Journal of Mathematical Analysis
ISSN: 2217-3412, URL: http://ilirias.com/jma
Volume 7 Issue 3(2016), Pages 13-20.

A VARIATION ON STRONGLY LACUNARY WARD CONTINUITY

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Abstract. In this paper, the concept of a strongly lacunary δ-quasi-Cauchy sequence is investigated. A real valued function f defined on a subset A of $\mathbb{R}$, the set of real numbers, is called strongly lacunary delta ward continuous on A if it preserves strongly lacunary delta quasi-Cauchy sequences of points in A, i.e. $(f(\alpha_k))$ is a strongly lacunary delta quasi-Cauchy sequence whenever (α_k) is a strongly lacunary delta quasi-Cauchy sequences of points in A, where a sequence (α_k) is called strongly lacunary delta quasi-Cauchy if $(\Delta\alpha_k)$ is a strongly lacunary quasi-Cauchy sequence where $\Delta^2\alpha_k = \alpha_{k+2} - 2\alpha_{k+1} + \alpha_k$ for each positive integer k. It turns out that the set of strongly lacunary delta ward continuous functions is a closed subset of the set of continuous functions.

1. Introduction

The concept of continuity and any concept involving continuity play a very important role not only in pure mathematics but also in other branches of sciences involving mathematics especially in computer science, information theory, economics, and biological science.

Buck ([4]) introduced Cesaro continuity in 1946. Thereafter, a number of authors Posner ([39]), Iwinski ([34]), Srinivasan ([41]), Antoni ([1]), Antoni and Salat ([2]), Spigel and Krupnik ([40]) have studied A-continuity defined by a regular summability matrix A. Some authors, Öztürk ([38]), Das and Savaş ([33]), Borsik and Salat ([3]) have studied A-continuity for methods of almost convergence or for related methods. Connor and Grosse-Erdman ([11]) have given sequential definitions of continuity for real functions calling G-continuity instead of A-continuity by means of a sequential method, or a method of sequential convergence, and their results cover the earlier works related to A-continuity where a method of sequential convergence, or briefly a method, is a linear function G defined on a linear subspace of all sequences of points in $\mathbb{R}$ denoted by c_G, into $\mathbb{R}$. A sequence $\boldsymbol{\alpha} = (\alpha_k)$ is said to be G-convergent to ℓ if $\boldsymbol{\alpha} \in c_G$,then $G(\boldsymbol{\alpha}) = \ell$. In particular, lim denotes the limit function $\lim \boldsymbol{\alpha} = \lim_k \alpha_k$ on the linear space c, where c denotes the space of convergent sequences. On the other hand, Çakallı has introduced a generalization

2010 *Mathematics Subject Classification.* Primary: 40A05 Secondaries: 40G15; 26A05, 26A15.
Key words and phrases. lacunary statistically convergence, strongly lacunary convergence, quasi-Cauchy sequences, continuity.

Submitted April 19, 2016. Published April 28, 2016.

of compactness ([12]), a generalization of connectedness ([19]), via a method of sequential convergence (see also [24] and [37]).

In recent years, using the same idea, many kinds of continuities were introduced and investigated, not all but some of them we state in the following: slowly oscillating continuity ([13]), quasi-slowly oscillating continuity ([10]), Δ-quasi-slowly oscillating continuity ([20], and [18]), ward continuity ([15]), δ-ward continuity ([14]), statistical ward continuity ([16], [17]), lacunary statistical ward continuity ([8]), ρ-statistically ward continuity ([7]), λ-statistically ward continuity ([27]), and $N_\theta-$ward continuity ([6], [23]). Investigation of some of these kinds of continuities lead some authors to find conditions on the domain of a function for some characterizations of uniform continuity of a real function in terms of sequences in the above manner ([43, Theorem 8], [15, Theorem 7], and [5, Theorem 1]).

The notion of strongly lacunary or N_θ convergence was introduced, and studied by Freedman, Sember, and M. Raphael in [31] in the sense that a sequence (α_k) of points in $\mathbb{R}$ is N_θ convergent to an $L \in \mathbb{R}$ if $\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} |\alpha_k - L| = 0$, and which is denoted by $N_\theta - \lim \alpha_k = L$, where $I_r = (k_{r-1}, k_r]$, and $k_0 \neq 0$, $h_r : k_r - k_{r-1} \to \infty$ as $r \to \infty$ and $\theta = (k_r)$ is an increasing sequence of positive integers. Throughout this paper, it is assumed that $\liminf_r \frac{k_r}{k_{r-1}} > 1$. The sums of the form $\sum_{k_{r-1}+1}^{k_r} |\alpha_k|$ frequently occur, and will often be written for convenience as $\sum_{k\in I_r} |\alpha_k|$.

The purpose of this paper is to investigate the notion of strongly lacunary δ ward continuity and prove interesting theorems.

2. Strongly lacunary δ quasi Cauchy sequences

A function defined on a subset A of $\mathbb{R}$ is called strongly lacunary continuous or N_θ continuous if it preserves N_θ convergent sequences of points in A, i.e. $(f(\alpha_k))$ is N_θ convergent whenever (α_k) is an N_θ convergent sequence of points in A. A function defined on a subset A of $\mathbb{R}$ is strongly lacunary continuous if and only if it is ordinary continuous. A function defined on a subset A of $\mathbb{R}$ is called strongly lacunary ward continuous or N_θ-ward continuous if it preserves N_θ-quasi-Cauchy sequences of points in A, i.e. $(f(\alpha_k))$ is N_θ-quasi-Cauchy whenever (α_k) is an N_θ-quasi-Cauchy sequence of points in A (see [23]), where a sequence (α_k) of points in $\mathbb{R}$ is called strongly lacunary quasi-Cauchy, or N_θ-quasi-Cauchy if $(\Delta\alpha_k)$ is N_θ-convergent to 0 ([6, 23]). ΔN_θ will denote the set of N_θ-quasi-Cauchy sequences of points in $\mathbb{R}$.

Example 1. *([9]) The sequence of Fibonacci numbers has a quite nice property when it is considered as a lacunary sequence. Lacunary sequential method obtained by the sequence of Fibonacci numbers is a regular method, i.e. $\theta = (k_r)$ is the lacunary sequence defined by writing $k_0 = 0$ and $k_r = F_{r+2}$ where (F_r) is the Fibonacci sequence, i.e. $F_1 = 1$, $F_2 = 1$, $F_r = F_{r-1} + F_{r-2}$ for $r \geq 3$. For this lacunary sequence $\theta = (k_r)$, a real valued function defined on a subset of $\mathbb{R}$ is strongly lacunary sequentially continuous if and only if it is ordinary sequentially continuous ([6]).*

Definition 1. *A sequence (α_k) of points in $\mathbb{R}$ is called strongly lacunary δ-quasi-Cauchy, or N_θ-δ-quasi-Cauchy if $(\Delta(\alpha_k))$ is an N_θ quasi-Cauchy sequence, i.e.*

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} |\Delta^2 \alpha_k| = 0$$

where $\Delta^2\alpha_k = \alpha_{k+2} - 2\alpha_{k+1} + \alpha_k$ for each positive integer k.

Now we give some interesting examples that show importance of the interest.

Example 2. *Let n be a positive integer. In a group of n people, each person selects at random and simultaneously another person of the group. All of the selected persons are then removed from the group, leaving a random number $n_1 < n$ of people which form a new group. The new group then repeats independently the selection and removal thus described, leaving $n_2 < n_1$ persons, and so forth until either one person remains, or no persons remain. Denote by α_n the probability that, at the end of this iteration initiated with a group of n persons, one person remains. Then the sequence $(\alpha_1, \alpha_2, , \alpha_n, ...)$ is a strongly lacunary delta quasi-Cauchy sequence (see also* [44]*).*

Example 3. *In a group of k people, k is a positive integer, each person selects independently and at random one of three subgroups to which to belong, resulting in three groups with random numbers k_1, k_2, k_3 of members; $k_1 + k_2 + k_3 = k$. Each of the subgroups is then partitioned independently in the same manner to form three sub subgroups, and so forth. Subgroups having no members or having only one member are removed from the process. Denote by α_k the expected value of the number of iterations up to complete removal, starting initially with a group of k people. Then the sequence $(\alpha_1, \frac{\alpha_2}{2}, \frac{\alpha_3}{3}, ..., \frac{\alpha_n}{n}, ...)$ is a bounded non-convergent strongly lacunary delta quasi-Cauchy sequence (*[35]*).*

We note that any N_θ-quasi Cauchy sequence is also $N_\theta - \delta$-quasi Cauchy, but the converse is not always true as it can be seen by considering the sequence (n). $\Delta^2 N_\theta$ will denote the set of $N_\theta - \delta$-quasi-Cauchy sequences of points in $\mathbb{R}$.

Now we give the definition of $N_\theta - \delta$-ward compactness of a subset of $\mathbb{R}$.

Definition 2. *A subset A of $\mathbb{R}$ is called strongly lacunary δ ward (or $N_\theta - \delta$-ward) compact if any sequence of points in A has an $N_\theta - \delta$ quasi-Cauchy subsequence, i.e. if whenever $\boldsymbol{\alpha} = (\alpha_k)$ is a sequence of points in A there is a subsequence $\boldsymbol{\beta} = (\beta_k) = (\alpha_{k_k})$ of $\boldsymbol{\alpha}$ with $N_\theta - \lim_{k\to\infty} \Delta^2 \boldsymbol{\beta}_k = 0$.*

Firstly, we note that any finite subset of $\mathbb{R}$ is $N_\theta - \delta$-ward compact, the union of finite number of $N_\theta - \delta$-ward compact subsets of $\mathbb{R}$ is $N_\theta - \delta$-ward compact and the intersection of any $N_\theta - \delta$-ward compact subsets of $\mathbb{R}$ is $N_\theta - \delta$-ward compact. Furthermore any subset of a $N_\theta - \delta$-ward compact set is $N_\theta - \delta$-ward compact and any bounded subset of $\mathbb{R}$ is $N_\theta - \delta$-ward compact. Any compact subset of $\mathbb{R}$ is also $N_\theta - \delta$-ward compact, and the converse is not always true. For example, any open interval is $N_\theta - \delta$-ward compact, which is not compact. On the other hand, the set $\mathbb{N}$ is not $N_\theta - \delta$-ward compact. We note that any slowly oscillating compact subset of $\mathbb{R}$ is $N_\theta - \delta$-ward compact (see also [13], and [20] for the results on slowly oscillating compactness), and any quasi-slowly oscillating compact subset of $\mathbb{R}$ is $N_\theta - \delta$-ward compact where a subset A of $\mathbb{R}$ is called quasi-slowly oscillating compact (see [10]) if whenever $\boldsymbol{\alpha} = (\alpha_n)$ is a sequence of points in A, there is a quasi-slowly oscillating subsequence $\boldsymbol{\beta} = (\beta_{n_k})$ of $\boldsymbol{\alpha}$.

Now we give the definition of strongly lacunary δ-ward continuity in the following.

Definition 3. *A function defined on a subset A of $\mathbb{R}$ is called strongly lacunary δ-ward continuous or $N_\theta - \delta$-ward continuous if it preserves $N_\theta - \delta$-quasi-Cauchy sequences of points in A, i.e. $(f(\alpha_k))$ is $N_\theta - \delta$-quasi-Cauchy whenever (α_k) is an $N_\theta - \delta$-quasi-Cauchy sequence of points in A.*

We note that the sum of two $N_\theta - \delta$-ward continuous functions is $N_\theta - \delta$-ward continuous, and the composite of two $N_\theta - \delta$-ward continuous functions is $N_\theta - \delta$-ward continuous, but the product of $N_\theta - \delta$-ward continuous functions need not be $N_\theta - \delta$-ward continuous. The function defined by $f(x) = x$ is clearly $N_\theta - \delta$-ward continuous whereas the product $f(x).f(x) = x^2$ is not $N_\theta - \delta$-ward continuous.

In connection with $N_\theta - \delta$-quasi-Cauchy sequences and convergent sequences the problem arises to investigate the following types of continuity of functions on $\mathbb{R}$:

$(\delta N_\theta \delta)$: $(\alpha_n) \in \Delta^2 N_\theta \Rightarrow (f(\alpha_n)) \in \Delta^2 N_\theta$
$(\delta N_\theta \delta c)$: $(\alpha_n) \in \Delta^2 N_\theta \Rightarrow (f(\alpha_n)) \in c$
(ΔN_θ): $(\alpha_n) \in \Delta N_\theta \Rightarrow (f(\alpha_n)) \in \Delta N_\theta$
(N_θ): $(\alpha_n) \in N_\theta \Rightarrow (f(\alpha_n)) \in N_\theta$
(c)**:** $(\alpha_n) \in c \Rightarrow (f(\alpha_n)) \in c$
$(c\delta N_\theta \delta)$**:** $(\alpha_n) \in c \Rightarrow (f(\alpha_n)) \in \Delta^2 N_\theta$

We see that $(\delta N_\theta \delta)$ is $N_\theta - \delta$-ward continuity of f, (N_θ) is N_θ-sequential continuity of f, and (c) is the ordinary continuity of f. It is easy to see that $(\delta N_\theta \delta c)$ implies $(\delta N_\theta \delta)$, and $(c\delta N_\theta \delta)$ does not imply $(\delta N_\theta \delta c)$; and $(\delta N_\theta \delta)$ implies $(c\delta N_\theta \delta)$, and $(c\delta N_\theta \delta)$ does not imply $(\delta N_\theta \delta)$; $(\delta N_\theta \delta c)$ implies (c), and (c) does not imply $(\delta N_\theta \delta c)$.

Now we give the implication $(\delta N_\theta \delta)$ implies (ΔN_θ), i.e. any $N_\theta - \delta$-ward continuous function is N_θ-ward continuous.

Theorem 2.1. *If f is $N_\theta - \delta$-ward continuous on a subset A of $\mathbb{R}$, then it is N_θ-ward continuous on A.*

Proof. Assume that f is an $N_\theta - \delta$-ward continuous function on A. Let (α_n) be any N_θ-quasi-Cauchy sequence of points in A. Then the sequence

$$(\alpha_1, \alpha_1, \alpha_2, \alpha_2, ..., \alpha_{n-1}, \alpha_{n-1}, \alpha_n, \alpha_n, ...)$$

is also N_θ-quasi-Cauchy. Hence it is $N_\theta - \delta$-quasi-Cauchy. As f is $N_\theta - \delta$-ward continuous, the sequence

$$(f(\alpha_1), (f(\alpha_1), f(\alpha_2), f(\alpha_2), ..., f(\alpha_{n-1}), f(\alpha_{n-1}), f(\alpha_n), f(\alpha_n), ...)$$

is $N_\theta - \delta$-quasi-Cauchy. It follows from this that the sequence $(f(\alpha_n))$ is N_θ-quasi-Cauchy. This completes the proof of the theorem. □

Corollary 2.2. *If a function f defined on an interval A is $N_\theta - \delta$-ward continuous, then it is uniformly continuous.*

Proof. The proof immediately follows from Theorem 2.1, and from [23, Theorem 3] so is omitted. □

Corollary 2.3. *If f is $N_\theta - \delta$-ward continuous on a subset A of $\mathbb{R}$, then it is continuous on A.*

Proof. The proof follows from Theorem 2.1, and so is omitted. □

It should be noted that any strongly lacunary delta ward continuous function is statistically continuous, N_θ-continuous, I-sequentially continuous for any non-trivial admissible ideal I, and G-sequentially continuous for any regular subsequential sequential method G.

Theorem 2.4. *$N_\theta - \delta$-ward continuous image of any $N_\theta - \delta$-ward compact subset of $\mathbb{R}$ is $N_\theta - \delta$-ward compact.*

Proof. Assume that f is an $N_\theta - \delta$-ward continuous function on a subset A of $\mathbb{R}$, and A is an $N_\theta - \delta$-ward compact subset of $\mathbb{R}$. Let (β_n) be any sequence of points in $f(A)$. Write $\beta_n = f(\alpha_n)$ where $\alpha_n \in A$ for each positive integer n. $N_\theta - \delta$-ward compactness of A implies that there is a subsequence $(\gamma_k) = (\alpha_{n_k})$ of (α_n) with $N_\theta \lim_{k\to\infty} \Delta^2 \gamma_k = 0$. Write $(t_k) = (f(\gamma_k))$. As f is $N_\theta - \delta$-ward continuous, $(f(\gamma_k))$ is $N_\theta - \delta$-quasi-Cauchy. Thus we have obtained a subsequence (t_k) of the sequence $(f(\alpha_n))$ with $N_\theta - \lim_{k\to\infty} \Delta^2 t_k = 0$. Thus $f(A)$ is $N_\theta - \delta$-ward compact. This completes the proof of the theorem. □

Corollary 2.5. *$N_\theta - \delta$-ward continuous image of any G-sequentially connected subset of $\mathbb{R}$ is G-sequentially connected.*

The proof follows from the preceding theorem.

Corollary 2.6. *$N_\theta - \delta$-ward continuous image of any bounded subset of $\mathbb{R}$ is bounded.*

The proof follows from [6, Theorem 3.3].

Corollary 2.7. *$N_\theta - \delta$-ward continuous image of a G-sequentially compact subset of $\mathbb{R}$ is $N_\theta - \delta$-ward compact for any regular subsequential method G.*

As far as ideal continuity is considered, we note that any $N_\theta - \delta$-ward continuous function is I-sequentially continuous for any admissible ideal I ([22]).

It follows from [23, Theorem 1] that $(f(\alpha_k))$ is $N_\theta - \delta$-quasi Cauchy whenever (α_k) is a quasi-Cauchy sequence of points in a subset A of $\mathbb{R}$ if f is uniformly continuous on A.

It is a well known result that the uniform limit of a sequence of continuous functions is continuous. This is also true in case of $N_\theta - \delta$-ward continuity, i.e. uniform limit of a sequence of $N_\theta - \delta$-ward continuous functions is $N_\theta - \delta$-ward continuous.

Theorem 2.8. *If (f_n) is a sequence of $N_\theta - \delta$-ward continuous functions on a subset A of $\mathbb{R}$, and (f_n) is uniformly convergent to a function f, then f is $N_\theta - \delta$-ward continuous on A.*

Proof. Let (α_k) be any $N_\theta - \delta$-quasi-Cauchy sequence of points in A, and let ε be any positive real number. By the uniform convergence of (f_n), there exists an $n_1 \in \mathbf{N}$ such that $|f(x) - f_k(x)| < \frac{\varepsilon}{6}$ for $n \geq n_1$ and every $x \in E$. As f_{n_1} is $N_\theta - \delta$-ward continuous on A, there exists an $n_2 \in \mathbf{N}$ such that for $r \geq n_2$

$$\frac{1}{h_r} \sum_{k \in I_r} |f_{n_1}(\alpha_{k+2}) - 2f_{n_1}(\alpha_{k+1}) + f_{n_1}(\alpha_k)| < \frac{\varepsilon}{2}.$$

Now write $n_0 = max\{n_1, n_2\}$. Thus for $r \geq n_0$ we have
$\frac{1}{h_r} \sum_{k \in I_r} |f(\alpha_{k+2}) - 2f(\alpha_{k+1}) + f(\alpha_k)| = \frac{1}{h_r} \sum_{k \in I_r} |f(\alpha_{k+2}) - 2f(\alpha_{k+1}) + f(\alpha_k) - [f_{n_1}(\alpha_{k+2}) - 2f_{n_1}(\alpha_{k+1}) + f_{n_1}(\alpha_k)] + [f_{n_1}(\alpha_{k+2}) - 2f_{n_1}(\alpha_{k+1}) + f_{n_1}(\alpha_k)]|$

$\leq \frac{1}{h_r}\sum_{k\in I_r}|f(\alpha_{k+2}) - f_{n_1}(\alpha_{k+2})| + \frac{1}{h_r}\sum_{k\in I_r}| - 2f(\alpha_{k+1}) + 2f_{n_1}(\alpha_{k+1})| + \frac{1}{h_r}\sum_{k\in I_r}|f(\alpha_k) - f_{n_1}(\alpha_k)| + \frac{1}{h_r}\sum_{k\in I_r}|f_{n_1}(\alpha_{k+2}) - 2f_{n_1}(\alpha_{k+1}) + f_{n_1}(\alpha_k)|$
$< \frac{\varepsilon}{6} + \frac{\varepsilon}{6} + \frac{\varepsilon}{6} + \frac{\varepsilon}{2} = \varepsilon$. Hence

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}|f(\alpha_{k+2}) - 2f(\alpha_{k+1}) + f(\alpha_k)| = 0.$$

Thus f preserves $N_\theta - \delta$-quasi-Cauchy sequences. This completes the proof of the theorem. □

Theorem 2.9. *The set of strongly lacunary delta ward continuous functions on a subset A of $\mathbb{R}$ is a closed subset of the set of continuous functions on A. i.e. $\overline{\Delta^2 N_\theta(A)} = \Delta^2 N_\theta(A)$ where $\overline{\Delta^2 N_\theta(A)}$ denotes the set of all cluster points of $\Delta^2 N_\theta(A)$.*

Proof. Let f be an element in $\overline{\Delta^2 N_\theta(A)}$. Then there exists a sequence (f_n) of points in $\Delta^2 N_\theta(A)$ such that $\lim_{n\to\infty} f_n = f$. Then there is an $n_1 \in \mathbf{N}$ such that $|f(x) - f_k(x)| < \frac{\varepsilon}{6}$ for $n \geq n_1$ and every $x \in E$. To show that f is strongly lacunary statistically δ ward continuous, consider a strongly lacunary δ quasi-sequence (α_n) of points in A. As f_{n_1} is $N_\theta - \delta$-ward continuous on A, there exists an $n_2 \in \mathbf{N}$ such that for $r \geq n_2$

$$\frac{1}{h_r}\sum_{k\in I_r}|f_{n_1}(\alpha_{k+2}) - 2f_{n_1}(\alpha_{k+1}) + f_{n_1}(\alpha_k)| < \frac{\varepsilon}{2}.$$

Now write $n_0 = max\{n_1, n_2\}$. Thus for $r \geq n_0$ we have
$\frac{1}{h_r}\sum_{k\in I_r}|f(\alpha_{k+2}) - 2f(\alpha_{k+1}) + f(\alpha_k)| = \frac{1}{h_r}\sum_{k\in I_r}|f(\alpha_{k+2}) - 2f(\alpha_{k+1}) + f(\alpha_k) - [f_{n_1}(\alpha_{k+2}) - 2f_{n_1}(\alpha_{k+1}) + f_{n_1}(\alpha_k)] + [f_{n_1}(\alpha_{k+2}) - 2f_{n_1}(\alpha_{k+1}) + f_{n_1}(\alpha_k)]|$
$\leq \frac{1}{h_r}\sum_{k\in I_r}|f(\alpha_{k+2}) - f_{n_1}(\alpha_{k+2})| + \frac{1}{h_r}\sum_{k\in I_r}| - 2f(\alpha_{k+1}) + 2f_{n_1}(\alpha_{k+1})| + \frac{1}{h_r}\sum_{k\in I_r}|f(\alpha_k) - f_{n_1}(\alpha_k)| + \frac{1}{h_r}\sum_{k\in I_r}|f_{n_1}(\alpha_{k+2}) - 2f_{n_1}(\alpha_{k+1}) + f_{n_1}(\alpha_k)|$
$< \frac{\varepsilon}{6} + \frac{\varepsilon}{6} + \frac{\varepsilon}{6} + \frac{\varepsilon}{2} = \varepsilon$. Hence

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}|f(\alpha_{k+2}) - 2f(\alpha_{k+1}) + f(\alpha_k)| = 0.$$

This completes the proof of the theorem.

□

Corollary 2.10. *The set of strongly lacunary delta ward continuous functions on a subset A of $\mathbb{R}$ is a complete subset of the set of continuous functions on A.*

3. Conclusion

In this paper, the concept of a strongly lacunary δ-quasi-Cauchy sequence is investigated. In this investigation, we proved interesting theorems related to strongly lacunary δ ward continuity, and some other kinds of continuities. For a further study, we suggest to investigate strongly lacunary δ-quasi-Cauchy sequences of fuzzy points (see [21], [30], and [32] for the definitions and related concepts in fuzzy setting), and we suggest to investigate strongly lacunary δ-quasi-Cauchy double sequences of fuzzy points (see [25], [29], and [36] for the concepts in double case). Another suggestion for another further study is to introduce and give investigations of strongly lacunary δ-quasi-Cauchy sequences of points in cone normed spaces (see [28], [26], and [42], for basic concepts in topological vector space valued cone metric,

and cone normed spaces). However due to the change in settings, the definitions and methods of proofs will not always be analogous to those of the present work.

Acknowledgments. We acknowledge that the statements of some results in this paper were presented at the International Conference on Advancements in Mathematical Sciences in 2015 and extended abstract appeared in Proceedings of the conference ([9]).

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