (2.5)}\big)
\end{aligned}$$

Since (CP) is KT-ρ-(η,ξ,θ)-invex, we have

$$H(x,u) - H(\bar{x},\bar{u}) \geq 0, \ \forall \ (x,u) \in K,$$

therefore $(\bar{x},\bar{u})$ is an optimal solution for (CP) and the proof is complete. □

KT-ρ-(η,ξ,θ)-invexity of (CP) is not only sufficient in order that a Kuhn-Tucker point be an optimal solution, but also a necessary condition which will be established in the next theorem.

Theorem 3.3. *If all Kuhn-Tucker points are optimal solution for* (CP) *then* (CP) *is* KT-ρ-(η,ξ,θ)*-invex.*

Proof. Let $(\bar{x},\bar{u}) \in K$ be a Kuhn-Tucker point, then there exist piecewise smooth functions $\lambda\colon I \to \mathbb{R}^k$ and $\mu\colon I \to \mathbb{R}^n$ such that conditions (2.4)-(2.7) are satisfied. Since all Kuhn-Tucker points are optimal solutions then, for all $(\bar{x},\bar{u}) \in K$,

$$H(x,u) - H(\bar{x},\bar{u}) \geq 0. \tag{3.1}$$

Suppose on the contrary (CP) is not KT-ρ-(η,ξ,θ)-invex, that is,

$$\begin{aligned}
&\int_a^b \big[\big\{f_x(t,\bar{x},\bar{u}) + \lambda(t)^T g_x(t,\bar{x},\bar{u}) + \mu(t)^T h_x(t,\bar{x},\bar{u})\big\}\eta(t,\bar{x},x,\bar{u},u,\lambda,\mu) \\
&-\mu(t)^T \dot{\eta}(t,\bar{x},x,\bar{u},u,\lambda,\mu) \\
&+\big\{f_u(t,\bar{x},\bar{u}) + \lambda(t)^T g_u(t,\bar{x},\bar{u}) + \mu(t)^T h_u(t,\bar{x},\bar{u})\big\}\xi(t,\bar{x},x,\bar{u},u,\lambda,\mu)\big]dt \\
&+\rho\|\theta(t,\bar{x},x,\bar{u},u,\lambda,\mu)\|^2 \geq 0
\end{aligned}$$

$$\Rightarrow H(x,u) - H(\bar{x},\bar{u}) \ngeq 0.$$

This contradicts to (3.1), hence (CP) is KT-ρ-(η,ξ,θ)-invex. □

Theorem 3.4 can be stated by combining Theorem 3.2 and Theorem 3.3.

Theorem 3.4. (CP) *is* KT-ρ-(η,ξ,θ)*-invex if and only if all Kuhn-Tucker points are optimal solutions for* (CP).

In order to state the following theorem with Fritz John type optimality conditions, Chandra *et al.* [5] needed the equality constraints to be locally solvable (see [7]).

For this purpose, they considered

$$Q'(\bar{x},\bar{u}) = [D - H_x(\bar{x},\bar{u}), -H_u(\bar{x},\bar{u})],$$

to be surjective; that is, it is necessary to assume that the differential equation

$$Dp(t) - h_x(t,\bar{x}(t),\bar{u}(t))p(t) - h_u(t,\bar{x}(t),\bar{u}(t))q(t) = z(t)$$

can be solved for piecewise smooth $p(\cdot)$ and piecewise continuous $q(\cdot)$, with the boundary conditions $p(a) = 0 = p(b)$, for any $z(\cdot)$.

Theorem 3.5. *If* $(\bar{x}, \bar{u}) \in K$ *is Fritz John point, then there exist* $\tau \in \mathbb{R}$ *and piecewise smooth functions* $\lambda\colon I \to \mathbb{R}^k$ *and* $\mu\colon I \to \mathbb{R}^n$ *such that*

$$\tau f_x(t,\bar{x},\bar{u}) + \lambda(t)^T g_x(t,\bar{x},\bar{u}) + \mu(t)^T h_x(t,\bar{x},\bar{u}) + \dot{\mu}(t) = 0, \tag{3.2}$$

$$\tau f_u(t,\bar{x},\bar{u}) + \lambda(t)^T g_u(t,\bar{x},\bar{u}) + \mu(t)^T h_u(t,\bar{x},\bar{u}) = 0, \tag{3.3}$$

$$\lambda(t)^T g(t,\bar{x},\bar{u}) = 0, \tag{3.4}$$

$$(\tau,\lambda(t)) \geq 0,\ \ (\tau,\lambda(t)) \neq 0, \tag{3.5}$$

for all $t \in I$*, except at discontinuities.*

The following definition is necessary to extend the results of Arana *et al.* [3].

Definition 3.6. *The control problem* (CP) *is said to be* FJ-ρ-(η,ξ,θ)*-invex at the point* $(\bar{x},\bar{u}) \in K$ *if for all* $(x,u) \in K$*, and for all* $\lambda\colon I \to \mathbb{R}^k$*, which satisfies* $\lambda(t)^T g(t,\bar{x},\bar{u}) = 0$*,* $(\tau,\lambda(t)) \geq 0$*, and* $\mu\colon I \to \mathbb{R}^n$ *piecewise smooth functions, there exist differentiable vector functions* $\eta(t,\bar{x},x,\bar{u},u,\tau,\lambda,\mu)$ *and* $\xi(t,\bar{x},x,\bar{u},u,\tau,\lambda,\mu)$*; and a vector function* $\theta(t,\bar{x},x,\bar{u},u,\tau,\lambda,\mu)$ *with*

$$\rho\|\theta(t,\bar{x},x,\bar{u},u,\tau,\lambda,\mu)\|^2 - \mu(t)\eta(t,\bar{x},x,\bar{u},u,\tau,\lambda,\mu)\Big|_a^b \geq 0,$$

$\rho \in \mathbb{R}$ *such that*

$$\begin{aligned}
&H(x,u) - H(\bar{x},\bar{u}) < 0 \quad \Rightarrow \\
&\int_a^b \big[\big\{\tau f_x(t,\bar{x},\bar{u}) + \lambda(t)^T g_x(t,\bar{x},\bar{u}) + \mu(t)^T h_x(t,\bar{x},\bar{u})\big\}\eta(t,\bar{x},x,\bar{u},u,\tau,\lambda,\mu) \\
&-\mu(t)^T \dot{\eta}(t,\bar{x},x,\bar{u},u,\tau,\lambda,\mu) \\
&+\big\{\tau f_u(t,\bar{x},\bar{u}) + \lambda(t)^T g_u(t,\bar{x},\bar{u}) + \mu(t)^T h_u(t,\bar{x},\bar{u})\big\}\xi(t,\bar{x},x,\bar{u},u,\tau,\lambda,\mu)\big]dt \\
&+\rho\|\theta(t,\bar{x},x,\bar{u},u,\tau,\lambda,\mu)\|^2 < 0.
\end{aligned}$$

The control problem (CP) *is said to be* FJ-ρ-(η,ξ,θ)*-invex if it is for all* $(\bar{x},\bar{u}) \in K$*.*

Theorem 3.7. (CP) *is* FJ-ρ-(η,ξ,θ)*-invex if and only if all Fritz John points are optimal solutions for* (CP)*.*

Proof. To prove the necessity, given that (CP) is FJ-ρ-(η,ξ,θ)-invex, let $(\bar{x},\bar{u})$ be a Fritz John point, i.e. there exist $\lambda\colon I \to \mathbb{R}^k$, $\mu\colon I \to \mathbb{R}^n$ and $\tau \in \mathbb{R}$ satisfying (3.2)-(3.5).

Now

$$\begin{aligned}
&\int_a^b \big[\big\{\tau f_x(t,\bar{x},\bar{u}) + \lambda(t)^T g_x(t,\bar{x},\bar{u}) + \mu(t)^T h_x(t,\bar{x},\bar{u})\big\}\eta(t,\bar{x},x,\bar{u},u,\tau,\lambda,\mu) \\
&-\mu(t)^T \dot{\eta}(t,\bar{x},x,\bar{u},u,\lambda,\mu) \\
&+\big\{\tau f_u(t,\bar{x},\bar{u}) + \lambda(t)^T g_u(t,\bar{x},\bar{u}) + \mu(t)^T h_u(t,\bar{x},\bar{u})\big\}\xi(t,\bar{x},x,\bar{u},u,\tau,\lambda,\mu)\big]dt \\
&+\rho\|\theta(t,\bar{x},x,\bar{u},u,\tau,\lambda,\mu)\|^2 \\
= &\int_a^b \big[\big\{\tau f_x(t,\bar{x},\bar{u}) + \lambda(t)^T g_x(t,\bar{x},\bar{u}) + \mu(t)^T h_x(t,\bar{x},\bar{u}) + \dot{\mu}(t)\big\}\eta(t,\bar{x},x,\bar{u},u,\tau,\lambda,\mu) \\
&+\big\{\tau f_u(t,\bar{x},\bar{u}) + \lambda(t)^T g_u(t,\bar{x},\bar{u}) + \mu(t)^T h_u(t,\bar{x},\bar{u})\big\}\xi(t,\bar{x},x,\bar{u},u,\tau,\lambda,\mu)\big]dt \\
&+\rho\|\theta(t,\bar{x},x,\bar{u},u,\tau,\lambda,\mu)\|^2 - \mu(t)\eta(t,\bar{x},x,\bar{u},u,\tau,\lambda,\mu)\Big|_a^b \ \big(\text{by integration by parts}\big) \\
= &\ \rho\|\theta(t,\bar{x},x,\bar{u},u,\tau,\lambda,\mu)\|^2 - \mu(t)\eta(t,\bar{x},x,\bar{u},u,\tau,\lambda,\mu)\Big|_a^b . \ \big(\text{using (3.2) and (3.3)}\big)
\end{aligned}$$

Since (CP) is FJ-ρ-(η,ξ,θ)-invex, we have

$$H(x,u) - H(\bar{x},\bar{u}) \geq 0,\ \forall\ (x,u) \in K,$$

therefore $(\bar{x},\bar{u})$ is an optimal solution for (CP).

To prove the converse, let $(\bar{x},\bar{u}) \in K$ be a Fritz John point, then there exist piecewise smooth functions $\lambda\colon I \to \mathbb{R}^k$, $\mu\colon I \to \mathbb{R}^n$ and $\tau \in \mathbb{R}$ such that conditions (3.2)-(3.5) are satisfied. Since all Fritz John points are optimal solutions then for all $(\bar{x},\bar{u}) \in K$,

$$H(x,u) - H(\bar{x},\bar{u}) \geq 0. \tag{3.6}$$

Suppose on the contrary that (CP) is not FJ-ρ-(η,ξ,θ)-invex, that is,

$$\begin{aligned} &\int_a^b \big[\big\{\tau f_x(t,\bar{x},\bar{u}) + \lambda(t)^T g_x(t,\bar{x},\bar{u}) + \mu(t)^T h_x(t,\bar{x},\bar{u})\big\}\eta(t,\bar{x},x,\bar{u},u,\tau,\lambda,\mu) \\ &-\mu(t)^T \dot{\eta}(t,\bar{x},x,\bar{u},u,\lambda,\mu) \\ &+\big\{\tau f_u(t,\bar{x},\bar{u}) + \lambda(t)^T g_u(t,\bar{x},\bar{u}) + \mu(t)^T h_u(t,\bar{x},\bar{u})\big\}\xi(t,\bar{x},x,\bar{u},u,\tau,\lambda,\mu)\big]dt \\ &+\rho\|\theta(t,\bar{x},x,\bar{u},u,\tau,\lambda,\mu)\|^2 \geq 0 \\ \Rightarrow\quad & H(x,u) - H(\bar{x},\bar{u}) \ngeq 0, \end{aligned}$$

which contradicts to (3.6), hence CP is FJ-ρ-(η,ξ,θ)-invex. □

4. Conclusion

In this paper, the conditions of KT-ρ-(η,ξ,θ)-invexity and FJ-ρ-(η,ξ,θ)-invexity of control problems are introduced. Also it is proved that KT-ρ-(η,ξ,θ)-invexity and FJ-ρ-(η,ξ,θ)-invexity are necessary and sufficient conditions for Kuhn-Tucker and Fritz John points respectively, to be an optimal solution of the control problem. This result extends the characterization result recently given by Arana *et al.* [2, 3]. Variational problems, control variational problems and symmetric variational problems with KT-ρ-(η,ξ,θ)-invexity and FJ-ρ-(η,ξ,θ)-invexity could be investigated further.

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