Journal of Mathematical Analysis
ISSN: 2217-3412, URL: http://ilirias.com/jma
Volume 7 Issue 3(2016), Pages 33-40.

ON CONSERVATIVE DISTANCE MAPS

K. ABODAYEH[1] AND W. SHATANAWI[1,2]

Abstract. In this paper, we introduce the notion of quasi-2-normed spaces with power r, and we prove that the Mazur-Ulam problem holds in linear quasi-2-normed spaces with power r. our result generalize the main result of Chu [3].

1. Introduction

A mapping T from a metric space X into a metric space Y is called an isometry map if T satisfies $d_Y(T(x), T(y)) = d_X(x, y)$ for all $x, y \in X$, where $d_X(.,.)$ and $d_Y(.,.)$ denote the metrics in the spaces X and Y, respectively. For some $r > 0$, suppose that T preserves the distance r ; that is, for all x, y in X with $d_X(x, y) = r$, we have $d_Y(T(x), T(y)) = r$. Then r is called a conservative (or preserved) distance for the mapping T. One of problems in analysis is whether the existence of a single conservative distance for some map T from X into Y implies that T is isometry. This problem is called the Aleksandrov problem. The Aleksandrov problem has been studied by many authors (see [1, 5, 7, 9, 10, 11]). The map T is called affine if T is linear up to translation. Mazur and Ulam [8], proved that every isometry T from a real normed space X onto another real normed space Y is affine. While Baker [2] proved that an isometry map from a real normed linear space X into a strictly convex real normed linear space Y is affine. Chu et al. [3], proved the Mazur-Ulam problem for a collinearity preserving mapping T in a linear 2-normed space. For related works we refer the reader to [1, 2, 3, 4, 5, 7, 8, 12, 13, 16]. Moreover, Alaca and others [17, 18]studied the Mazur-Ulam problem for different spaces. For more study about geometric nonlinear analysis, we refer to [19]. Recently Shatanawi and Postolache [12] proved Mazur-Ulam theorem for probabilistic 2-normed spaces. In this paper, we introduce the notion of a quasi-2-normed space X with power r and we prove the Mazur-Ulam problem for a quasi-2-normed space with power r.

2. Basic Concepts.

Definition 2.1. [14] *Let X be a real linear space with $\dim X > 1$ and $||.,.|| : X \times X \to \mathbf{R}$ a function. Then $(X, ||.,.||)$ is called a linear 2-normed space if:*

(1) $||x_1, x_2|| = 0$ *iff x_1 and x_2 are linearly dependent.*

2000 *Mathematics Subject Classification.* 46B20.

Key words and phrases. Isometry Maps, Mazur-Ulam theorem, 2-normed spaces, linearly dependent.

©2016 Universiteti i Prishtinës, Prishtinë, Kosovë.

Submitted April 5, 2016. Published May 12, 2016.

(2) $||x_1, x_2|| = ||x_2, x_1||$.
(3) $||\beta x_1, x_2|| = |\beta|\,||x_1, x_2||$.
(4) $||x_1 + x_2, y|| \leq ||x_1, y|| + ||x_2, y||$

for all $\beta \in \mathbf{R}$ *and* $y, x_1, x_2 \in X$. *The function* $||.,.||$ *is called a 2-norm function.*

Definition 2.2. [3] *Let* X *be a real linear space and* x, y, z *mutually disjoint elements of* X. *Then* x, y *and* z *are said to be collinear if*

$$y - z = t(x - z).$$

for some real number t.

Definition 2.3. [6] *Let* X *be a linear space over the real field* $\mathbf{R}$. *A real function* $||x||$ *of* x *is called a quasi-norm map with the power* r *if it satisfies the following conditions:*

(1) $||x + y|| \leq ||x|| + ||y||$
(2) $||\lambda x|| = |\lambda|^r ||x||$ *for* $\lambda \in \mathbf{R}$ *and* $x \in X$,$0 < r \leq 1$.
(3) $||x|| = 0$ *iff* $x = 0$.

3. Main Results.

To proceed in our work, and achieve our results we introduce the following definitions:

Definition 3.1. *Let* X *be a real linear space with* $\dim X \geq 2$ *and* $r \in (0, 1]$. *A real function*

$$||.,.|| : X \times X \to \mathbf{R}$$

is called quasi-2-norm with power r *if it satisfies the following conditions:*

(1) $||x_1, x_2|| = 0$ *iff* x_1 *and* x_2 *are linearly dependent,*
(2) $||x_1, x_2|| = ||x_2, x_1||$,
(3) $||\beta x_1, x_2|| = |\beta|^r\,||x_1, x_2||$, *and*
(4) $||x_1 + x_2, y|| \leq ||x_1, y|| + ||x_2, y||$

for all $\beta \in \mathbf{R}$ *and* $y, x_1, x_2 \in X$. *The space* $(X, ||.,.||)$ *is called a quasi-2-normed space with power* r.

Definition 3.2. *Let* X *and* Y *be linear quasi-2-normed spaces, and* α *be a non zero scalar. The map* $T : X \to Y$ *is called* α*-2-isometry if*

$$||T(x) - T(c), T(y) - T(c)|| = |\alpha|||x - c, y - c||$$

holds for all $x, y, c \in X$.

Now we introduce an example of a quasi-2-normed space which is not a 2-normed space.

Example 3.1. *Define* $||.,.|| : \mathbf{R^2} \times \mathbf{R^2} \to \mathbf{R}^+$ *by*

$$||(x_1, x_2), (y_1, y_2)|| = \sqrt[3]{|x_1 y_2 - x_2 y_1|}.$$

Then $(\mathbf{R^2}, ||.,.||)$ *is a quasi-2-normed space with power* $\frac{1}{3}$ *which is not a 2-normed space.*

From now on, unless otherwise stated, we let X and Y be linear quasi-2-normed spaces with $\dim(X) \geq 2$.
One can see the following lemma directly from the definition of quasi-2-normed spaces.

Lemma 3.1. *For $x_1, x_2 \in X$ and $\alpha \in \mathbf{R}$, we have $||x_1, \alpha x_1 + x_2|| = ||x_1, x_2||$.*

The following lemma is crucial in proving our next result.

Lemma 3.2. *Given two distinct elements x_1 and x_2 in X. Let*

$$u = \frac{t_1 x_1 + t_2 x_2}{t_1 + t_2}$$

for some scalars t_1, t_2 not all are 0. Then u satisfies the following inequalities:

(1) $||x_2 - u, x_2 - c|| = \frac{|t_1|^r}{|t_1+t_2|^r}||x_1 - c, x_2 - c||$, *and*
(2) $||x_1 - u, x_1 - c|| = \frac{|t_2|^r}{|t_1+t_2|^r}||x_1 - c, x_2 - c||$,

for $c \in X$ with $||x_1 - c, x_2 - c|| \neq 0$

Proof. To prove 1, choose $c \in X$ with $||x_1 - c, x_2 - c|| \neq 0$. So

$$\begin{aligned}
||x_2 - u, x_2 - c|| &= \left|\left|x_2 - \frac{t_1 x_1 + t_2 x_2}{t_1 + t_2}, x_2 - c\right|\right| \\
&= \left|\left|\frac{(t_1 + t_2)x_2 - (t_1 x_1 + t_2 x_2)}{t_1 + t_2}, x_2 - c\right|\right| \\
&= \frac{1}{|t_1 + t_2|^r}||t_1 x_2 - t_1 x_1, x_2 - c|| \\
&= \frac{1}{|t_1 + t_2|^r}||t_1 x_2 - t_1 c + t_1 c - t_1 x_1, x_2 - c|| \\
&= \frac{1}{|t_1 + t_2|^r}||t_1(x_2 - c) + t_1(c - x_1), x_2 - c|| \\
&= \frac{|t_1|^r}{|t_1 + t_2|^r}||x_1 - c, x_2 - c||.
\end{aligned}$$

Similarly we can proof 2. □

Our next main result is essential in proving our main result.

Theorem 3.1. *Given two distinct elements x_1 and x_2 in X. Let*

$$A = \left\{ \frac{t_1 x_1 + t_2 x_2}{t_1 + t_2} : t_1, t_2 \text{ are scalars not both are } 0 \right\}.$$

Then $u = \frac{x_1+x_2}{2}$ is the only unique element in A satisfying the following inequalities:

(1) $||x_1 - u, x_1 - c|| = \frac{1}{2^r}||x_1 - c, x_2 - c||$, *and*
(2) $||x_2 - u, x_2 - c|| = \frac{1}{2^r}||x_1 - c, x_2 - c||$

for $c \in X$ with $||x_1 - c, x_2 - c|| \neq 0$.

Proof. Choose $t_1 = 1$ and $t_2 = 1$ in Lemma 3.2,we get that u satisfies

(1) $||x_1 - u, x_1 - c|| = \frac{1}{2^r}||x_1 - c, x_2 - c||$, and
(2) $||x_2 - u, x_2 - c|| = \frac{1}{2^r}||x_1 - c, x_2 - c||$.

To prove the uniqueness, let v be an element in A such that v satisfying:

(1) $||x_1 - v, x_1 - c|| = \frac{1}{2^r}||x_1 - c, x_2 - c||$, and
(2) $||x_2 - v, x_2 - c|| = \frac{1}{2^r}||x_1 - c, x_2 - c||$

for $c \in X$ with $||x_1 - c, x_2 - c|| \neq 0$. Since $v \in A$, there exist scalars λ_1 and λ_2 not both are 0 such that

$$v = \frac{\lambda_1 x_1 + \lambda_2 x_2}{\lambda_1 + \lambda_2}.$$

As similar arguments of proof Lemma 3.2, we get that v satisfying the following inequalities:

(1) $||x_2 - v, x_2 - c|| = \frac{|\lambda_1|^r}{|\lambda_1+\lambda_2|^r}||x_1 - c, x_2 - c||$, and

(2) $||x_1 - v, x_1 - c|| = \frac{|\lambda_2|^r}{|\lambda_1+\lambda_2|^r}||x_1 - c, x_2 - c||$,

for $c \in X$ with $||x_1 - c, x_2 - c|| \neq 0$. Therefore we get that

$$\frac{1}{2^r} = \frac{|\lambda_1|^r}{|\lambda_1 + \lambda_2|^r} = \frac{|\lambda_2|^r}{|\lambda_1 + \lambda_2|^r}.$$

Hence

$$2^r|\lambda_1|^r = 2^r|\lambda_2|^r = |\lambda_1 + \lambda_2|^r \text{ and so } |\lambda_1| = |\lambda_2|.$$

Therefore

$$2^r|\lambda_1|^r = |\lambda_1 + \lambda_2|^r \leq (|\lambda_1| + |\lambda_2|)^r = 2^r|\lambda_1|^r.$$

Hence

$$|\lambda_1 + \lambda_2| = |\lambda_1| + |\lambda_2|.$$

Hence λ_1 and λ_2 are both positive or negative, and in both cases we get that $v = u$. □

Our next result is the following lemma.

Lemma 3.3. *Given two distinct elements x_1 and x_2 in X. Let*

$$u = \frac{t_1 x_1 + t_2 x_2}{t_1 + t_2}$$

for some scalars t_1, t_2 not all are 0. Then u satisfies the following inequalities:

(1) $||x_1 - c, x_2 - u|| = \frac{|t_1|^r}{|t_1+t_2|^r}||x_1 - c, x_2 - c||$, *and*

(2) $||x_1 - u, x_2 - c|| = \frac{|t_2|^r}{|t_1+t_2|^r}||x_1 - c, x_2 - c||$,

for some $c \in X$ with $||x_1 - c, x_2 - c|| \neq 0$

Proof. To prove 1, choose $c \in X$ with $||x_1 - c, x_2 - c|| \neq 0$. So

$$\begin{aligned} ||x_1 - c, x_2 - u|| &= \left|\left|x_1 - c, x_2 - \frac{t_1 x_1 + t_2 x_2}{t_1 + t_2}\right|\right| \\ &= \left|\left|x_1 - c, \frac{(t_1 + t_2)x_2 - (t_1 x_1 + t_2 x_2)}{t_1 + t_2}\right|\right| \\ &= \frac{1}{|t_1 + t_2|^r}||x_1 - c, t_1 x_2 - t_1 x_1||. \end{aligned}$$

Let $w = -t_1 c + t_1 c$. Then

$$t_1 x_2 - t_1 x_1 = t_1 x_2 - t_1 x_1 + w = t_1(x_2 - c) - t_1(x_1 - c).$$

Hence

$$||x_1 - c, x_2 - u|| = \frac{1}{|t_1 + t_2|^r}||x_1 - c, t_1(x_2 - c)||,$$

and therefore

$$||x_1 - c, x_2 - u|| = \frac{|t_1|^r}{|t_1 + t_2|^r}||x_1 - c, x_2 - c||.$$

We prove 2 by similar argument to 1. □

The proof of the following our result is the same as the proof of Theorem 3.1.

Theorem 3.2. *Given two distinct elements x_1 and x_2 in X. Let*

$$A = \left\{ \frac{t_1x_1 + t_2x_2}{t_1 + t_2} : t_1, t_2 \text{ are scalars not both are } 0 \right\}.$$

Then $u = \frac{x_1+x_2}{2}$ is the only unique element in A satisfying the following inequalities:

(1) $||x_1 - c, x_2 - u|| = \frac{1}{2^r}||x_1 - c, x_2 - c||$, *and*
(2) $||x_1 - u, x_2 - c|| = \frac{1}{2^r}||x_1 - c, x_2 - c||$

for $c \in X$ with $||x_1 - c, x_2 - c|| \neq 0$.

As direct application to Theorem 3.1 and Theorem 3.2, we get the following result:

Corollary 3.1. *Given $x_1, x_2 \in X$. Let*

$$A = \left\{ \frac{t_1x_1 + t_2x_2}{t_1 + t_2} : t_1, t_2 \text{ are scalars not both are } 0 \right\}.$$

Then $u = \frac{x_1+x_2}{2}$ is the only unique element in A satisfying

$$||x_1-c, x_2-u|| = ||x_1-u, x_2-c|| = ||x_1-u, x_1-c|| = ||x_2-u, x_2-c|| = \frac{1}{2^r}||x_1-c, x_2-c||,$$

for $c \in X$ with $||x_1 - c, x_2 - c|| \neq 0$.

The following our result is crucial in proving our main result.

Lemma 3.4. *Let T be α-2-isometry from a linear quasi-2-normed space X with power r into a linear quasi-2-normed space Y with power r. Let x_1 and x_2 be two distinct element in X and let*

$$A = \left\{ \frac{t_1T(x_1) + t_2T(x_2)}{t_1 + t_2} : t_1, t_2 \text{ are not both } 0 \right\}.$$

Then

$$T\left(\frac{x_1 + x_2}{2}\right) \in A.$$

Proof. Since

$$\left|\left|x_1 - \frac{x_1 + x_2}{2}, x_1 - x_2\right|\right| = \frac{1}{2^r}||x_1 - x_2, x_1 - x_2|| = 0,$$

we have

$$\left|\left|T(x_1) - T\left(\frac{x_1 + x_2}{2}\right), T(x_1) - T(x_2)\right|\right| = 0.$$

So we conclude that

$$T(x_1) - T\left(\frac{x_1 + x_2}{2}\right) \text{ and } T(x_1) - T(x_2)$$

are linearly dependents. Therefore there exists a scalar λ such that

$$T(x_1) - T\left(\frac{x_1 + x_2}{2}\right) = \lambda(T(x_1) - T(x_2)).$$

Hence

$$T\left(\frac{x_1+x_2}{2}\right)=(1-\lambda)T(x_1)+\lambda T(x_2).$$

Let $t_1=1-\lambda$ and $t_2=\lambda$. Then

$$T\left(\frac{x_1+x_2}{2}\right)=\frac{t_1T(x_1)+t_2T(x_2)}{t_1+t_2}.$$

So

$$T\left(\frac{x_1+x_2}{2}\right)\in A.$$

□

We have furnished all necessary background to introduce and prove our main result.

Theorem 3.3. *Every α-2-isometry between quasi-2-normed spaces X and Y with power r is affine.*

Proof. Let $T:X\to Y$ be α-2-isometry. Then $S(x)=T(x)-T(0)$ is also α-2-isometry. So we may assume that $T(0)=0$. So it suffices to prove that T is linear. Let x and y be two distinct elements in X, and $u=\frac{x+y}{2}$. Since $\dim X\geq 2$, there is $c\in X$ such that $||x-c,y-c||\neq 0$. Now

$$\begin{aligned}
||T(x)-T(u),T(x)-T(c)|| &= |\alpha|||x-u,x-c|| \\
&= |\alpha|\left|\left|x-\frac{x+y}{2},x-c\right|\right| \\
&= |\alpha|\frac{1}{2^r}||x-c,y-c|| \text{ from Theorem 3.1 Part 1} \\
&= \frac{1}{2^r}||T(x)-T(c),T(y)-T(c)||.
\end{aligned}$$

On other hand

$$\begin{aligned}
||T(y)-T(u),T(y)-T(c)|| &= |\alpha|||y-u,y-c|| \\
&= |\alpha|\left|\left|y-\frac{x+y}{2},y-c\right|\right| \\
&= |\alpha|\frac{1}{2^r}||y-c,y-c|| \text{ from Theorem 3.1 Part 2} \\
&= \frac{1}{2^r}||T(y)-T(c),T(y)-T(c)||.
\end{aligned}$$

By Lemma 3.4, we get that

$$T(u)=T\left(\frac{x+y}{2}\right)\in\left\{\frac{t_1T(x)+t_2T(y)}{t_1+t_2}:t_1,t_2 \text{ are not all } 0\right\}.$$

Hence by Theorem 3.1, we conclude that

$$T(u)=T\left(\frac{x+y}{2}\right) = \frac{T(x)+T(y)}{2}. \tag{1}$$

For $x\in X$ and $\alpha\in\mathbf{R}^+-\{0\}$, we have

$0=\alpha^r||x,x||=||\alpha x,x||=||\alpha x-0,x-0||=||T(\alpha x)-T(0),T(x)-T(0)||=||T(\alpha x),T(x)||.$

So $T(\alpha x)$ and $T(x)$ are linearly dependent. Hence there is $k \in \mathbf{R}$ such that $T(\alpha x) = kT(x)$. Choose $y \in X$ such that $||x, y|| \neq 0$. Then

$$\alpha^r||x, y|| = ||\alpha x, y|| = ||T(\alpha x), T(y)|| = ||kT(x), T(y)|| = |k|^r||T(x), T(y)|| = |k|^r||x, y||,$$

and hence $\alpha = |k|$. Claim: $k = \alpha$. To prove our claim, if $k = -\alpha$, then

$$\begin{aligned} |\alpha - 1|^r||x, y|| &= ||(\alpha - 1)x, y|| \\ &= ||\alpha x - x, y - x|| \\ &= ||T(\alpha x) - T(x), T(y) - T(x)|| \\ &= || - \alpha T(x) - T(x), T(y) - T(x)|| \\ &= (\alpha + 1)^r||T(x), T(y) - T(x)|| \\ &= (\alpha + 1)^r||T(x), T(y)|| \\ &= (\alpha + 1)^r||x, y||. \end{aligned}$$

So $|\alpha - 1|^r = (\alpha + 1)^r$, and hence $\alpha = 0$ which is a contradiction. Hence $k = \alpha$ and so that $T(\alpha x) = \alpha T(x)$ for all $\alpha \in \mathbf{R}^+ - \{0\}$. Similarly, we can show that $T(\alpha x) = \alpha T(x)$ for all $\alpha \in \mathbf{R}^- - \{0\}$. Given two distinct elements x and y in X. Since

$$T(x + y) = T\left(\frac{2x + 2y}{2}\right)$$

by Inequality (1) we get that

$$T(x + y) = \frac{T(2x) + T(2y)}{2} = \frac{2T(x) + 2T(y)}{2} = T(x) + T(y).$$

If $x = y$, then $T(x + y) = T(2x) = 2T(x) = T(x) + T(y)$. So T is affine. □

As an application of Theorem 3.3, we have the following corollary.

Corollary 3.2. [3] *Assume that $T(x), T(y)$ and $T(z)$ are collinear when x, y and z are collinear. If T is a 2-isometry map from a 2-normed space X into a 2-normed space Y, then T is affine.*

References

[1] A.D. Aleksandrov, Mappings of Families of Sets, Soviet Math. Dokl. 11, 116-120 (1970).
[2] J.A. Baker, Isometries in Normed Spaces, Amer. Math. Monthly 655-658 (1971).
[3] H.Y. Chu, On the Mazur-Ulam Problem in Linear 2-Normed Spaces, J. Math. Anal. Appl. 327, 1041-1045 (2007).
[4] H.Y. Chu, S.H. Ku, D.S. Kang, Characterizations on 2-Isometries, J. Math. Anal. Appl. 340, 621-628 (2008).
[5] H.Y. Chu, C.G. Park, W.G. Park, The Aleksandrov Problem in Linear 2-Normed Spaces, J. Math. Anal. Appl. 289, 666-672 (2004).
[6] T. Konda, On Quasi Normed Space 1, Proc. Japan Acad. 35, 340-587 (1959).
[7] Y.M. Ma, The Aleksandrov Problem for Unit Distance Preserving Mapping, Acta Math. Sci. Ser. B Engl. Ed. 20, 359-364 (2000).
[8] S. Mazur, S. Ulam, Sur Les Transformationes Isométriques D'espacesvectoriels Normés, C. R. Acad. Sci. Paris 194, 945-948 (1932).
[9] B. Mielnik, Th.M. Rassias, On the Aleksandrov Problem of Conservative Distances, Proc. Amer. Math. Soc. 116, 1115-1118 (1992).
[10] Th.M. Rassias, On the A.D. Aleksandrov Problem of Conservative Distances and the Mazur-Ulam Theorem, Nonlinear Anal. 47, 2597-2608 (2001).
[11] Th.M. Rassias, P. S̃emrl, On the Mazur-Ulam Problem and the Aleksandrov Problem for Unit Distance Preserving Mappings, Proc. Amer. Math. Soc. 118, 919-925 (1993).

[12] W Shatanawi, W, Postolache, M: Mazur-Ulam theorem for probabilistic 2-normed spaces, J. Nonlinear Sci. Appl. 8 (2015), 1228-1233.
[13] H. Gunaway, O. Neswan and E. Sukaesih, Fixed point theorems on bounded sets in an n-normed space, J. of Mathematical Analysis, 6 No. 3 (2015), 51-58.
[14] Y.J. Cho, P.C.S. Lin, S.S. Kim, A. Misiak, Theory of 2- inner product spaces, Nova Science, New York, 2001.
[15] C. Park, Generalized quasi-Banach spaces and quasi-(2-p)-normed spaces, Journal of the Chungcheong Matematical Society, 19 No. 2, (2006) 197–206.
[16] M. Kir and M. Acikgoz, A study involving the completion of a quasi-2-normed spaces, International Journal of Analysis, (2013) Article ID: 512372, 4 pages.
[17] C. Alaca, A new perspective to the Mazur-Ulam problem in 2- fuzzy 2-normed linear spaces. Iranian J. Fuzzy Syst. 7, (2010) 109-119.
[18] C. Park and C. Alaca, An introduction to 2-fuzzy n-normed linear spaces and a new perspective to the Mazur-Ulam problem. J. Inequal. Appl. 2012, 14 (2012). doi:10.1186/1029-242X-2012-14.
[19] Y. Benyamini and J. Lindenstrauss, Geometric Nonlinear Functional Analysis, Vol. 1, Colloq. Publ. 48, Amer. Math. Soc., Providence, 2000.

[1]Department of Mathematics and general courses, Prince Sultan University, Riyadh, Saudi Arabia.,

[2]Department of Mathematics, Faculty of Science, Hashemite University, Zarqa, Jordan.

E-mail address: kamal@psu.edu.sa, wshatanawi@psu.edu.sa