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ON THE CONVEX COMBINATIONS OF SLANTED HALF-PLANE HARMONIC MAPPINGS

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Abstract. The main result in this paper shows that convex combinations of slanted half-plane harmonic mappings are convex.

1. Introduction and Preliminaries

Let $\mathcal{H}$ denote the class of all *complex-valued harmonic mappings* f in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ normalized by $f(0) = f_z(0) - 1 = 0$. Let $\mathcal{S}_H$ be the subclass of $\mathcal{H}$ consisting of univalent and sense-preserving functions. Such functions can be written in the form $f = h + \overline{g}$, where

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} b_n z^n$$

are analytic in $\mathbb{D}$ and the Jacobian $J_f(z) = |h'(z)|^2 - |g'(z)|^2 > 0$. Equivalently, $h' \neq 0$ in the unit disk and the *analytic complex dilatation* ω of f defined by $\omega = g'/h'$ satisfies $|\omega(z)| < 1$, $z \in \mathbb{D}$. The classical family $\mathcal{S}$ of analytic, univalent and normalized functions in $\mathbb{D}$ is a subclass of $\mathcal{S}_H$ with $g(z) \equiv 0$. The family of all functions $f \in \mathcal{S}_H$ with the additional property that $f_{\overline{z}}(0) = 0$ is denoted by $\mathcal{S}_H^0$. Further information about planar harmonic mappings can be found in the book [5].

A function $f \in \mathcal{S}_H$ is said to be *convex* if it maps the unit disk onto a convex domain. If the intersection of $f(\mathbb{D})$ with every line parallel to the one passing through the origin and $e^{i\theta}$ $(0 \leq \theta < \pi)$ is connected, we say that f is *convex in the direction* θ. It is obvious that any convex function is convex in *every* direction.

The shear construction introduced by Clunie and Sheil-Small in [2] provides a way of producing univalent harmonic functions in the unit disk to regions which are convex in one direction. More speciffically, the following result is an easy consequence of [2, Thm. 5.3].

Theorem A. *A harmonic function $f = h + \overline{g}$, locally univalent in $\mathbb{D}$, is a univalent mapping of onto a domain convex in the direction θ if and only if $h - e^{2i\theta} g$ is an analytic univalent mapping in $\mathbb{D}$ onto a domain convex in the direction θ.*

As usual, we use $\mathcal{C}_H^0$ to denote the set of all sense-preserving convex harmonic mappings $f = h + \overline{g}$ in $\mathbb{D}$ satisfying $h(0) = g(0) = g'(0) = 1 - h'(0) = 0$. Indeed, in

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what follows, all the functions considered here will satisfy these normalizations so that we won't mention this again.

Of particular importance among the functions in $\mathcal{C}_H^0$ (because of their relationship with certain extremal problems in this class, for instance) is the family of functions that map the unit disk onto the half-plane $H = \{w \in \mathbb{C} : \Re(w) > -1/2\}$. It is known (see [3, Cor. 3.2]) that any sense-preserving univalent function $f = h + \overline{g}$ with $f(\mathbb{D}) = H$ satisfies

$$h(z) + g(z) = \frac{z}{1-z}, \qquad z \in \mathbb{D}. \tag{1.1}$$

We say that a normalized sense-preserving harmonic mapping $f = h + \overline{g}$ belongs to $\mathcal{F}_0$ if and only if (1.1) holds. Note that since the function $l(z) = z/(1-z)$ is convex in the unit disk, by Theorem A we see that any function $f \in \mathcal{F}_0$ will be univalent in the unit disk and also convex in the vertical direction (this is, convex in the direction $\pi/2$). In fact, a direct application of [1, Lemma 1], which proves that any *sense-preserving* univalent harmonic mapping $f = h + \overline{g}$ satisfying (1.1) is convex, shows that any function in $f \in \mathcal{F}_0$ maps the unit disk onto a convex domain.

If a function $f \in \mathcal{C}_H^0$ maps $\mathbb{D}$ onto the half-plane $\{\Re(e^{i\gamma}w) > -1/2\}$ for some $0 \le \gamma < 2\pi$, then f is said to be a *slanted half-plane mapping with parameter* γ. Obviously, any slanted half-plane mapping with parameter $\gamma = 0$ maps the unit disk onto H. Moreover, f is a slanted half-plane mapping with parameter γ if and only if $\widehat{f}(z) = e^{i\gamma} f(e^{-i\gamma}z)$ satisfies $\widehat{f}(\mathbb{D}) = H$. Therefore, any such slanted half-plane mapping $f = h + \overline{g}$ satisfies

$$h(z) + e^{-2i\gamma} g(z) = \frac{z}{1 - e^{i\gamma}z}, \qquad z \in \mathbb{D}. \tag{1.2}$$

We will use $\mathcal{F}_\gamma$ to denote the family of sense-preserving harmonic mappings $f = h + \overline{g}$ in the unit disk satisfying (1.2). Again, using Theorem A, we see that any function $f \in \mathcal{F}_\gamma$ is necessarily univalent in $\mathbb{D}$ and also convex in certain direction that depends on γ. Note that using the same (invertible) correspondence $f \to \widehat{f}$ mentioned above between slanted half-plane mappings with parameter γ and functions that map the unit disk onto H one can prove that $f \in \mathcal{F}_\gamma$ if and only if $\widehat{f} \in \mathcal{F}_0$. We refer the reader to [1, 4, 6, 7] for recent investigations in the subclass $\mathcal{F}_\gamma$.

A common way to try to construct new functions with a given property is to take convex combinations of functions satisfying the desired condition. More concretely, given the functions $f_j = h_j + \overline{g_j} \in \mathcal{S}_H^0$ and the real numbers t_j with $0 \le t_j \le 1$, $(j = 1, 2, \ldots, n)$, we construct a new harmonic mapping

$$F = \sum_{j=1}^{n} t_j f_j = \sum_{j=1}^{n} t_j h_j + \sum_{j=1}^{n} t_j \overline{g_j} = H + \overline{G}, \qquad \sum_{j=1}^{n} t_j = 1, \tag{1.3}$$

with dilatation $\omega = G'/H'$.

Wang *et al.* [8] proved a sufficient condition for the convex combination of the form $F = tf_1 + (1-t)f_2$ with $h_j + g_j = z/(1-z)$, $j = 1, 2$, to be univalent and convex in the horizontal direction.

The aim of this paper is to show that as long as the functions f_j belong to $\mathcal{F}_\gamma$, the corresponding function F as in (1.3) will map the unit disk onto a convex

domain, thus generalizing the main results obtained in [8]. The following lemma will be fundamental to prove this result.

Lemma 1.1. *Let k_j $(j = 1, 2)$ be analytic in $\mathbb{D}$ with $|k_j(z)| < 1$ and $k_j(0) = 0$. Then, in the unit disk,*

$$\Re\left(\frac{1 - k_1\overline{k_2}}{(1 + e^{-2i\beta}k_1)(1 + e^{2i\beta}\overline{k_2})}\right) > 0$$

for all $\beta \in \mathbb{R}$.

Proof. Note that the functions $z \to 1/(1 + z)$ and $z \to z/(1 - z)$ map the unit disk $\mathbb{D}$ onto $\{w \in \mathbb{C} : \Re(w) > 1/2\}$ and $\{w \in \mathbb{C} : \Re(w) > -1/2\}$, respectively. Hence, for any given $\lambda = 1$ and any function k analytic in $\mathbb{D}$ with $|k| < 1$, we have

$$\Re\left(\frac{1}{1 + \overline{\lambda k}}\right) > \frac{1}{2} \qquad \text{and} \qquad \Re\left(\frac{-\lambda k}{1 + \lambda k}\right) > -\frac{1}{2}.$$

Since

$$\frac{1 - k_1\overline{k_2}}{(1 + e^{-2i\beta}k_1)(1 + e^{2i\beta}\overline{k_2})} = \frac{1}{1 + \overline{e^{-2i\beta}k_2}} - \frac{e^{-2i\beta}k_1}{1 + e^{-2i\beta}k_1},$$

the result easily follows. □

2. Main Result

We now state the main theorem.

Theorem 2.1. *Any convex combination of functions in $\mathcal{F}_\gamma$ is convex. Therefore, any convex combination of slanted half-plane mappings with parameter γ is convex.*

Proof. For $j = 1, 2, \ldots, n$, let t_j be real numbers (with $0 \le t_j \le 1$ and such that $\sum_{j=1}^n t_j = 1$) and $f_j = h_j + \overline{g_j} \in \mathcal{F}_\gamma$; this is,

$$h_j + e^{-2i\gamma}g_j = \frac{z}{1 - e^{i\gamma}z}, \qquad 0 \le \gamma < 2\pi, \quad z \in \mathbb{D}.$$

Then, the mappings $\widehat{f_j} = \widehat{h}_j + \overline{\widehat{g_j}}$ defined by $\widehat{f_j}(z) = e^{i\gamma}f_j(e^{-i\gamma}z) \in \mathcal{F}_0$, hence

$$\widehat{h}_j(z) + \widehat{g}_j(z) = \frac{z}{1 - z}. \tag{2.1}$$

Moreover, the function

$$\widehat{F} = \sum_{j=1}^n t_j\widehat{f_j} = \widehat{H} + \overline{\widehat{G}}$$

satisfies $\widehat{F}(z) = e^{i\gamma}F(e^{-i\gamma}z)$, where F is as in (1.3). Therefore, to show that F is convex, it suffices to check that $\widehat{F}$ is convex. Since

$$\widehat{H}(z) + \widehat{G}(z) = \frac{z}{1 - z},$$

we see that according to [1, Lemma 1], the result will follow once it is proved that $\widehat{F}$ is locally univalent.

To prove that $\widehat{F}$ is locally univalent, let us use ω_j to denote the dilatation of $\widehat{f_j}$ and define

$$\Phi = \left|\sum_{j=1}^n \frac{t_j}{1 + \omega_j}\right|^2 - \left|\sum_{j=1}^n \frac{t_j\omega_j}{1 + \omega_j}\right|^2.$$

We find that

$$\begin{aligned}\Phi &= \left(\sum_{j=1}^{n}\frac{t_j}{1+\omega_j}\right)\left(\sum_{j=1}^{n}\frac{t_j}{1+\overline{\omega_j}}\right)-\left(\sum_{j=1}^{n}\frac{t_j\omega_j}{1+\omega_j}\right)\left(\sum_{j=1}^{n}\frac{t_j\overline{\omega_j}}{1+\overline{\omega_j}}\right)\\ &= \sum_{j=1}^{n}\sum_{k=1}^{n}\frac{t_jt_k}{(1+\omega_j)(1+\overline{\omega_k})}-\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{t_jt_k\omega_j\overline{\omega_k}}{(1+\omega_j)(1+\overline{\omega_k})}\\ &= \sum_{j=1}^{n}\sum_{k=1}^{n}\frac{t_jt_k(1-\omega_j\overline{\omega_k})}{(1+\omega_j)(1+\overline{\omega_k})}\\ &= 2\sum_{j=1}^{n}\sum_{k<j}^{n}\Re\left(\frac{t_jt_k(1-\omega_j\overline{\omega_k})}{(1+\omega_j)(1+\overline{\omega_k})}\right)+\sum_{j=1}^{n}\frac{t_j^2(1-|\omega_j^2|)}{|1+\omega_j|^2}.\end{aligned}$$

Since ω_j are analytic and $|\omega_j|<1$ $(j=1,2,\ldots,n)$, by Lemma 1.1 we see that $\Phi>0$ in the unit disk. Then, on the one hand,

$$\left|\sum_{j=1}^{n}\frac{t_j}{1+\omega_j}\right|>0 \tag{2.2}$$

and, on the other hand

$$\frac{\left|\sum_{j=1}^{n}\frac{t_j\omega_j}{1+\omega_j}\right|}{\left|\sum_{j=1}^{n}\frac{t_j}{1+\omega_j}\right|}<1. \tag{2.3}$$

Now, using (2.1), we have

$$\widehat{h}_j'(z)=\frac{1}{\big(1+\omega_j(z)\big)(1-z)^2},$$

so that

$$\widehat{H}'(z)=\sum_{j=1}^{n}\frac{t_j}{\big(1+\omega_j(z)\big)(1-z)^2}$$

and by (2.2) we obtain that $\widehat{H}'\neq 0$ in the unit disk. Moreover, the modulus of the dilatation ω of $\widehat{F}$ equals

$$|\omega|=\left|\frac{\sum_{j=1}^{n}t_j\widehat{g}_j'}{\sum_{j=1}^{n}t_j\widehat{h}_j'}\right|==\left|\frac{\sum_{j=1}^{n}t_j\omega_j\widehat{h}_j'}{\sum_{j=1}^{n}t_j\widehat{h}_j'}\right|=\left|\frac{\sum_{j=1}^{n}\frac{t_j\omega_j}{1+\omega_j}}{\sum_{j=1}^{n}\frac{t_j}{1+\omega_j}}\right|.$$

Hence, by (2.3), we get that $|\omega(z)|<1$ for all $z\in\mathbb{D}$. This shows that $\widehat{F}$ is locally univalent in $\mathbb{D}$ and ends the proof of the theorem. □

To finish this paper, we state the following corollary of Theorem A and the proof of Theorem 2.1.

Corollary 2.2. *Let $f_j=h_j+\overline{g_j}\in\mathcal{S}_H^0$ satisfy*

$$h_j-e^{-2i\gamma}g_j=\frac{z}{1-e^{i\gamma}z},\qquad 0\le\gamma<2\pi,\quad j=1,2,\ldots,n.$$

Then, the function F defined by (1.3) *is convex in the direction $-\gamma$.*

Proof. The function $\widehat{F} = \widehat{H} + \overline{\widehat{G}}$ defined as in the proof of Theorem 2.1 satisfies $\widehat{H} - \widehat{G} = z/(1-z)$. A similar procedure as the one employed above shows that $\widehat{F}$ is locally univalent in the unit disk. Hence, by Theorem A, $\widehat{F}$ is convex in the horizontal direction. This is equivalent to the fact that the function F itself is convex in the direction $-\gamma$. □

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