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CONFORMAL CHANGE OF DOUGLAS SPACE OF SECOND KIND WITH (α, β)-METRIC

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Abstract. The Douglas space of second kind with an (α, β)-metric is defined by Il Yong Lee [5]. In this paper, we prove the Douglas space of second kind with an (α, β)-metric is conformally transformed to a Douglas space of second kind. Further, we find the condition that, conformal change of Finsler space with Randers and Kropina metric is of Douglas space of second kind.

1. Introduction

The Finsler space $F^n = (M^n, L(x, y))$ is said to have an (α, β)-metric if L is a positively homogeneous function of degree one in two variables $\alpha = \sqrt{(a_{ij}(x)y^i y^j)}$ and $\beta = b_i(x)y^i$. The notion of Douglas space was introduced by S. Bacso and M. Matsumoto as a generalization of Beewald space from viewpoint of geodesic equations. M. Matsumoto [9] has found the condition that, the Finsler space with some (α, β)-metrics to be Douglas space. Recently, I. Y. Lee [5] defined a Douglas space of second kind and he proved that Finsler space with Matsumoto metric to be a Douglas space of second kind. It is remarkable that a Finsler space with (α, β)-metric is a Douglas space of second kind if the tensor B^i_{hjk} vanishes identically [5].

On the other hand conformal theory of Finsler space was introduced by M. S. Kneblman in 1929 and this theory has been investigated in detail by M. Hashiguchi [4], later on Y. D. Lee [6] and B. N. Prasad [12] found conformally invariant tensors in the Finsler space with (α, β)-metric. In [10], conformal transformation of Douglas space with special (α, β)-metric have been studied by S. K. Narasimhamurthy. And also in [11] we have derived the necessary and sufficient condition for Douglas spaces with Douglas space with (α, β)-metric under conformal β change.

The purpose of the present paper is to develop the condition for a Douglas space of second kind under conformal transformation (Theorem 4.2). Further we proved that Douglas space of second kind with Randers metric and Kropina metric is conformally transformed to Douglas space of second kind (Theorem 5.1 and Theorem 5.2).

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2. Preliminaries

Let $F^n = (M^n, L)$ be an n-dimensional Finsler space, where M^n be a differentiable manifold of dimension n and $L(x,y)$ (where $y^i = \dot{x}^i$) is the fundamental function defined on the manifold $TM \setminus \{0\}$ of nonzero tangent vectors. We assume that $L(x,y)$ is positive and the metric tensor $g_{ij}(x) = \frac{1}{2}\dot{\partial}_j \dot{\partial}_i L^2$ is positive definite, where $\dot{\partial}_i = \partial/\partial y^i$.

The geodesics of an n-dimensional Finsler space $F^n = (M^n, L)$ are given by the system of differential equations [4]

$$\frac{d^2x^i}{dt^2}y^j - \frac{d^2x^j}{dt^2}y^i + 2(G^i y^j - G^j y^i) = 0, \quad y^i = \frac{dx^i}{dt},$$

in a parameter t. The function $G^i(x,y)$ is given by

$$2G^i(x,y) = g^{ij}(y^r \dot{\partial}_j \partial_r F - \partial_j F) = \gamma^i_{jk} y^j y^k,$$

where $\partial_i = \frac{\partial}{\partial x^i}$, $F = \frac{L^2}{2}$ and $g^{ij}(x,y)$ be the inverse of Finsler metric tensor $g_{ij}(x,y)$. According to [9], F^n is of Douglas type if

$$D^{ij} = G^i(x,y)y^i - G^j(x,y)y^i \tag{2.1}$$

are homogeneous polynomials in (y^i) of degree three.

Further differentiating (2.1) by y^m and contracting m and j in the obtained equation, we have

$$D^{im}_m = (n+1)G^i - G^m_m y^i. \tag{2.2}$$

Thus F^n is said to be a Douglas space of the second kind if and only if (2.2) are homogeneous polynomials in (y^i) of degree two.

Definition 2.1. *If a Finsler space F^n is said to be Douglas space of second kind if it satisfies the condition that $D^{im}_m = (n+1)G^i - G^m_m y^i$ be homogeneous polynomials in (y^i) of degree two.*

A Finsler metric $L(x,y)$ is called an (α,β)-metric, where L is a positively homogeneous function $L(\alpha,\beta)$ of degree one in two variables $\alpha(x,y) = \sqrt{a_{ij}(x)y^i y^j}$ and $\beta = b_i(x)y^i$. The space $R^n = (M^n, \alpha)$ is called the associated Riemannian space with F^n. We use the following symbols [8]:

$$\begin{aligned} r_{ij} &= \frac{1}{2}(b_{i;j}) + b_{j;i}, \quad s_{ij} = \frac{1}{2}(b_{i;j} - b_{j;i}), \\ s^i_j &= a^i r s_{rj}, \quad s_j = b_r s^r_j. \end{aligned}$$

Further we have a Finsler space of an (α,β)-metric is said to be a Douglas space of the second kind if and only if

$$B^{im}_m = (n+1)B^i - B^m_m y^i, \tag{2.3}$$

are homogeneous polynomials in (y^i) of degree two, where B^m_m is given by [5]. Furthermore differentiating the above with respect to y^h, y^j and y^k, we get

$$B^{im}_{hjkm} = B^i_{hjk} = 0.$$

Thus, we have

Definition 2.2. *If a Finsler space F^n with (α,β)-metric is said to be Douglas space of second kind if it satisfies the condition that $B^{im}_m = (n+1)B^i - B^m_m y^i$ be homogeneous polynomials in (y^i) of degree two.*

3. Douglas space of second kind with (α, β)-metric

In this section, we deal with the condition that a Finsler space with an (α, β)-metric be a Douglas space of the second kind.

Now we consider the function $G^i(x, y)$ of F^n with an (α, β)-metric. According to [7], $G^i(x, y)$ can be written as

$$\begin{aligned} 2G^i &= \gamma^i_{00} + 2B^i, \\ B^i &= (\alpha L_\beta / L_\alpha) s^i_0 + C^* \left\{ \frac{\beta L_\beta}{\alpha L} y^i - \frac{\alpha L_{\alpha\alpha}}{L_\alpha} \left(\frac{1}{\alpha} y^i - \frac{\alpha}{\beta} b^i \right) \right\}, \end{aligned} \tag{3.1}$$

where we put

$$\begin{aligned} C^* &= \frac{\alpha\beta(r_{00} L_\alpha - 2\alpha s_0 L_\beta)}{2(\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha})}, \\ \gamma^2 &= b^2\alpha^2 - \beta^2. \end{aligned} \tag{3.2}$$

Since $\gamma^i_{00} = \gamma^i_{jk}(x) y^i y^j$ are homogeneous polynomials in (y^i) of degree two, equation (3.1) yields

$$B^{ij} = \frac{\alpha L_\beta}{L_\alpha}(s^i_0 y^j - s^j_0 y^i) + \frac{\alpha^2 L_{\alpha\alpha}}{\beta L_\alpha} C^* (b^i y^j - b^j y^i). \tag{3.3}$$

by means of (2.1) and (3.3), we have the following lemma[9]:

Lemma 3.1. *A Finsler space F^n with an (α, β)-metric is a Douglas space if and only if $B^{ij} = B^i y^j - B^j y^i$ are hp(3).*

Further differentiating (3.3) by y^m and contracting m and j in the obtained equation, we obtain

$$\begin{aligned} B^{im}_m = {} & \frac{(n+1)\alpha L_\beta}{L_\alpha} s^i_0 + \frac{\alpha\{(n+1)\alpha^2 \Omega L_{\alpha\alpha} b^i + \beta\gamma^2 A y^i\}}{2\Omega^2} r_{00} \\ & - \frac{\alpha^2\{(n+1)\alpha^2 \Omega L - \beta L - \alpha\alpha b^i + B y^i\}}{L_\alpha \Omega^2} s_0 - \frac{\alpha^3 L_{\alpha\alpha} y^i}{\Omega} r_0, \end{aligned} \tag{3.4}$$

where, we put

$$\begin{aligned} \Omega &= (\beta^2 l_\alpha + \alpha\gamma^2 L_{\alpha\alpha}), \\ A &= \alpha L_\alpha L_{\alpha\alpha\alpha} + 3L_\alpha L_{\alpha\alpha} - 3\alpha(L_{\alpha\alpha})^2, \\ B &= \alpha\beta\gamma^2 L_\alpha L_\beta L_{\alpha\alpha\alpha} + \beta\{(3\gamma^2 - \beta^2)L - \alpha - 4\alpha\gamma^2 L_{\alpha\alpha}\} L_\beta L_{\alpha\alpha} + \Omega L L_{\alpha\alpha}. \end{aligned} \tag{3.5}$$

We use the following result [5];

Theorem 3.2. *The necessary and sufficient condition for a Finsler space F^n with an (α, β)-metric to be a Douglas space of the second kind is that, B^{im}_m are homogeneous polynomials in (y^m) of degree two, where B^{im}_m is given by (3.4) and (3.5), provided that $\Omega \neq 0$.*

4. Conformal change of Douglas space of second kind with (α, β)-metric

In the present section, we derive the condition on conformal change, so that a Douglas space of second kind is conformally transformed to a Douglas space of second kind .

Let $F^n = (M^n, L)$ and $\overline{F}^n = (M^n, \overline{L})$ be two Finsler space on the same underlying manifold M^n. If we have a function $\sigma(x)$ in each coordinate neighbourhoods of M^n such that $\overline{L}(x,y) = e^\sigma L(x,y)$, then F^n is called conformal to $\overline{F}^n$ and the change $L \to \overline{L}$ of metric is called conformal change.

As to (α,β)-metrics, $\overline{L} = e^\sigma L(\alpha,\beta)$, is equivalent to $\overline{L} = L(e^\sigma\alpha, e^\sigma\beta)$ by homogeneity. Therefore, a conformal change of (α,β)-metric is expressed as $(\alpha,\beta) \to (\overline{\alpha},\overline{\beta})$, where $\overline{\alpha} = e^\sigma\alpha$, $\overline{\beta} = e^\sigma\beta$. Therefore, we have

$$\begin{aligned} &\overline{a}_{ij} = e^{2\sigma}a_{ij}, \quad \overline{b}_i = e^\sigma b_i \\ &\overline{a}_{ij} = e^{-2\sigma}a^{ij}, \quad \overline{b}^i = e^{-\sigma}b^i \end{aligned} \tag{4.1}$$

and $b^2 = a^{ij}b_ib_j = \overline{a}^{ij}\overline{b}_i\overline{b}_j$. Thus we state the following:

Proposition 4.1. *A Finsler space with (α,β)-metric the length b of b_i with respect to the Riemannian metric α is invariant under any conformal change of (α,β)-metric.*

From (4.1), it follows that the conformal change of Cristoffel symbols is given by [4];

$$\overline{\gamma}^i_{jk} = \gamma^i_{jk} + \delta^i_j\sigma_k + \delta^i_k\sigma_j - \sigma^i a_{jk}, \tag{4.2}$$

where $\sigma_j = \partial_j\sigma$ and $\sigma^i = a^{ij}\sigma_j$.

From (4.1) and (4.2), we have the following identities:

$$\begin{aligned} \overline{\nabla}_j\overline{b}_i &= e^\sigma(\nabla_j b_i + \rho a_{ij} - \sigma_i b_j), \\ \overline{r}_{ij} &= e^\sigma[r_{ij} + \rho a_{ij} - \frac{1}{2}(b_i\sigma_j + b_j\sigma_i)], \quad \overline{s}_{ij} = e^\sigma[s_{ij} + \frac{1}{2}(b_i\sigma_j - b_j\sigma_i)], \quad (4.3) \\ \overline{s}^i_j &= e^{-\sigma}[s^i_j + \frac{1}{2}(b^i\sigma_j - b_j\sigma^i)], \qquad \overline{s}_j = s_j + \frac{1}{2}(b^2\sigma_j - \rho b_j), \end{aligned}$$

where $\rho = \sigma_r b^r$.

From (4.2) and (4.3), we can easily obtain the following:

$$\begin{aligned} \overline{\gamma}^i_{00} &= \gamma^i_{00} + 2\sigma_0 y^i - \alpha^2\sigma_j, \qquad \overline{r}_{00} = e^\sigma(r_{00} + \rho\alpha^2 - \sigma_0\beta), \quad (4.4) \\ \overline{s}^i_0 &= e^{-\sigma}[s^i_0 + \frac{1}{2}(\sigma s_0 b^i - \beta\sigma^i)], \quad \overline{s}_0 = s_0 + \frac{1}{2}(\sigma_0 b^i - \rho\beta). \end{aligned}$$

Next, we find the conformal change of B^{ij} given in (3.3), since $\overline{L}(\alpha,\beta) = e^\sigma L(\alpha,\beta)$, and

$$\overline{L}_{\overline{\alpha}} = L_\alpha, \quad \overline{L}_{\overline{\alpha}\ \overline{\alpha}} = e^{-\sigma}L_{\alpha\alpha}, \quad \overline{L}_{\overline{\beta}} = L_\beta, \quad \overline{\gamma}^2 = e^{2\sigma}\gamma^2. \tag{4.5}$$

By using (3.2), (4.4), (4.5) and lemma (3.1), we obtain

$$\overline{C}^* = e^\sigma(C^* + D^*),$$

where

$$D^* = \frac{\alpha\beta[(\beta\alpha^2 - \sigma_0\beta)L_\alpha - \alpha(b^2\sigma_0 - \rho\beta)L_\beta]}{2(\beta^2 L_\alpha + \alpha r^2 L_{\alpha\alpha})}.$$

Hence under the conformal change B^{ij} can be written as:

$$\begin{aligned} \overline{B}^{ij} &= \frac{\alpha L_\beta}{L_\alpha}(s^i_0 y^j - s^j_0 y^i) + \frac{\alpha^2 L_{\alpha\alpha}}{\beta L_\alpha}C^*(b^i y^j - b^j y^i) \\ &\quad + \left(\frac{\alpha\sigma_0 L_\beta}{L_\alpha} + \frac{\alpha^2 L_{\alpha\alpha}}{\beta L_\alpha}D^*\right)(b^i y^j - b^j y^i) - \frac{\alpha\beta L_\beta}{2L_\alpha}(\sigma^i y^j - \sigma^j y^i), \\ &= B^{ij} + C^{ij}, \end{aligned}$$

where

$$C^{ij} = \left(\frac{\alpha\sigma_0 L_\beta}{L_\alpha} + \frac{\alpha^2 L_{\alpha\alpha}}{\beta L_\alpha} D^*\right)(b^i y^j - b^j y^i) - \frac{\alpha\beta L_\beta}{2L_\alpha}(\sigma^i y^j - \sigma^j y^i).$$

From the equation (3.5), it is clear that

$$\overline{\Omega} = e^{2\sigma}\Omega, \quad \overline{A} = e^{-\sigma}A, \quad \overline{B} = e^{2\sigma}B.$$

Now we apply conformal transformation to B^{im}_m, we obtain

$$\overline{B}^{im}_m = B^{im}_m + K^{im}_m, \tag{4.6}$$

where

$$\begin{aligned} 2K^{im}_m &= \frac{(n+1)\alpha L_\beta}{L_\alpha}(\sigma_0 b^i - \beta\sigma^i) + \alpha\left\{\frac{(n+1)\alpha^2\Omega L_{\alpha\alpha} b^i + \beta r^2 A y^i}{\Omega^2}\right\}(\rho\alpha^2 - \sigma_0\beta) \\ &- \left[\frac{\alpha^2\{(n+1)\alpha^2\Omega L_\beta L_{\alpha\alpha} b^i + B y^i\}}{L_\alpha \Omega^2} - \frac{\alpha^3 l_{\alpha\alpha} y^i}{\Omega}\right](b^2\sigma_0 - \rho\beta). \end{aligned} \tag{4.7}$$

Thus, we state the following:

Theorem 4.2. *The necessary and sufficient condition for a conformal change of Douglas space of the second kind of Finsler space to be a Douglas space of the second kind, is that $K^{im}_m(x)$ are homogenous polynomial in (y^m) of degree two.*

5. Examples

In this section , we consider two important examples of (α,β)-metric such as Randers metric $L = \alpha + \beta$ and Kropina metric $L = \frac{\alpha^2}{\beta}$:

5.1. **Randers metric.** Consider the Randers metric, i.e., $L = \alpha + \beta$, so that $L_\alpha = L_\beta = 1$ and $L_{\alpha\alpha} = L_{\alpha\alpha\alpha} = 0$. Hence, from (4.7) we get

$$K^{im}_m = \frac{n+1}{2}\alpha(\sigma_0 b^i - \beta\sigma^j). \tag{5.1}$$

Since α is irrational function in y^i from (5.1) it follows that K^{im}_m is $hp(2)$ if and only if

$$\sigma_0 b^i - \beta\sigma^i = 0, \quad K^{im}_m = 0.$$

From the above equation we have $\sigma_k b^i - b_k\sigma^i = 0$ i.e., $b_i\sigma_j - b_j\sigma_i = 0$ which gives $\sigma_i = \frac{\rho}{b^2}b_i$.

Conversely if $\sigma_i = \frac{\rho}{b^2}b^i$, then $\sigma_0 = \frac{\rho}{b^2}\beta$ and (5.1) will give $K^{im}_m = 0$. Hence, (4.6) gives $\overline{B}^{im}_m = B^{im}_m$. Hence, we state the following

Theorem 5.1. *The Douglas space of second kind with Randers metric is conformally transformed to a Douglas space of second kind if and only if $\sigma_i = \frac{\rho}{b^2}b_i$.*

5.2. **Kropina metric.** In case, we consider Kropina metric i.e., $L = \frac{\alpha^2}{\beta}$, so that $L_\alpha = \frac{2\alpha}{\beta}$, $\quad L_{\alpha\alpha} = \frac{2}{\beta}$, $\quad L_{\alpha\alpha\alpha} = 0$ and $L_\beta = -\frac{\alpha^2}{\beta^2}$. Hence, from (3.5) we have

$$\Omega = 2\frac{b^2\alpha^3}{\beta}, \quad A = 0, \quad B = 8\frac{b^2\alpha^5}{\beta^3}.$$

Thus, from (4.7), K^{im}_m is reduces to

$$K^{im}_m = \frac{n+1}{2}\left(\frac{b^i}{b^2}\rho\alpha^2 + \alpha^2\sigma^i - \frac{b^i}{b^2}\sigma_i y^i\beta\right), \tag{5.2}$$

which shows that K_m^{im} is $hp(2)$. Hence

Theorem 5.2. *The Douglas space of second kind with Kropina metric is conformally transformed to a Douglas space of second kind.*

6. Conclusion

The theories of Finsler space with (α, β)-metric have contributed to the development of Finsler geometry [8], and Douglas space with (α, β)-metric have been treated by some authors [2], [5]. Conformal change is one of the important transforms which preserves the angle and this theory is developed in 1929 and studied by many author.

The present work includes a brief presentation of the Finsler space of Douglas type, the properties of Douglas space of second kind is mainly based on the work of Il Young Lee. Here we present the condition that the conformally changed Finsler space with an (α, β)-metric be a Douglas space of second kind. Finally we able to give example for conformally changed Douglas space of second kind, as Randers metric and Kropina metric.

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