COMPUTING QUANDLE COLOURINGS

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Abstract. In previous work, results were obtained relating two different knot invariants, the Alexander polynomial on the one hand, and the number of quandle colourings using any finite linear Alexander quandle, on the other. Given such a quandle, specified by two coprime integers n and m, the number of colourings of a knot diagram is given by counting the solutions of a matrix equation of the form $AX = 0 \mod n$, where A is the m-dependent colouring matrix. The same matrix A determines the Alexander polynomial of the knot. Our previous results were based in part on computations using an algorithm to reduce A to echelon form, and in part on proving properties of the matrix equations in their reduced form. When two knots have different Alexander polynomials, and their reduced colouring matrices are upper triangular, we proved that there exists a specific quandle which distinguishes them by colourings, and conjectured that this would be true without the condition on the reduced form of the colouring matrices. In the present article we address this issue from a new perspective, using the fact that all colouring matrices are triangularizable when certain conditions on m and n are met, and find further support for our conjecture. A description of an improved version of the algorithm is also presented.

1. Introduction

The theory of knots has a long and rich history, giving rise to a large number of different invariants coming from many diverse perspectives. For this reason it is very important to understand the interconnections between different knot invariants, normally not an easy task. In [4], using a combination of calculational and theoretical approaches, we established results that relate two knot invariants, namely the classical Alexander polynomial on the one hand, and the number of quandle colourings using any finite linear Alexander quandle, on the other.

The number of quandle colourings of a knot diagram is a well known invariant of a knot, introduced independently in [13] and [19] – see also [5] for a recent survey. An interesting class of quandles are the finite linear Alexander quandles, which are given by two coprime integers n and m. Thus we can consider the information contained in the number of such quandle colourings for arbitrary choices of n and m. A separate invariant of the knot is its Alexander polynomial [1], and in [4]

we clarify a number of points about the precise relationship between these two invariants.

Given a finite linear Alexander quandle, the number of quandle colourings of a knot diagram is given by counting the solutions of a matrix equation of the form $AX = 0$, where the entries of the colouring matrix A are Laurent polynomials in m, and the corresponding linear equations are all taken modulo n. Thus a natural strategy is to try and reduce A to echelon form. We devised an algorithm (using the Mathematica programming environment) which does this in such a way as to preserve a property of the matrix A, namely that its columns add up to zero. Applying this algorithm to the colouring matrices for all prime knots with up to 10 crossings we found that all the reduced matrices were of just three distinct and highly specific types, two of which were upper triangular with mostly 1's on the diagonal (which we call Type I and Type II), and the third of which was non-triangular, but upper block triangular, with mainly 1's on the diagonal along with a single 2 by 2 non-triangular block. There were only 12 out of 249 prime knots with up to 10 crossings that gave non-triangular reduced matrices.

Using theoretical methods, we were then able to prove general formulae for the number of solutions of $AX = 0$ as a function of n and m, when the reduced matrix is of Type I or Type II. Since a different knot invariant, the Alexander polynomial, is given by the determinant of a minor of the same matrix A, the formulae that we obtained for the Type I or Type II reduced colouring matrices either involve the Alexander polynomial of the knot, or factors of the Alexander polynomial, both evaluated at m. This opened the way for proving general results which relate two different invariants: the Alexander polynomial, and the number of quandle colourings using any finite linear Alexander quandle [4].

When two knots have different Alexander polynomials, and their colouring matrices can be triangularized (not necessarily only as Type I or Type II matrices), we proved that they can be distinguished by colourings using a suitable finite linear Alexander quandle. In [4] we conjectured that, in general, when two knots have different Alexander polynomials, it is possible to distinguish them with finite linear Alexander quandle colourings, irrespective of whether their colouring matrices can be triangularized. In the present article we analyse this issue further, and find additional results which support our conjecture. We also describe the algorithm which was used to reduce the colouring matrices A.

The structure of this article is as follows. In section 2 we recall the necessary background for quandles [13,19] and quandle colourings of knot diagrams [2,5,11,12, 18,20], focussing on the case of finite linear Alexander quandles. We also introduce the colouring matrix A associated to a knot diagram, and its role both in computing the number of quandle colourings and in getting the Alexander polynomial [1,17].

In section 3 we describe our computations which reduce A to specific echelon or upper triangular forms, for all prime knots with up to 10 crossings. We observe that precisely three patterns occur for the reduced matrices: two types of upper triangular matrices (which we call Type I and Type II) and a non-triangular form which is which is block upper triangular containing a single non-triangular 2×2 block. When the colouring matrix of a knot diagram is equivalent to a Type I or Type II upper triangular matrix, the number of quandle colourings, using a finite linear Alexander quandle specified by two coprime integers m and n, obeys a general formula.

In our main Section 4 we recall some results from [4] relating the Alexander polynomial and the number of colourings invariant, and based on the general formulae for Type I or Type II reduced colouring matrices. We focus on the issue of whether two knots with different Alexander polynomials can be distinguished by linear quandle colourings. For knots whose colouring matrix can be reduced to triangular form, we showed in [4] that this is indeed the case, and conjectured that it is true in general. We note that there are instances of knots whose colouring matrix can definitely not be reduced to triangular form by means of the allowed operations, which was one of the main results of [4]. We analyse these cases from a new perspective, namely that when the parameter m is fixed then all matrices can be triangularized. This does not contradict that fact that there are non-triangularizable matrices for general m. It means that in such cases there is not one single triangular matrix that accounts for all values of m but it allows for different matrices for different values of m. We discuss conditions on the entries of these non-triangular matrices that allow for triangularization. This analysis lends further support to our conjecture above.

In the appendix we describe the algorithm used to simplify the colouring matrices and its input, the Gauss codes for knots up to 10 crossings.

2. Background

In this section we recall the definition of a quandle and the notion of quandle coloring of a diagram. Below we also present the notions of finite Alexander quandle, Alexander polynomial and linear finite Alexander quandle. The number of colourings in a linear finite Alexander quandle is the number of solutions of a system of equations that can be written in matrix form. This matrix is called the colouring matrix. The determinant of the matrix obtained by removing the last row and the last column of this colouring matrix is a knot invariant, the Alexander polynomial.

2.1. **Quandles and colourings.** Colourings of the arcs of oriented knot diagrams with elements of a quandle generalize *mod p* labellings of the arcs, that, in turn, generalize the colorability invariant of R. Fox (with $p = 3$ colours). They are also a generalization of arc labellings of oriented knot diagrams with group elements (see, for instance, [17]). At each crossing the quandle elements labelling the arcs are related by the quandle operation $*$. The number of colourings is a knot invariant since different diagrams of the same knot have the same number of colourings using a given quandle. Indeed, the definition of a quandle consists of precisely those properties of the binary operation $*$ that ensure that colourings are preserved under the Reidemeister moves.

Definition 1 (Quandle). *A quandle is a set X endowed with a binary operation, denoted $*$, such that:*

(a) for any $a \in X, a * a = a$ f (b) for any a and $b \in X$, there is a unique $x \in X$ such that $a = x * b$. This element x is denoted by $a *' b$.

(c) for any a, b and $c \in X, (a * b) * c = (a * c) * (b * c)$

We may use the elements of a quandle to colour the arcs of a knot diagram.

Definition 2 (Quandle colouring of a knot diagram). *Let X be a fixed finite quandle, K a knot (assumed to be oriented), $\overrightarrow{D}$ a diagram of K and $R_{\overrightarrow{D}}$ the set of arcs*

of $\overrightarrow{D}$. A quandle colouring of a diagram $\overrightarrow{D}$ is a map $C : R_{\overrightarrow{D}} \longrightarrow X$ such that, at each crossing:
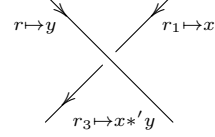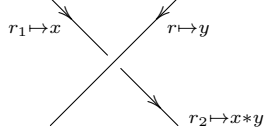
i.e. if $C(r_1) = x$ and $C(r) = y$, then $C(r_2) = x * y$ for the crossing on the left, and if $C(r_1) = x$ and $C(r) = y$, then $C(r_3) = x *' y$ for the crossing on the right.

Since the knot quandle, coming from the knot itself, is a classifying invariant for knots (introduced independently by Joyce and Matveev – see [13] and [19]), the number of quandle colorings associated to a knot diagram, for any fixed quandle, is a knot invariant.

Theorem 3. *Let X be a fixed finite quandle, K a knot and D and D' oriented diagrams of K. Then the number of colourings $C : R_D \longrightarrow X$ is equal to the number of colourings $C' : R_{D'} \longrightarrow X$.*

For a more complete discussion of the results above and related topics see [5, 12, 14, 18, 20].

2.2. **Finite Alexander Quandles.** Finite Alexander quandles are the special case of quandles of the form $\mathbb{Z}_n[t, t^{-1}]/h(t)$ where n is an integer and $h(t)$ is a monic polynomial in t. These quandles have as elements equivalence classes of Laurent polynomials with coefficients in $\mathbb{Z}_n$, where two polynomials are equivalent if their difference is divisible by $h(t)$. The quandle operation is

$$a * b = ta + (1 - t)b. \tag{1}$$

Note that this means equality of quandle elements, i.e. equivalence classes of Laurent polynomials. Recall that $c = a *' b$ is defined to mean the same as $a = c * b$. From this it follows easily that

$$a *' b = t^{-1}a + (1 - t^{-1})b. \tag{2}$$

For finite Alexander quandles the colouring condition at each crossing states that the label of the emerging arc is expressed as a linear combination of the labels of the other two arcs. Therefore one uses matrices to organize the colouring conditions (equations).

For that purpose we need an enumeration of the arcs and the crossings. Any enumeration will do, but to fix ideas we describe one possibility. We choose a starting arc, labelled 1, and use an enumeration that assigns $i + 1$ to the emerging arc where i is the number assigned to the incoming arc (see figure below), except for the last crossing when the emerging arc is already labelled (by 1). For crossings we use the enumeration suggested by the enumeration of arcs, i.e. the k-th crossing is the one with under arc also labelled k.
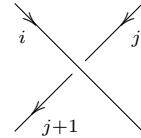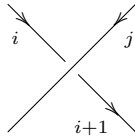
Let X_k be the label (in the quandle) of arc k. Then the colouring conditions of Definition 2, using (1) and (2), applied to the figure below:

are, respectively,

$$X_k = tX_i + (1-t)X_j$$
$$X_k = t^{-1}X_j + (1-t^{-1})X_i$$

The second condition $X_k = X_j *' X_i$ is equivalent to $X_j = X_k * X_i$, i.e. $X_j = tX_k + (1-t)X_i$, and thus the colouring conditions for the crossings in the figure above can also be expressed as:

$$tX_i + (1-t)X_j - X_k = 0$$
$$(1-t)X_i - X_j + tX_k = 0$$

It may happen that two of the arcs labelled i, j, k at a crossing are actually the same arc, e.g. $i = j$. In this case, the corresponding terms in the equations above are combined.

Thus, given an oriented diagram D of a knot K, we can write the colouring conditions as a matrix equation

$$AX = 0$$

where X is the vector of colouring unknowns $(X_1, X_2, \ldots, X_i, \ldots)$ and each row in the matrix A represents a colouring condition for one crossing in D.

Obviously the number of colourings of a diagram in a linear Alexander quandle is the number of solutions of $AX = 0$. We will call the matrix A a *colouring matrix*. For example,

$$\begin{bmatrix} -1 & t & 0 & 1-t \\ 1-t & t & -1 & 0 \\ 0 & 1-t & -1 & t \\ -1 & 0 & 1-t & t \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

is the matrix equation corresponding to the following diagram of the knot 4_1 (the figure-8 knot):
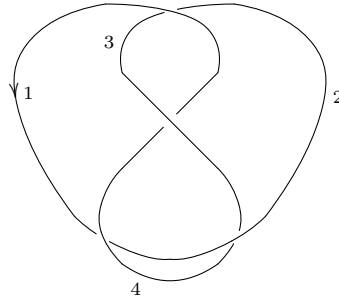
Remark 4. *We note for future reference that the sum of the columns of any colouring matrix is the zero column (this is obvious from the coefficients in the colouring conditions).*

2.3. **The Alexander Polynomial.** The Alexander polynomial $\text{Alex}_K(t)$ of a knot K is a knot invariant that may be computed in a number of ways [1]. In Livingston [17] a $N \times N$ matrix, where N is the number of crossings (and arcs) in the diagram is defined. This matrix is exactly the same as the colouring matrix A from the previous subsection [4], assuming we use the same enumerations for arcs and crossings. The only difference is the orientation convention used in the two procedures. In any case, the Alexander polynomial, to be defined next, is independent of all choices, including the choice of orientation for the diagram.

The Alexander polynomial is essentially obtained by removing the final column and row of A to obtain a reduced matrix A_r, which is non-singular, and taking its determinant. However different choices for the enumerations and orientation may lead to different polynomials, which however are always related by multiplying by a sign or an integer power of t. Thus the Alexander polynomial in the definition to follow is, in fact, an equivalence class of polynomials.

Definition 5 (From [17])**.** *The $(N-1) \times (N-1)$ matrix A_r obtained by removing the final row and column from the $N \times N$ matrix A described above is called the Alexander matrix of K. The determinant of the Alexander matrix is called the Alexander polynomial of K, regarded up to equivalence, where two polynomials are equivalent if they are obtained from each other by multiplying by a sign and/or by an integer power of t. It is customary to normalize the Alexander polynomial [8] by choosing the representative with no negative powers of t and a positive constant term.*

Example 6. *Applying the definition to the colouring matrix for the knot 4_1 from the previous subsection, the Alexander polynomial is $-1 + 3t - t^2$, or in normalized form $1 - 3t + t^2$.*

3. Computations with Linear Finite Alexander Quandles

From now on we will be concentrating on quandle colourings using a special class of quandles, called linear finite Alexander quandles. These are finite Alexander quandles (see subsection 2.2), of the form $\mathbb{Z}_n[t, t^{-1}] / (t - m)$, where n and m are integers and n, m are coprime. Recall that the elements are equivalence classes of Laurent polynomials having the same remainder when divided by $t - m$. Obviously the polynomial t is in the same equivalence class as the constant polynomial m, since $t = (t - m) + m$. Similarly t^{-1} is equivalent to m^{-1} (the inverse of m in $\mathbb{Z}_n$), since $t^{-1} - m^{-1} = -m^{-1}t^{-1}(t - m)$. (Note that m is invertible since $\gcd(m, n) = 1$). It follows that any polynomial is equivalent to some number in $\mathbb{Z}_n$ and that one can identify $\mathbb{Z}_n[t, t^{-1}] / (t - m)$ with $\mathbb{Z}_n$. The quandle operation can be written as $a * b = ma + (1 - m)b \pmod{n}$ and $a *' b = m^{-1}a + (1 - m^{-1})b \pmod{n}$.

Thus the colourings of any knot diagram with elements of a linear finite Alexander quandle are the solutions of the matrix equation

$$AX = 0,$$

where A is the colouring matrix of subsection 2.2 with t replaced by m, X is the vector of colouring unknowns $(X_1, X_2, \ldots, X_i, \ldots)$, belonging to $\mathbb{Z}_n$, and equalities hold in $\mathbb{Z}_n$ (i.e. equality mod n).

In this section we summarize our work towards giving explicit formulas for the number of colourings, which originated in the results of our computations. Firstly we simplify the original matrix to an equivalent one. The simplified matrices are "as triangular as possible" (see below) and they are related to the original one by an equivalence relation that preserves the number of colourings.

Definition 7. *Let A and B be $N \times N$ matrices with entries in $\mathbb{Z}_n[m, m^{-1}]$. We say B is equivalent to A iff the following two conditions hold:*

1) *B is obtained from A by a sequence of the following operations:*
   a) multiplication of a row by m, m^{-1} or -1; b) replacing a row by its sum with some row; c) swapping two rows; d) swapping two columns,
2) *$|A_r| = |B_r|$, up to equivalence.*

It is easy to check that this is indeed an equivalence relation and that it preserves the number of colourings and also the property that the sum of the columns of the matrix is the zero vector, as well as preserving the Alexander polynomial. The first property is useful to derive general expressions for the number of colourings for certain classes of knots. We have written several algorithms in Mathematica that reduce the colouring matrix to a standard echelon form. The latest version is presented in the appendix.

Applying the algorithm to the prime knots with up to 10 crossings, we reduced their colouring matrices to three kinds of echelon form, which we call Type I, Type II and non-triangular. For the two first types a general expression for the number of colourings can be found.

Type I matrices: These are upper triangular, with 1's on the diagonal except in the penultimate row where the entry is denoted $\alpha(m)$, and in the last row which has all entries equal to zero. The entry $\alpha(m)$ is the normalized Alexander polynomial of the knot for t equal to m.

<div align="center">Type I</div>

$$
\begin{bmatrix}
1 & \lambda_{12}(m) & \cdots & & \cdots & & \lambda_{1N}(m) \\
0 & \ddots & \ddots & & \cdots & & \vdots \\
t \ \vdots & 0 & 1 & \lambda_{N-2 \ N-1}(m) & & \lambda_{N-2 \ N}(m) \\
\vdots & \vdots & 0 & \alpha(m) & & -\alpha(m) \\
0 & 0 & \cdots & & \cdots & & 0
\end{bmatrix}
$$

In this case, it is easy to find a general expression for the number of linear quandle colourings, using an arbitrary linear quandle, as the following proposition shows.

Proposition 8. *Let K be a Type I knot, and Q be the linear finite Alexander quandle $Q = \mathbb{Z}_n[t, t^{-1}]/(t - m)$. Then $C_Q(K)$, the number of Q-colourings of K, is*

$$C_Q(K) = n \times gcd(Alex_K(m), n).$$

Proof. The proof is straightforward and uses the linear congruence theorem. Details can be found in [4]. Recall that the Alexander polynomial of a type I matrix is $\alpha(m)$. ☐

Type II matrices: These are upper triangular, with 1's on the diagonal except in the antepenultimate and penultimate rows, where the diagonal entries are

denoted $\alpha_1(m)$ and $\alpha_2(m)$, respectively, and in the last row which has all entries equal to zero. The product $\alpha_1(m)\alpha_2(m)$ is the normalized Alexander polynomial of the knot for t equal to m.

Type II

$$\begin{bmatrix} 1 & \lambda_{12}(m) & \cdots & & \cdots & & \cdots & & \lambda_{1N}(m) \\ 0 & \ddots & \ddots & & \cdots & & \cdots & & \vdots \\ \vdots & 0 & 1 & \lambda_{N-3\ N-2}(m) & \lambda_{N-3\ N-1}(m) & & \lambda_{N-3\ N}(m) \\ \vdots & \vdots & 0 & \alpha_1(m) & \beta_1(m) & & -(\alpha_1(m)+\beta_1(m)) \\ \vdots & \vdots & \vdots & 0 & \alpha_2(m) & & -\alpha_2(m) \\ 0 & 0 & 0 & \cdots & \cdots & & 0 \end{bmatrix}$$

For Type II knots the process of calculating the number of solutions is similar to that of Type I knots. In this case, apart from the equations coming from the final two rows of the Type II matrix a third equation has to be considered.

Proposition 9. *Let K be a Type II knot, and Q be the linear finite Alexander quandle $Q = \mathbb{Z}_n[t, t^{-1}]/(t-m)$. Then $C_Q(K)$, the number of Q-colourings of K, is*

$$C_Q(K) = n \times gcd(\alpha_2(m), n) \times gcd(\beta_1(m) \frac{n}{gcd(\alpha_2(m), n)}, gcd(\alpha_1(m), n)).$$

Proof. See [4].   □

Non-triangular matrices: These are just like Type II matrices, except for replacing the triangular 2×2 array

$$\begin{bmatrix} \alpha_1(m) & \beta_1(m) \\ 0 & \alpha_2(m) \end{bmatrix} \tag{3}$$

with a non-triangular 2×2 array, which has determinant equal to the normalized Alexander polynomial. In these cases we were unable to find a general formula for the number of colourings. This, however, does not mean that the simplified matrices can not be used to calculate colourings. For example, for the knot 9_{35} our program gave as output the following non-triangular echelon matrix:

$$\begin{bmatrix} 1 & h_{1j} & \cdots & & \cdots & & \cdots & & h_{1r} \\ 0 & \ddots & \ddots & & \cdots & & \cdots & & \vdots \\ \vdots & 0 & 1 & h_{r-3\ r-2} & h_{r-3\ r-1} & & h_{r-3\ r} \\ \vdots & \vdots & 0 & 2-m & -1-m & & -1+2m \\ \vdots & \vdots & \vdots & -3 & -2+7m & & 5-7m \\ 0 & 0 & 0 & 0 & 0 & & 0 \end{bmatrix}$$

If we choose $m = 2$ and $n = 3$, we get a matrix with the 3 final rows consisting only of zeros, after reducing modulo 3. Thus, for this case, the number of colourings is equal to $3^3 = 27$, since we can choose the final 3 unknowns, X_7, X_8 and X_9, freely in $\mathbb{Z}_3$, and the remaining unknowns are uniquely determined after this choice.

We now summarize the results of our computations.

Type I The vast majority of colouring matrices for knots with up to 10 crossings could be reduced to a matrix of Type I. We do not list these 216 knots, or the special entry $\alpha(m)$, since it is just the corresponding normalized Alexander polynomial with t equal to m. For up to 9 crossings these are listed in Livingston [17]. For up to 10 crossings they are given in shorthand form in Rolfsen [22] or Kawauchi [16], e.g. for the knot 6_3, Rolfsen gives $[5 - 3 + 1$, meaning that the Alexander polynomial is $m^{-2} - 3m^{-1} + 5 - 3m + m^2$ or in normalized form $1 - 3m + 5m^2 - 3m^3 + m^4$.

Type II There were 21 colouring matrices that could be reduced to Type II matrices, namely those coming from the following knots:

$$8_{18}; 9_{37}; 9_{40}; 9_{46}; 10_{61}; 10_{63}; 10_{65}; 10_{74}; 10_{75}; 10_{98}; 10_{99}; 10_{103};$$

$$10_{106}; 10_{122}; 10_{123}; 10_{140}; 10_{142}; 10_{144}; 10_{147}; 10_{155}; 10_{164}$$

Non-triangular Finally there were 12 colouring matrices that could be reduced to the non-triangular echelon form described above. These came from the following knots:

$$9_{35}; 9_{38}; 9_{41}; 9_{47}; 9_{48}; 9_{49}; 10_{69}; 10_{101}; 10_{108}; 10_{115}; 10_{157}; 10_{160}$$

In [4] we list the characteristic 2 by 2 arrays of Type II and non-triangular matrices.

4. COMPARING COLOURINGS AND THE ALEXANDER POLYNOMIAL

The results of the previous section open the way for making comparisons between two separate knot invariants: the Alexander polynomial, on the one hand, and the number of quandle colourings for any linear Alexander quandle, on the other. For example, it is a direct consequence of proposition 8 that two Type I knots, i.e. knots whose colouring matrix can be reduced to Type I form, with the same Alexander polynomial, cannot be distinguished by linear quandle colourings. In [4], using propositions 8 and 9, other cases were found for which Type I or Type II knots with the same Alexander polynomial can be distinguished by linear quandle colourings.

However, we want to focus here on the question of whether knots with different Alexander polynomials can be distinguished by linear quandle colourings. In [4], we showed that, given two knots with triangularizable colouring matrices but different Alexander polynomials, a linear finite Alexander quandle can be exhibited that distinguishes the two knots by the number of colourings.

Proposition 10. *Let K_1 and K_2 be knots with different Alexander polynomials $Alex_{K_1}(m) \neq Alex_{K_2}(m)$. Assume furthermore that their colouring matrices are both equivalent to a triangular matrix with only zeros in the final row. Then there is a linear finite Alexander quandle that distinguishes them by counting colourings.*

Proof. See [4]. $\qquad\square$

Note that this holds for any two knots with colouring matrices that are equivalent to a matrix of triangular form, which may be of a more general type than Type I or Type II.

We conjectured in [4], that any two knots with different Alexander polynomials can be distinguished by colourings, irrespective of whether their colouring matrix can be triangularized. If all colouring matrices could be triangularized this would follow immediately from the result above. However, we also showed that there are colouring matrices that are definitely not triangularizable (using the operations of Definition 7). This exploits the fact that some knots have Alexander polynomials that cannot be properly factorized.

Proposition 11. *The colouring matrices of the knots 9_{35}, 9_{47}, 9_{48}, 9_{49} and 10_{157} cannot be triangularized.*

Proof. For the full proof see [4]. We sketch the idea. Since these knots have non-properly factorizable Alexander polynomials, we know that, if their colouring matrix is equivalent to a triangular form, then that form must be Type I. Hence the number of colourings using any linear Alexander quandle $\mathbb{Z}_n[t, t^{-1}] \;/\; (t - m)$ must be given by the Type I formula $n \times \gcd(\mathrm{Alex}_K(m), n)$ of proposition 8. The proof consists now in exhibiting for each such knot K a quandle for which the true number of colourings is not equal to $n \times \gcd(\mathrm{Alex}_K(m), n)$. The specific quandles for this purpose are given by $m = 2$ and $n = 3$ for 9_{35}, 9_{47} and 9_{48}; $m = 4, n = 5$ for 9_{49}; and $m = 6, n = 7$ for 10_{157}. $\square$

The fact that there are knots with colouring matrices that cannot be converted to equivalent triangular matrices is an interesting starting point for further research.

In fact, for fixed m, any matrix can be further simplified (see [3]). This suggests that, for the non-triangular matrices, although there is no general equivalent triangular matrix for all values of m, there will be different triangular matrices depending on certain conditions on the entries of the colouring matrix. Consider the simplified colouring matrix[1] for 9_{35} where the first row was multiplied by -1:

$$\begin{pmatrix} m - 2 & 1 + m \\ -3 & -2 + 7m \end{pmatrix}$$

It is easy to further simplify this matrix in the case where $m - 2$ has an inverse[2], i.e. for those quandles where m is coprime with n and furthermore $m - 2$ is coprime with n. There will be infinitely many such pairs. Then we can multiply the first row by the inverse $(m - 2)_n^{-1}$ of $m - 2$:

$$\begin{pmatrix} 1 & (m - 2)_n^{-1}(1 + m) \\ -3 & -2 + 7m \end{pmatrix}$$

This matrix can be further simplified by adding 3 times the first row to the second row:

$$\begin{pmatrix} 1 & (m - 2)_n^{-1}(1 + m) \\ 0 & -2 + 7m + 3(m - 2)_n^{-1}(1 + m) \end{pmatrix}$$

This matrix is of Type I form, but the determinant is not the same as that of the original matrix. However we may now multiply the second row by $m - 2$ obtaining a triangular matrix of Type I form with the correct determinant, and that has the same number of colourings as the original one using any quandle such that $m - 2$ is coprime with n. Moreover, for those quandles, the number of colourings is given by the Type I expression $C_Q(K) = n \times \gcd(\mathrm{Alex}_K(m), n)$.

$$\begin{pmatrix} 1 & (m - 2)_n^{-1}(1 + m) \\ 0 & (-2 + 7m)(m - 2) + 3(1 + m) \end{pmatrix}$$

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[1]For simplicity we display only the significant part of that matrix.

[2]We could also choose the second row and consider those quandles where 3 is coprime with n.

The previous discussion suggests that, even if it is not in general possible to find a triangular matrix for all colouring matrices, there will be a triangular matrix that is equivalent to the original one for an infinite number of finite Alexander quandles. If that is the case then we conjecture that the proof of proposition 10 can be adjusted and a suitable quandle can be found that distinguishes the given knots, even when the colouring matrices are not triangular. Note also that the simplification presented above can be easily adapted to all other non triangular matrices of prime knots up to ten crossings since their significant part is also a 2×2 matrix. Of course, the procedure has to be generalized to any non-triangular matrix. But it is also interesting to note that we can apply similar ideas to convert Type II matrices into Type I with the advantage that, for those quandles where this operation applies, the number of colourings will depend only on the Alexander polynomial of the knot.

It is interesting to review proposition 11 with the previous remarks in mind. In fact we show that the colouring matrices of knots 9_{35}, 9_{47}, 9_{48}, 9_{49} and 10_{157} cannot be triangularized by finding a quandle that is a "counter-example" to that property. The significant part of their reduced matrices follows.

$$9_{35} : \begin{pmatrix} 2 - m & -1 - m \\ -3 & -2 + 7m \end{pmatrix}$$

$$9_{47} : \begin{pmatrix} -1 + 4m - m^2 & -2 - m - m^2 + m^3 \\ 2 - 7m & 3 + 4m + 2m^2 - m^4 \end{pmatrix}$$

$$9_{48} : \begin{pmatrix} 2 - m & 2 - 8m + 7m^2 - m^3 \\ 3 & 3 - 10m + 2m^2 + 5m^3 - m^4 \end{pmatrix}$$

$$9_{49} : \begin{pmatrix} -2 - m + m^2 & 3 - m - m^2 - 2m^3 \\ 3 - 2m & -3 + 3m + m^2 \end{pmatrix}$$

$$10_{157} : \begin{pmatrix} 4 - 3m & -7 + 12m - 9m^2 + 6m^3 - m^4 \\ -1 + m^2 & 2 - 3m + m^2 - m^3 \end{pmatrix}$$

For the knot 9_{35} that quandle is specified by $m = 2$ and $n = 3$. We have also seen above that it is possible to triangularize the colouring matrix if $m - 2$ is coprime with n or n is coprime with 3. This is not contradictory since $2 - 2 = 0$ is not coprime with 3 and 3 is not coprime with 3. Similar considerations can be made for the other knots. The colouring matrix for 9_{47} can be triangularized if $-1 + 4m - m^2$ or $2 - 7m$ is coprime with n. The counter example quandle is again $m = 2$ and $n = 3$. In this case $-1 + 4m - m^2 = 3$ and $2 - 7m = -12$ are both not coprime with 3. The colouring matrix for 9_{48} can be triangularized if $2 - m$ or 3 are coprime with n so this case is identical to the one for 9_{35}. The colouring matrix for 9_{49} can be triangularized if $-2 - m + m^2$ or $3 - 2m$ are coprime with n. In this case the quandle used was $m = 4, n = 5$. Indeed $-2 - m + m^2 = 10$ and $3 - 2m = -5$ are not coprime with 5. Finally, the colouring matrix for 10_{157} can be triangularized if $4 - 3m$ or $-1 + m^2$ are coprime with n. For the counter example quandle $m = 6, n = 7$ one has $4 - 3m = -14$ and $-1 + m^2 = 35$ that again are not coprime with 7.

We also know that the knots 10_{69}, 10_{101}, 10_{115} and 10_{160} have Alexander polynomials that do not factorize but we could not find a quandle that is a counter example. It would be interesting to review these cases using this new information: either there is a counter example quandle or we can show that such colouring matrices are equivalent to triangular colouring matrices.

In conclusion, there are colouring matrices that are not triangularizable in the sense that there is no equivalent triangular matrix for all values of m and n. But, in general, all colouring matrices can be triangularized for infinite values of m and n. This suggests that proposition 10 can be generalized and that knots with different Alexander polynomials can always be distinguished by finite Alexander quandles.

5. Conclusions and Further Work

It seems likely that in future work we can follow the arguments outlined above to sharpen our conclusions, and prove that the number of colourings using any finite linear quandle, is at least as strong an invariant as the Alexander polynomial for distinguishing knots, with less or no restrictions on the type of reduced colouring matrix. With further improvements, the calculations we have performed might be extended to knot diagrams with more than 10 crossings. Note that our proofs of the general formulae for the number of colourings for Type I and Type II knots apply irrespective of the number of crossings, i.e. the size of the colouring matrix. Another natural direction for future work is to try and find general expressions for the number of colourings when the reduced colouring matrix is non-triangular, or of a more general triangular form.

Appendix A. The algorithm for reducing colouring matrices

In this section we describe the current version of the algorithm that computes the reduced colouring matrix. Its main function is called `f` and takes as input the Gauss code of a diagram of a knot. The Gauss codes were obtained from the diagrams in [22]. The Mathematica code and the Gauss codes are available on request (by email to the first author).

For example, `a10a1={-9,-8,-7,-6,10,-4,-3,-2,-1,5}` is the Gauss code for the knot 10_1 and the output of `f[a10a1]` is the following colouring matrix:

$$\begin{pmatrix}
1 & -m & 0 & 0 & 0 & 0 & 0 & -1+m & 0 & 0 \\
0 & 1 & -m & 0 & 0 & 0 & -1+m & 0 & 0 & 0 \\
0 & 0 & 1 & -m & 0 & -1+m & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -m & 0 & 0 & 0 & -1+m & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -\frac{1}{m} & -1+\frac{1}{m} \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -3+\frac{1}{m}+m & 2-\frac{1}{m}-m \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -5+\frac{2}{m}+2m & 4-\frac{2}{m}-2m \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -7+\frac{3}{m}+3m & 6-\frac{3}{m}-3m \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4-9m+4m^2 & -4+9m-4m^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

As stated previously, the operations used in this reduction process are of four types:

- multiplying rows by -1, m and m^{-1}
- adding rows
- swapping rows
- swapping columns

Our algorithm is optimized since it uses the polynomial greatest common divisor (GCD) of a column, starting with the first column. When there is a column such that its GCD occurs as an entry in the column, then it is trivial to annihilate other entries in that column, and achieve zero entries below the diagonal in that column by performing suitable row swaps. If this procedure is no longer possible, even by swapping columns, then the algorithm uses other strategies: look for a zero entry and two non-zero entries such that a linear combination of them is their GCD. By performing row operations the zero entry will become the GCD of those two polynomials, and that is then used to annihilate those non-zero entries. If a column has no zero entry, but has two entries one of which is a multiple of the other, the zero entry can be obtained by row operations. A further strategy uses one non-zero entry multiplied by a suitable power of m to lower the degree of another entry.

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